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ANALYTICAL MODEL FOR REGULAR DENSE ARRAYS OF PLANAR DIPOLE SCATTERERS

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Abstract—An analytical boundary condition for modeling the electromagnetic properties of planar regular dense arrays of dipole particles for oblique incidence of plane waves is developed. The regular array is assumed to be dense which means that the dipole particles are close to each other. The interaction between the dipole particles is taken into account by interaction constant. The expression for the interaction constant is written in analytical form and is used for developing a transmission-line model for arrays of planar dipole scatterers. The regular dense array is modeled as a shunt impedance which is different for TM and TE polarizations.

1. INTRODUCTION

Impedance boundary conditions have been used in many applications in electromagnetics such as considering multilayered structures, thin dielectric layers and surfaces [1, 2]. Numerical methods are used to study reflection properties of arrays of resonant particles [3–5]. In such complicated structures effective impedance boundary conditions which connect the averaged electric field and the averaged current simplify calculations. In this study an analytical model for dense regular dipole arrays is developed. In the calculation of the local field the approximate analytical full-wave theory for dense array is used. Analytical approximation of the local field leads to a simple boundary condition [6].

The reflection and transmission properties of plane waves in planar regular arrays of dipoles are studied. The distance between dipoles is denoted as a, and the dipoles are on a plane forming a planar dipole array. The reflection and transmission coefficients for oblique incident plane waves are considered. It is assumed that the distance between inclusions a is small enough compared to the wave length, thus, we deal with a dense array. Every particle is characterized by the polarizability dyadic $\overline{\overline{\alpha}}$. The dipole moment is obtained through the polarizability dyadic and the local electric field as

$$\mathbf{p} = \overline{\overline{\alpha}} \cdot \mathbf{E}_{\text{loc}} \tag{1}$$

The interaction between dipoles is taken into account as follows. The local field is a sum of the incident and interaction fields

$$\mathbf{E}_{\text{loc}} = \mathbf{E}_{\text{ext}} + \overline{\overline{\beta}} \cdot \mathbf{p} \tag{2}$$

where $\mathbf{E}_{\text{ext}}$ is the external field, and the interaction dyadic $\overline{\overline{\beta}}$ takes into account the effect of the other dipoles. In the local field $\mathbf{E}_{\text{loc}}$ which excites the reference dipole, the contribution from the distant dipoles and from near-by dipoles are separated. The effect of the distant dipoles located at distances larger than R is calculated through averaging. The distance R is written as $R = R_o = a/1.438$, which is a special number obtained from the quasi static value of the interaction field [7]. Then there are no individual particles except the reference particle inside the circle of radius R_o, as shown in Figures 1 and 2. For the normal incidence this interaction dyadic is proportional to the two-dimensional unit dyadic $\overline{\overline{I}}_t$: $\overline{\overline{\beta}} = \beta \overline{\overline{I}}_t$, where [6]

$$\beta \approx -j \frac{\omega}{S_o} \frac{\eta}{4} \left(1 - \frac{1}{jkR_0}\right) e^{-jkR_0} \tag{3}$$

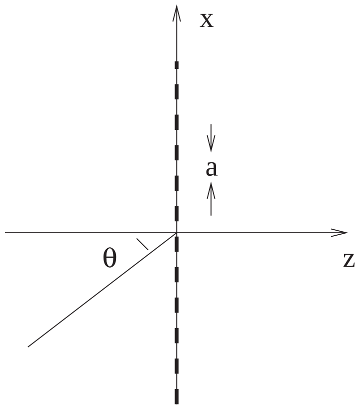

Figure 1. Planar dipole array, oblique plane wave incidence.

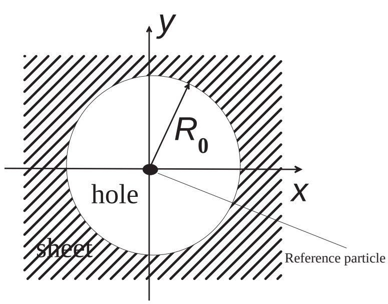

Figure 2. On the calculation of the local field, the effect of other dipoles are taken into account as averaged sheet.

with $\eta = \sqrt{\mu_0/\epsilon_0}$, $k = \omega\sqrt{\epsilon_0\mu_0}$ and $S_o = a^2$.

In this paper we consider a more general case, namely, obliquely incident plane waves. For planar dipole arrays the interaction dyadic is written in terms of parallel (TM) and perpendicular (TE) polarization. The imaginary parts of the polarizability factor and the interaction constant obey a certain relation which is obtained by considering the energy conservation requirement [8].

2. THEORY

Let us consider an array of dipole particles which are located on xy-plane (for example, disks or wire line dipoles on the xy-plane). Because the distance between the dipole particles is small compared to wave length, $ka < 1$, there is strong coupling between the dipoles. The interaction caused by all the other dipoles is taken into account through the interaction constant which for oblique incident field is different for TM and TE polarizations. For oblique incidence the interaction dyadic for a planar array is written in the form

$$\overline{\overline{\beta}} = \beta_{\|} \frac{\mathbf{KK}}{K^2} + \beta_{\perp} \frac{(\mathbf{u}_z \times \mathbf{K})(\mathbf{u}_z \times \mathbf{K})}{K^2} \tag{4}$$

where the vector $\mathbf{K}$ is the component of the wave vector on the array plane. The interaction constants for these two eigenpolarizations are obtained through integration; the effect of the other dipoles on array to the reference dipole are taken into account as averaging. The local field caused by the other dipoles is [7]

$$\mathbf{E}_{\text{loc}} = \frac{1}{4\pi\epsilon S_o} \int_R^{\infty} \int_0^{2\pi} \left[k^2 (\mathbf{n} \times \mathbf{p}) \times \mathbf{n} + [3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}] \left(\frac{1}{r^2} + \frac{jk}{r} \right) \right] e^{-jkr} e^{-jKr\cos\varphi}\, d\varphi dr \tag{5}$$

where $\mathbf{n} = \cos\varphi \mathbf{u}_x + \sin\varphi \mathbf{u}_y$ and $\mathbf{p} = p_x \mathbf{u}_x + p_y \mathbf{u}_y$. The x and y-components ($\mathbf{u}_x = \frac{\mathbf{K}}{K}$ and $\mathbf{u}_y = \frac{\mathbf{u}_z \times \mathbf{K}}{K}$) of the interaction dyadic can be identified from the above integral expression. Writing this expression as in (2) and (4), and omitting the terms with $\int_0^{2\pi} \sin\varphi \cos\varphi d\varphi = 0$, the interaction constants are

$$\beta_{\|} = \frac{1}{S_o 4\pi\epsilon} \int_R^{\infty} \int_0^{2\pi} \left[k^2 \sin^2\varphi + (3\cos^2\varphi - 1) \left(\frac{1}{r^2} + \frac{jk}{r} \right) \right] e^{-jkr} e^{-jKr\cos\varphi} dr d\varphi \tag{6}$$

and

$$\beta_{\perp} = \frac{1}{S_o 4\pi\epsilon} \int_R^{\infty} \int_0^{2\pi} \left[k^2 \cos^2\varphi + (3\sin^2\varphi - 1) \left(\frac{1}{r^2} + \frac{jk}{r} \right) \right] e^{-jkr} e^{-jKr\cos\varphi} dr d\varphi \tag{7}$$

Writing $\sin^2\varphi = \frac{1}{2}[1 - \cos 2\varphi]$ and $\cos^2\varphi = \frac{1}{2}[1 + \cos 2\varphi]$ and using the integral identity [9]

$$\int_0^{2\pi} e^{-j\gamma\cos x} \cos nx\, dx = (-j)^n 2\pi J_n(\gamma) \tag{8}$$

these double integral expressions are reduced to one-dimensional integral forms:

$$\begin{aligned}\beta_{\|} &= \frac{1}{4\pi\epsilon S_o}\int_R^{\infty}\pi\left[\left(k^2+\frac{1}{r^2}+\frac{jk}{r}\right)J_0(Kr)\right.\\ &\qquad\left.+\left(k^2-\frac{j3k}{r}-\frac{3}{r^2}\right)J_2(Kr)\right]e^{-jkr}dr \qquad (9)\end{aligned}$$

and

$$\begin{aligned}\beta_{\perp} &= \frac{1}{4\pi\epsilon S_o}\int_R^{\infty}\pi\left[\left(k^2+\frac{jk}{r}+\frac{1}{r^2}\right)J_0(Kr)\right.\\ &\qquad\left.-\left(k^2-\frac{j3k}{r}-\frac{3}{r^2}\right)J_2(Kr)\right]e^{-jkr}dr \qquad (10)\end{aligned}$$

After that, integrating by parts and using the identities for the Bessel functions with index $n = 0,\ 1,\ 2$

$$\frac{2nJ_n(x)}{x} = J_{n-1}(x)+J_{n+1}(x), \quad 2J'_n(x) = J_{n-1}(x)-J_{n+1}(x) \qquad (11)$$

leads finally after some algebra (see Appendix A) to the following quite simple-looking expressions

$$\begin{aligned}\beta_{\|} &= \frac{1}{2\epsilon S_o}\left[\left(-\left(\frac{1}{R}+jk\right)\frac{J_1(KR)}{KR}+\frac{J_0(KR)}{R}\right)e^{-jkR}\right.\\ &\qquad\left.-K^2\int_R^{\infty}\frac{J_1(Kr)}{Kr}e^{-jkr}dr\right] \qquad (12)\end{aligned}$$

and

$$\begin{aligned}\beta_{\perp} &= \frac{1}{2\epsilon S_o}\left[\left(\left(\frac{1}{R}+jk\right)\frac{J_1(KR)}{KR}-jkJ_0(KR)\right)e^{-jkR}\right.\\ &\qquad\left.+jkK\int_R^{\infty}J_1(Kr)e^{-jkr}dr\right] \qquad (13)\end{aligned}$$

Part of the remaining integrals can be written in closed form [9]

$$\int_R^{\infty}\frac{J_1(Kr)}{Kr}e^{-jkr}dr = -\frac{j}{k+\sqrt{k^2-K^2}}-\int_0^{R}\frac{J_1(Kr)}{Kr}e^{-jkr}dr \qquad (14)$$

and

$$\int_R^\infty J_1(Kr)e^{-jkr}dr = -\frac{K}{\sqrt{k^2-K^2}(k+\sqrt{k^2-K^2})} - \int_0^R J_1(Kr)e^{-jkr}dr \tag{15}$$

The remaining integrals are calculated approximatively. Because Kr is small in the integration range ($R = a/1.438$ and $K < k$ and $ka < 1$) the integrands can be approximated as $\frac{J_1(Kr)}{Kr} \approx \frac{1}{2}$ and $J_1(Kr) \approx \frac{Kr}{2}$, then

$$\int_0^R \frac{J_1(Kr)}{Kr} e^{-jkr}dr \approx \frac{1}{j2k}\left(1-e^{-jkR}\right) \tag{16}$$

and

$$\int_0^R J_1(Kr)e^{-jkr}dr \approx \frac{K}{2k^2}\left[(1+jkR)\,e^{-jkR}-1\right] \tag{17}$$

Finally, the interaction constants are

$$\begin{aligned}\beta_{\parallel} \;=\; & -\frac{jk}{2\epsilon S_o}\left[\left(\frac{J_1(KR)}{KR} - \frac{1}{jkR}\left[J_0(KR)-\frac{J_1(KR)}{KR}\right] - \frac{K^2}{2k^2}\right)e^{-jkR}\right.\\ & \left. -\frac{K^2}{k(k+\sqrt{k^2-K^2})} + \frac{K^2}{2k^2}\right]\end{aligned} \tag{18}$$

and

$$\begin{aligned}\beta_{\perp} \;=\; & -\frac{jk}{2\epsilon S_o}\left[\left(\left[J_0(KR)-\frac{J_1(KR)}{KR}\right]-\frac{1}{jkR}\frac{J_1(KR)}{KR}+\frac{K^2}{2k^2}(1+jkR)\right)\right.\\ & \left. e^{-jkR} + \frac{K^2}{\sqrt{k^2-K^2}(k+\sqrt{k^2-K^2})} - \frac{K^2}{2k^2}\right]\end{aligned} \tag{19}$$

It is easily seen that for normal incidence, $K \to 0$, both of these expressions reduce to expression (3). Another approximate expression for integrals (6), (7) was given in [10].

3. IMAGINARY PART OF $\overline{\overline{\beta}}$

As was shown in [6] and [8], for the normal incidence the imaginary part of the polarizability factor and the interaction constant obey a certain relation based on the power conservation requirements. The imaginary part of the interaction coefficients are considered in detail:

$$\begin{aligned}\mathrm{Im}\{\beta_\parallel\} &= -\frac{k}{2S_o\epsilon}\left[\frac{J_1(KR)}{KR}\cos kR + \left(J_o(KR) - \frac{J_1(KR)}{KR}\right)\frac{\sin kR}{kR}\right. \\ &\left. \quad -\frac{K^2}{k(k+\sqrt{k^2-K^2})} + \frac{K^2}{2k^2}(1-\cos kR)\right] \end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathrm{Im}\{\beta_\perp\} = -\frac{k}{2S_o\epsilon}&\left[\left(J_o(KR) - \frac{J_1(KR)}{KR}\right)\cos kR + \frac{J_1(KR)}{KR}\frac{\sin kR}{kR}\right. \\ &\left. +\frac{K^2}{\sqrt{k^2-K^2}(k+\sqrt{k^2-K^2})} - \frac{K^2}{2k^2}(1-\cos kR) + \frac{K^2}{2k^2}kR\sin kR\right] \end{aligned} \tag{21}$$

Next we denote $\cos\theta = \sqrt{k^2-K^2}/k$ and $K = k\sin\theta$. In the low frequency limit, $k \to 0$, these coefficients reduce to the form

$$\begin{aligned}\mathrm{Im}\{\beta_\parallel\} &= -\frac{k}{2S_o\epsilon}\left[\cos\theta - \left(\frac{3\cos^2\theta+1}{12} + \frac{\sin^2\theta}{8}\right)(kR)^2\right] \\ &= \left(\frac{3\cos^2\theta+5}{48\pi}\right)\frac{\pi R^2}{S_o}\eta\mu\epsilon\omega^3 - \frac{Z^{TM}\omega}{2S_o} \end{aligned} \tag{22}$$

and

$$\begin{aligned}\mathrm{Im}\{\beta_\perp\} &= -\frac{k}{2S_o\epsilon}\left[\frac{1}{\cos\theta} - \left(\frac{3\cos^2\theta+1}{12} + \frac{\sin^2\theta}{8}\right)(kR)^2\right] \\ &= \left(\frac{3\cos^2\theta+5}{48\pi}\right)\frac{\pi R^2}{S_o}\eta\mu\epsilon\omega^3 - \frac{Z^{TE}\omega}{2S_o} \end{aligned} \tag{23}$$

where the expressions for the wave impedances $Z^{TM} = \eta\cos\theta$ and $Z^{TE} = \eta/\cos\theta$ are used. One can observe that the first term, which is same for both polarizations, describes the dipole radiation in array for oblique incidence. The second term is a plane wave term for TM and TE polarizations, respectively.

The whole array radiates plane waves whose eigenvectors are $\frac{\mathbf{K}}{K}$ and $\frac{\mathbf{u}_z\times\mathbf{K}}{K}$. The planar polarizability dyadic is written in terms of the eigenvectors (here it is assumed that there is no coupling between the eigenpolarizations)

$$\overline{\overline{\alpha}} = \alpha_\parallel\frac{\mathbf{KK}}{K^2} + \alpha_\perp\frac{(\mathbf{u}_z\times\mathbf{K})(\mathbf{u}_z\times\mathbf{K})}{K^2} \tag{24}$$

Assuming that there is no absorption in the particles, every particle in the array radiates the same power as spent by the incident field on its excitation [8]. Based on this power conservation, the following relations between the imaginary parts should be valid

$$\mathrm{Im}\left\{\alpha_{\|}^{-1}-\beta_{\|}\right\}=\frac{Z^{TM}\omega}{2S_o},\qquad \mathrm{Im}\left\{\alpha_{\perp}^{-1}-\beta_{\perp}\right\}=\frac{Z^{TE}\omega}{2S_o} \tag{25}$$

This kind of relations written in terms of TM and TE eigenvectors are analogous to those for non-reciprocal inclusions for the normal incidence when written by using circularly polarized eigenvectors [8].

4. IMPEDANCE AND REFLECTION DYADIC

The total averaged electric field in the plane of a dipole array is the sum of the incident field and the plane-wave field created by the averaged current $\mathbf{J}$:

$$<\mathbf{E}_{\mathrm{total}}>=\mathbf{E}_{\mathrm{ext}}-\frac{\eta}{2}\mathbf{J} \tag{26}$$

With the polarizability dyadic $\overline{\overline{\alpha}}$ and the interaction dyadic $\overline{\overline{\beta}}$ we can write the total field in terms of the averaged current density. Writing the current density on the array as $\mathbf{J}=j\omega\mathbf{p}/a^2$, one obtains $<\mathbf{E}_{\mathrm{total}}>=\overline{\overline{Z}}_s\cdot\mathbf{J}$, where $\overline{\overline{Z}}_s$ is the shunt impedance dyadic of a regular array of dipoles. For the normal incidence with an isotropic polarizability of inclusions the impedance reads

$$\overline{\overline{Z}}_s=\frac{a^2}{j\omega}\mathrm{Re}\left\{\frac{1}{\alpha}-\beta\right\}\overline{\overline{I}}_t \tag{27}$$

For oblique incidence the shunt impedances are different for perpendicular (TE) and parallel (TM) polarizations.

The imaginary parts fulfill certain conditions based on the power conservation. Considering the real parts the impedance model for the dipole array is obtained. The real part of the interaction constants for two polarizations are

$$\begin{aligned}\mathrm{Re}\left\{\beta_{\|}\right\} &= \frac{k}{2\epsilon S_o}\left[\left(J_0(KR)-\frac{J_1(KR)}{KR}\right)\frac{\cos kR}{kR}\right.\\ &\quad\left.-\left(\frac{J_1(KR)}{KR}-\frac{K^2}{2k^2}\right)\sin kR\right]\end{aligned} \tag{28}$$

and

$$\mathrm{Re}\left\{\beta_{\perp}\right\} = \frac{k}{2\epsilon S_o}\left[\left(\frac{J_1(KR)}{KR}+\frac{K^2R^2}{2}\right)\frac{\cos kR}{kR}\right.$$

$$-\left(J_0(KR)-\frac{J_1(KR)}{KR}+\frac{K^2}{2k^2}\right)\sin kR\Bigg] \tag{29}$$

For perpendicular polarization the interaction coefficient at low frequencies can be compared with the accurate expression given in [7, p. 784]. The value of this model is

$$\beta_\perp \rightarrow \frac{1}{\pi\epsilon a^3}\left[\frac{\pi a}{4R_o}-\frac{12+\sin^2\theta}{16}\frac{k^2a^2}{1.438}+j\left(-\frac{ka}{\cos\theta}+\frac{3\cos^2\theta+5}{24}\frac{k^3a^3}{1.438^2}\right)\right] \tag{30}$$

In [7] the expression contains infinite double summations. Taking the first term in the infinite double sum, the expression for the interaction constant is

$$C \rightarrow \frac{1}{\pi\epsilon a^3}\left[\sum_{m=1}^{\infty}\frac{1}{m^3}-8\pi^2K_0(2\pi)-0.7224\,k^2a^2+j\left(-\frac{ka}{2\cos\theta}+\frac{k^3a^3}{6}\right)\right] \tag{31}$$

The two first terms together form the static interaction constant in planar infinite arrays of dipoles (definition of R_o) from which the value $R_o = a/1.438$ was obtained [7, p. 758, 784]. So, in the proposed interaction constant and in the expression in [7] the zeroth and first order terms are the same. In the second and third order terms there are small differences caused by that here planar dipole scatterers are considered, in [7] isotropic dipole array is considered.

The essential part in the expression for the impedance of the array comes from the real parts of the polarizability dyadic and the interaction dyadic

$$Z_{s\|} = \frac{a^2}{j\omega}\mathrm{Re}\{\alpha_\|^{-1}-\beta_\|\}, \qquad Z_{s\perp} = \frac{a^2}{j\omega}\mathrm{Re}\{\alpha_\perp^{-1}-\beta_\perp\} \tag{32}$$

In the lossless case the impedance values are imaginary.

As a specific example, let us consider an array of small circular conducting disks on xy-plane. The polarizability factor of a metal disk is $\alpha = \frac{16}{3}\epsilon_o r^3$ (a more accurate expression is $\alpha = \frac{16}{3}\epsilon_o r^3[1+\frac{k^2r^2}{15}]$) [7]. By using the expression for the impedance (32), the shunt impedance of a regular array of circular disks is written for the two polarizations as

$$\begin{aligned} Z_{s\|} \;=\; & \frac{\eta}{jka}\left[\frac{3}{16}\frac{a^3}{r^3}-\frac{a}{2R}\left[\left(J_0(KR)-\frac{J_1(KR)}{KR}\right)\cos kR\right.\right. \\ & \left.\left. -\;kR\left(\frac{J_1(KR)}{KR}-\frac{K^2}{2k^2}\right)\sin kR\right]\right] \end{aligned} \tag{33}$$

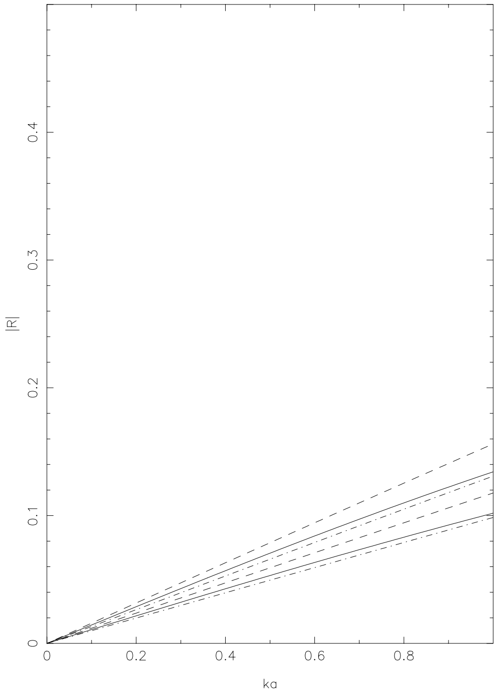

Figure 3. Reflection from the regular circular disk array for TE and TM polarizations, $r = 0.35a$ and $\theta = \pi/6$.

and

$$\begin{aligned} Z_{s\perp} &= \frac{\eta}{jka}\left[\frac{3}{16}\frac{a^3}{r^3} - \frac{a}{2R}\left[\left(\frac{J_1(KR)}{KR} + \frac{K^2R^2}{2}\right)\cos kR \right.\right. \\ &\qquad \left.\left. - kR\left(J_0(KR) - \frac{J_1(KR)}{KR} + \frac{K^2}{2k^2}\right)\sin kR\right]\right] \end{aligned} \tag{34}$$

In a quasistatic limit, $\omega \to 0$, the shunt impedance reduces to $Z_s = \frac{\eta}{jka}[\frac{3}{16}\frac{a^3}{r^3} - 0.36]$ for both polarizations. Naturally, the shunt impedance is capacitive. The reflection coefficients are obtained by using the transmission-line model. The array is seen as a capacitive shunt impedance between two transmission lines. The reflection coefficients for transverse fields are

$$R_\parallel = -\frac{1}{1 + \dfrac{2Z_{s\parallel}}{\eta\cos\theta}}, \quad R_\perp = -\frac{1}{1 + \dfrac{2Z_{s\perp}}{\eta/\cos\theta}} \tag{35}$$

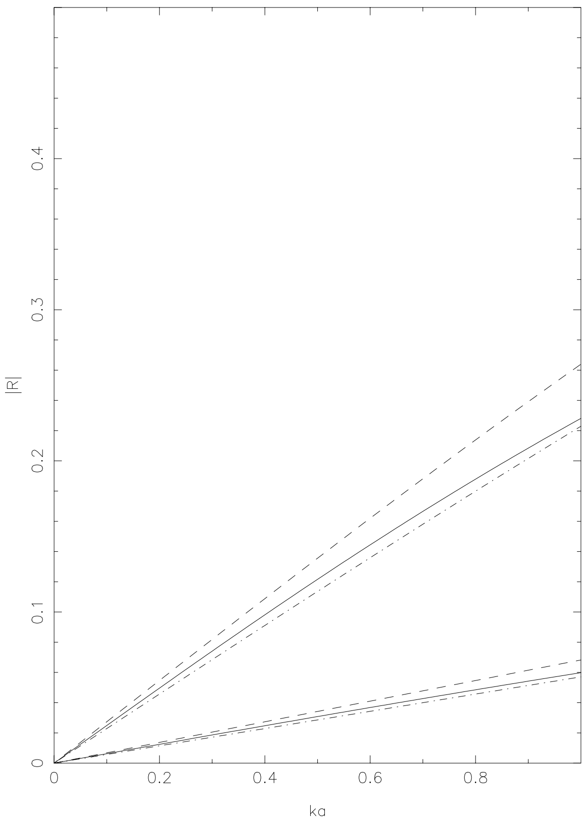

Figure 4. Reflection from the regular circular disk array for TE and TM polarizations, $r = 0.35a$ and $\theta = \pi/3$.

The transmission dyadic is $\overline{\overline{T}} = \overline{\overline{I}}_t + \overline{\overline{R}}$. In Figures 3 and 4 the reflection coefficients are illustrated as a function of the normalized period ka with certain fixed angles. In these examples the radius of the disk is $r = 0.35\,a$. Reflection for TE polarization is stronger than the reflection for TM polarization. In these Figures the effect of interaction is shown; the solid lines illustrate the reflection coefficient with the dynamic interaction, dashed line illustrate the reflection when taking quasistatic interaction and dotted dashed lines when neglecting the interaction between the dipoles in the array. It is seen that the interaction increases the reflection from the array, however, the quasistatic model for the dipole array seems to give too large values for the reflected fields.

5. CONCLUSION

For oblique incidence of plane waves the equivalent model is different for the two eigenpolarizations, perpendicular (TE) and parallel (TM) polarizations. In this case there is no coupling between TE and TM polarization. Finally, the suggested procedure leads to a model where the regular dense array is given as a shunt impedance, different for the two polarizations. The reflection coefficients are solved by using the transmission line model. As an example, the reflection from a dense array of metal circular disks is considered. The equivalent model for the array impedance is capacitive. The analytical model of a thin dense array of dipoles for oblique incidence can be used for studying surface waves propagating along the array and for material modeling of layered dense arrays of small particles.

APPENDIX A.

After successive integration by parts the integral in expression (9) for $\beta_{\|}$ can be obtained straightforwardly in the form

$$\begin{aligned} I_{\|} &= 4\pi\left|_{R}^{\infty}\left(\frac{1}{r}+jk\right)e^{-jkr}\frac{J_1(Kr)}{Kr}+\pi\int_{R}^{\infty}\left(\frac{3}{r^2}+\frac{j3k}{r}-k^2\right)\right. \\ &\quad e^{-jkr}\left[J_0(Kr)+J_2(Kr)\right]dr \end{aligned} \tag{A1}$$

By subtracting and adding the term $4\pi\int_{R}^{\infty}(\frac{1}{r^2}+\frac{jk}{r})e^{-jkr}J_0(Kr)dr$ and comparing the integral in (9), we have

$$I_{\|}=4\pi\left|_{R}^{\infty}\left(\frac{1}{r}+jk\right)e^{-jkr}\frac{J_1(Kr)}{Kr}+4\pi\int_{R}^{\infty}\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}J_0(Kr)dr-I_{\|}\right. \tag{A2}$$

from which after applying integration by parts, finally

$$\begin{aligned} I_{\|} &= 2\pi\left[\left(-\left(\frac{1}{R}+jk\right)\frac{J_1(KR)}{KR}+\frac{J_0(KR)}{R}\right)e^{-jkR}\right. \\ &\quad \left.-K^2\int_{R}^{\infty}\frac{J_1(Kr)}{Kr}e^{-jkr}dr\right] \end{aligned} \tag{A3}$$

For the perpendicular polarization, the integral expression in (10) is

$$I_{\perp} = \pi \int_{R}^{\infty} \left[\left(k^2 + \frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} J_0(Kr) - \left(k^2 - \frac{j3k}{r} - \frac{3}{r^2} \right) e^{-jkr} J_2(Kr) \right] dr \tag{A4}$$

Previously in (9)

$$I_{\|} = \pi \int_{R}^{\infty} \left[\left(k^2 + \frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} J_0(Kr) + \left(k^2 - \frac{j3k}{r} - \frac{3}{r^2} \right) e^{-jkr} J_2(Kr) \right] dr \tag{A5}$$

After eliminating the term $\int_{R}^{\infty} \left(k^2 - \frac{j3k}{r} - \frac{3}{r^2} \right) e^{-jkr} J_2(Kr) dr$,

$$I_{\perp} = 2\pi k^2 \int_{R}^{\infty} e^{-jkr} J_0(Kr) dr + 2\pi \int_{R}^{\infty} \left(\frac{1}{r^2} + \frac{jk}{r} \right) e^{-jkr} J_0(Kr) dr - I_{\|} \tag{A6}$$

and using (A2) one obtains

$$I_{\perp} = -2\pi \Big|_{R}^{\infty} \left(\frac{1}{r} + jk \right) e^{-jkr} \frac{J_1(Kr)}{Kr} + 2\pi k^2 \int_{R}^{\infty} e^{-jkr} J_0(Kr) dr \tag{A7}$$

which is after integrating by parts

$$I_{\perp} = 2\pi \left[\left(\left(\frac{1}{R} + jk \right) \frac{J_1(KR)}{KR} - jk J_0(KR) \right) e^{-jkR} + jkK \int_{R}^{\infty} J_1(Kr) e^{-jkr} dr \right] \tag{A8}$$

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