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WAVE BEAM PROPAGATION IN LAYERED MEDIA

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Abstract—Using rather general assumptions, wave beam propagation is considered in a medium constituted of two half-spaces with smoothly changing properties, these latter changing stepwise at the half-spaces' interface. Expressions for the beam-shape change in the course of propagation are obtained. General results are applied to a Gaussian beam propagating in a series chain, and to fields described by the Helmholtz equation.

1. INTRODUCTION

Investigations of wave beam propagation in homogeneous media are of interest both scientific and applied [1–3]. Sufficiently complete bibliography on this problem is given in [2]. However, investigations of this important problem are carried out as usual for homogeneous media. Meanwhile real media are frequently inhomogeneous, and this fact may result in serious changes of the features of the propagating wave beam. As far as we know, the only inhomogeneity previously considered, was presented by two isotropic homogeneous half-spaces, where reflection and refraction of the one-dimensional wave beam from the half-spaces' interface were studied [4]. Helmholtz's equation is usually used for describing the beam propagation. In a number of cases, however, e.g. in media with spatial dispersion, gyrotropic and anisotropic media, this equation becomes much more intricate, which seriously complicates the propagation picture. Those and related effects are of actual interest, and their consideration is the subject of this paper.

In the single assumption of smoothness of medium's properties, we have derived expressions describing the beam propagating in the near-zone (the Fresnel zone) and in the far-zone (the Fraunhofer zone). We have studied in detail the change of the beamwidth caused by the medium's inhomogeneity, and culculated also multiwave propagation in such media. The medium is considered to be isotropic with one-dimensional inhomogeneity along the z-coordinate. The wave beam propagation is described with an equation of quite general form, and the theory built here is applied to the wave propagation in a series chain.

2. GENERAL THEORY

We start with

$$\hat{H}\left\{z, \frac{\partial}{\partial \vec{r}}, \frac{\partial}{\partial z}\right\} E = 0, \tag{1}$$

where operator $\hat{H}$ acting on a field E is an arbitrary function of its arguments. Eq. (1) may be a differential, difference, integral with difference kernel, etc. A solution to the equation provides an integral

$$E(z, \vec{r}) = \int E_{k_\perp}^{(0)} U_{k_\perp}(z) e^{i\vec{k}_\perp \vec{r}} d\vec{k}_\perp, \tag{2}$$

where $\vec{k}_\perp$ is a vector with components k_x, k_y and $\vec{r}$ is a vector with components x, y; $U_{k_\perp}(z)$ is the sought after function. Substituting (2) into (1) yields an equation for $U_{k_\perp}(z)$:

$$\hat{H}\left\{z, i\vec{k}_\perp, \frac{\partial}{\partial z}\right\} U_{k_\perp} = 0. \tag{3}$$

Let us formulate the boundary conditions of the problem. Suppose that on the plane $z = 0$, a field distribution is given:

$$E(\vec{r}, 0) = E^{(0)}(\vec{r}), \tag{4}$$

where $E^{(0)}(\vec{r})$ is a known function. From this condition (4), the boundary condition for $U_{k_\perp}(z)$ is derived:

$$U_{k_\perp}(0) = 1. \tag{5}$$

Let the wave propagate in the positive direction, then for $U_{k_\perp}(z)$, Zommerfeld's radiation condition is fulfilled. As appears from Eqs. (2), (4), and (5), $E^{(0)}_{k_\perp}$ is the Fourier transform for $E^{(0)}(\vec{r})$:

$$E^{(0)}_{k_\perp} = \frac{1}{(2\pi)^2} \int E^{(0)}(\vec{r}) e^{-i\vec{k}_\perp \vec{r}} d\vec{k}_\perp. \tag{6}$$

To obtain specific results, an explicit functional form for $E^{(0)}(\vec{r})$ has to be given. The Gaussian wave beam is practically the most applicable one in the situation, therefore we assume

$$E^{(0)}(\vec{r}) = E_0 e^{-\frac{r^2}{w_0^2}} \tag{7}$$

and

$$E_{k_\perp} = E_0 \frac{w_0^2}{4\pi} e^{-\frac{w_0^2 k_\perp^2}{4}}, \tag{8}$$

here w_0 is the beamwidth in the $z = 0$ plane, which is the beamwaist in this case. Let us look for $U_{k_\perp}(z)$ in the form

$$U_{k_\perp}(z) = \rho_{k_\perp}(z) e^{iS_{k_\perp}(z)}. \tag{9}$$

Note that ρ is a dimensionless quantity and S is the phase.

Let $\hat{H}$ as a function of z be changing substantially on a distance L, which is, hence, a characteristic length of the change of medium's properties; let the correlations

$$\begin{gathered} \frac{1}{\rho(z)} \frac{\partial \rho(z)}{\partial z} \backsim \frac{1}{L}, \\ \frac{1}{S_{k_\perp}(z)} \frac{\partial S_{k_\perp}(z)}{\partial z} \backsim k, \\ \frac{1}{S_{k_\perp}(z)} \frac{\partial^2 S_{k_\perp}(z)}{\partial z^2} \backsim \frac{k}{L} \end{gathered} \tag{10}$$

be fulfilled. The wave amplitude also changes on the same length L, and $1/k$ is a characteristic distance over which the wave phase changes.

Hereinafter, we shall use a geometrical optics method JWKB for which the validity condition is $kL \gg 1$. To define S and ρ from Eq. (10) we have a system of equations:

$$H(i\vec{k}_\perp, ik_z, z) = 0, \quad \rho^2 v(i\vec{k}_\perp, ik_z, z) = \text{const.}, \tag{11}$$

where

$$k_z = \frac{\partial S_{k_\perp}(z)}{\partial z} \quad \text{and} \quad v = \frac{\partial H(z, i\vec{k}_\perp, ik_z)}{\partial k_z}.$$

The set of equations (11) is a system of ordinary equations, of which the first one is the eikonal equation and the second one is the continuity equation. A future publication will be dedicated to deducing these equations.

From the first equation of (11) we have found $k_z(i\vec{k}_\perp, z)$, then for S and ρ we obtain

$$S_{k_\perp}(z) = \int_0^z k_z(z, i\vec{k}_\perp)dz, \quad \rho_{k_\perp}(z) = \sqrt{\frac{v\big(0, i\vec{k}_\perp, ik_z(0)\big)}{v\big(z, i\vec{k}_\perp, ik_z(z)\big)}}. \tag{12}$$

The second expression in (12) has been obtained by choosing such a value of the "const." in the second equation of (11) that yields $\rho_{k_\perp}(0) = 1$. It follows from Eqs. (11) that $U_{k_\perp}(0) = 1$.

We consider here two limiting cases: near-field zone and far-field zone. In the near-zone, one can consider k_x and k_y to be small parameters and to expand $\rho(i\vec{k}_\perp, ik_z, z)$ and $S(i\vec{k}_\perp, ik_z, z)$ in corresponding series in the near neighborhood of the $\vec{k}_\perp = 0$ point. The needed criteria will be given further. $S_{k_\perp}$ and $\rho_{k_\perp}$ are even functions of $\vec{k}_\perp$ because the medium is isotropic and the substitution $\vec{k}_\perp$ for $-\vec{k}_\perp$ does not change these functions. After expanding $\rho(i\vec{k}_\perp, ik_z, z)$ and $S(i\vec{k}_\perp, ik_z, z)$ in series we get

$$\begin{aligned} &\rho_{k_\perp}(z) \approx \rho_0(z), \\ &S_{k_\perp}(z) = S_0(z) + \frac{1}{2}\left[\frac{\partial^2 S_0(z)}{\partial k_x^2}k_x^2 + 2\frac{\partial^2 S_0(z)}{\partial k_x \partial k_y}k_x k_y + \frac{\partial^2 S_0(z)}{\partial k_y^2}k_y^2\right]. \end{aligned} \tag{13}$$

For $\rho_{k_\perp}(z)$ one can restrict onself to the first expansion term, and for the phase S, because it is included in the exponent, to the second term.

Substituting (9) and (14) into (3) gives

$$E(\vec{r},z)=\frac{E_0 w_0^2 \rho_0(z) e^{iS_0(z)}}{\sqrt{D(z)}}\exp\left\{-\frac{1}{4a}\left[x^2+\frac{(ay-bx^2)}{D(z)}\right]\right\}, \tag{14}$$

here

$$D(z)=a(z)c(z)-b^2(z), \quad \text{where} \quad a=\frac{w_0^2}{4}-\frac{i}{2}\frac{\partial^2 S_0(z,i\vec{k}_\perp)}{\partial k_x^2},$$

$$b=-\frac{i}{2}\frac{\partial^2 S_0(z,i\vec{k}_\perp)}{\partial k_x \partial k_y}, \quad c=\frac{w_0^2}{4}-\frac{i}{2}\frac{\partial^2 S_0(z,i\vec{k}_\perp)}{\partial k_y^2}.$$

If in the $z=$ const. planes the medium is uniform and isotropic, i.e. $S_{k_\perp}$ is a function of only $k_\perp^2$, after some calculations we get an expression for the field:

$$E=Ae^{i\Phi}, \tag{15}$$

where A is the field amplitude and Φ is its phase:

$$A=\frac{E_0\rho_0(z)}{W(z)}\exp\left\{-\alpha z+\frac{r^2}{w_0^2 W(z)}\right\} \tag{16}$$

$$\Phi=S_0(z)-\arctan\left\{\frac{4S_0(z)}{w_0^2 W(z)}\right\}+\frac{4S_0'(z)r^2}{w_0^4}, \tag{17}$$

here

$$W(z)=1+\left[\frac{4S_0'(z)}{w_0^2}\right]^2$$

and α is the field attenuation due to dissipative processes. It is supposed that

$$\frac{\alpha z}{k}\ll 1, \quad S_0(z)=S_{k_\perp}(z)\big|_{k_\perp=0} \quad \text{and} \quad S_0'(z)=\left.\frac{dS_{k_\perp}(z)}{dk_\perp^2}\right|_{k_\perp=0}.$$

Note that

$$w^2(z)=w_0^2 W(z) \tag{18}$$

is the effective beam width related to its diffractional spreading in the medium. The same is the origin of the two last terms in (17).

For the far-field zone, the integral in formula (2) can be calculated using the saddle-point technique [5], which results in

$$A = \frac{E_0 w_0^2}{4\left|D^{1/2}\left(\vec{k}_{\perp S}\right)\right|} \exp\left\{-\alpha z - \frac{w_0^2 \vec{k}_{\perp S}^2}{4}\right\}, \tag{19}$$

$$\Phi = S_{k_{\perp S}}(z) + \vec{k}_{\perp S}\vec{r}, \tag{20}$$

where vector $\vec{k}_{\perp S}$ can be found from the equation

$$\vec{r} + \frac{\partial S\left(z, \vec{k}_{\perp S}\right)}{\partial \vec{k}_{\perp S}} = 0. \tag{21}$$

3. BEAMS TRANSMITTED AND REFLECTED FROM THE INTERFACE

We shall suppose the interface to be the $z = 0$ plane and the beam source located in a $z = z_0$ plane of a medium filling the positive half-space. In this half-space the field is described by Eq. (1), and in the negative half-space by the same equation with another H function that will be referred to as H'. Thus, passing the $z = 0$ plane, the medium properties change abruptly. As boundary conditions relating the field in the positive half-space with that of the negative half-space, we shall take the continuity condition on the $z = 0$ plane for the field E and its derivative with respect to z:

$$E(+0) = E(-0), \quad \frac{\partial E(+0)}{\partial z} = \frac{\partial E(-0)}{\partial z}. \tag{22}$$

In the positive half-space, the field is a sum of two fields: incident E_i and reflected E_R, and in the negative half-space the field is E_T. Setting aside the details of the standard calculation procedure, we obtain:

$$E_i = \int E_{k_\perp} \rho_{k_\perp}(z, z_0) \exp\left\{i\left[S_{k_\perp}(z, z_0) + \vec{k}_\perp \vec{r}\right]\right\} d\vec{k}_\perp \tag{23}$$

with $\rho(z, z_0)$ and $S(z, z_0)$ that should satisfy the following boundary conditions, respectively

$$\rho_{k_\perp}(z_0, z_0) = 1 \quad \text{and} \quad S_{k_\perp}(z_0, z_0) = 0; \tag{24}$$

these requirements lead to the following expressions for $\rho_{k_\perp}(z, z_0)$ and $S_{k_\perp}(z, z_0)$:

$$\rho_{k_\perp}(z, z_0) = \sqrt{\frac{v\left[i\vec{k}_\perp, ik_z(z_0)\right]}{v_z\left[i\vec{k}_\perp, ik_z(z)\right]}}, \quad S_{k_\perp}(z, z_0) = \int\limits_{z_0}^{z} k_z(z) dz. \tag{25}$$

Then

$$E_R = \int E_{k_\perp} \rho_{k_\perp R}(z) \exp\{i[S_{k_\perp R}(z) + \vec{k}_\perp \vec{r}]\} d\vec{k}_\perp, \tag{26}$$

and

$$E_T = \int E_{k_\perp} \rho_{k_\perp T}(z) \exp\{i[S_{k_\perp T}(z) + \vec{k}_\perp \vec{r}]\} d\vec{k}_\perp. \tag{27}$$

In the two last expressions:

$$\begin{aligned}
\rho_{k_\perp R}(z) &= \frac{k_z(0,\vec{k}_\perp) - k'_z(0,\vec{k}_\perp)}{k_z(0,\vec{k}_\perp) + k'_z(0,\vec{k}_\perp)} \sqrt{\frac{v_{k_\perp}(z_0)}{v_{k_\perp}(z)}}, \\
S_{k_\perp R}(z) &= S_{k_\perp}(0, z_0) - S_{k_\perp}(z, 0), \\
\rho_{k_\perp T}(z) &= \frac{2k_z(0,\vec{k}_\perp)}{k'_z(0,\vec{k}_\perp) - k_z(0,\vec{k}_\perp)} \sqrt{\frac{v_{k'_\perp}(0) v_{k_\perp}(z_0)}{v_{k_\perp}(0) v_{k'_\perp}(z)}}, \quad \text{and} \\
S_{k_\perp T}(z) &= S_{k_\perp T}(z, 0) + S_{k_\perp T}(0, z).
\end{aligned}$$

One can see that the formulas for reflected and transmitted beams, up to the designation of the variables, coincide with formulas for the field in an unbounded space, therefore all the results obtained for a beam in an unbounded space are applicable to the reflected and transmitted beams.

4. TWO-WAVE PROBLEM

Up to now, it has been supposed that the first of Eqs. (11) has a unique solution. However, there are cases of actual physical interest, when several solutions satisfy this equation, that is we are dealing with the propagation of several waves. Suppose now that there are two solutions denoted as $k^{(1,2)}(z, k_\perp)$. For the solutions of the problem to be single-valued, there must be—in addition to the boundary condition (4)—one more boundary condition over the field derivative in the $z = 0$ plane, namely

$$\frac{\partial E_z(0)}{\partial z} = \mathfrak{F}(\vec{r}). \tag{28}$$

The fields $E^{(1,2)}$ can be determined by a formula analogous to (2):

$$E^{(1,2)}(\vec{r}) = \int E^{(1,2)}_{k_\perp} U^{(1,2)}_{k_\perp}(z) e^{i\vec{k}_\perp \vec{r}} d\vec{k}_\perp, \tag{29}$$

where

$$E_{k_\perp}^{(1)} = \frac{k_{zk_\perp}^{(2)}(0)E_{k_\perp}^{(0)} + i\mathfrak{F}_{k_\perp}}{k_{zk_\perp}^{(2)}(0) - k_{zk_\perp}^{(1)}(0)}, \quad E_{k_\perp}^{(2)} = -\frac{iE_{k_\perp}^{(0)} + k_{zk_\perp}^{(1)}(0)E_{k_\perp}^{(0)}}{k_{zk_\perp}^{(2)}(0) - k_{zk_\perp}^{(1)}(0)},$$

and $U_{k_\perp}^{(1,2)}$ is determined like in the case with a unique solution.

5. INTEGRAL AND DIFFERENCE EQUATIONS CONVERTED INTO DIFFERENTIAL ONES

Now we shall demonstrate how an integral equation and a finite-difference one can be converted into equations of the form (1). Consider as an example a finite-difference equation

$$\sum_{n=-\infty}^{\infty} A_n(z)E\big(\vec{\Re} + \vec{\Re}_n\big) = 0. \tag{30}$$

Note that since this paragraph's rasults are valid as well for a three-dimensional non-uniform medium, here and below we use $\vec{\Re}$, a three-dimensional vector with the x, y, z components, and $\vec{\Re}_n$, a constant vector. As is well known, $E\big(\vec{\Re} + \vec{\Re}_n\big) = \exp\big(\vec{\Re}_n \frac{\partial}{\partial \vec{\Re}}\big)$, which being substituted into (30) yields Eq. (1), where

$$\hat{H}\left(z, \frac{\partial}{\partial \vec{\Re}}\right) = \sum_{n=-\infty}^{\infty} A_n(z) \exp\left(\vec{\Re}_n \frac{\partial}{\partial \vec{\Re}}\right). \tag{31}$$

The following integral equation can also get the form (1):

$$\sum_{n=-\infty}^{\infty} B_n(z) \int K_n\big(\vec{\Re} - \vec{\Re}'\big)E\big(\vec{\Re}'\big)d\vec{\Re}' = 0. \tag{32}$$

The Fourier transform for $K_n\big(\vec{\Re} - \vec{\Re}'\big)$ is

$$K_n\big(\vec{\Re} - \vec{\Re}'\big) = \frac{1}{(2\pi)^3} \int Q_n(\vec{\varkappa}) \exp\big[i\vec{\varkappa}\big(\vec{\Re} - \vec{\Re}'\big)\big] d\vec{\varkappa}, \tag{33}$$

which can be rewritten as follows:

$$K_n\big(\vec{\Re} - \vec{\Re}'\big) = \frac{1}{(2\pi)^3} \hat{Q}_n \left(\frac{1}{i} \frac{\partial}{\partial \vec{\Re}}\right) \int \exp\big[i\vec{\varkappa}\big(\vec{\Re} - \vec{\Re}'\big)\big] d\varkappa, \tag{34}$$

or, using the definition of the δ-function,

$$K_n(\vec{\Re} - \vec{\Re}') = \hat{Q}_n \left(\frac{1}{i} \frac{\partial}{\partial \vec{\Re}} \right) \delta(\vec{\Re} - \vec{\Re}'). \tag{35}$$

By substituting (35) into (32) we arrive again at Eq. (1), where $\hat{H}$ assumes a form

$$\hat{H} \left(z, \frac{\partial}{\partial \vec{\Re}} \right) = \sum_{n=-\infty}^{\infty} B_n(z) \hat{Q}_n \left(\frac{1}{i} \frac{\partial}{\partial \vec{\Re}} \right). \tag{36}$$

6. BEAM PROPAGATION IN A SERIES CHAIN

Now we will use the theory developed above to solve a specific problem of wave beam propagation in a periodic series chain. The corresponding equation is written as

$$k^2(z)E + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E + \frac{\eta}{d^2} [E(z+d) + E(z-d) - 2E(z)] = 0, \tag{37}$$

where d is the chain period and η is the coupling coefficient of the chain's adjacent knots. Using the above procedure converts Eq. (37) into form (1) with

$$\hat{H} = k^2(z) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{4\eta}{d^2} \left(\sinh \frac{d}{2} \frac{\partial}{\partial z} \right)^2. \tag{38}$$

We shall look for the solution of Eq. (38) as an expression of the form (2), where $U_{k_\perp}(z)$ is defined with formula (9). In the geometrical optics approximation, the Hamilton-Jacoby equation and continuity equation are written as follows:

$$q^2(z) - \frac{4\eta}{d^2} \sin^2 \frac{d}{2} \frac{\partial S}{\partial z} = 0, \tag{39}$$

$$\rho^2 v = j, \tag{40}$$

here

$$v = 2\eta^{1/2} q(z) \sqrt{1 - \frac{q^2(z) d^2}{4\eta}}, \tag{41}$$

where $q(z) = \sqrt{k^2(z) - k_\perp^2}$. The boundary conditions for Eqs. (39) and (40) as they were formulated above, are: $\rho(0) = 1$, $S(0) = 0$. By taking these boundary conditions into consideration, we get for S and ρ:

$$S_{k_\perp} = \frac{2}{d} \int_0^z \arcsin \frac{q(z)d}{2} dz, \quad \rho_{k_\perp} = \sqrt{\frac{q(z)}{q(0)} \left(\frac{1 - \frac{q^2(z)d^2}{4\eta}}{1 - \frac{q^2(0)d^2}{4\eta}} \right)^{1/2}}. \tag{42}$$

Consider now the beam's field in the near-zone, which is defined by expressions (15)–(17). For that, one should determine S_0 and S_0'. Calculations give

$$\begin{aligned} S_0(z) &= \frac{1}{2d} \int_0^z \arcsin \frac{k(z)d}{2} dz, \\ S_0' &= -\frac{1}{2} \int_0^z \frac{dz}{k(z)\sqrt{1 - \frac{k^2(z)d^2}{4\eta}}}, \\ \rho_0 &= \sqrt{\frac{k(z)}{k(0)} \left(\frac{1 - \frac{k^2(z)d^2}{4\eta}}{1 - \frac{k^2(0)d^2}{4\eta}} \right)^{1/2}}, \\ W(z) &= 1 + \left(\frac{2}{w_0^2} \int_0^z \frac{dz}{k(z)\sqrt{1 - \frac{k^2(z)d^2}{4\eta}}} \right)^2. \end{aligned} \tag{43}$$

For the vacuum or an homogeneous medium, k is independent of z and

$$\begin{aligned} S_0 &= \frac{2z}{d} \arcsin \frac{kd}{2}, \\ S_0' &= -\frac{z}{2k\sqrt{1 - \frac{k^2d^2}{4\eta}}}, \\ \rho_0 &= 1, \\ W(z) &= 1 + \left(\frac{2z}{w_0^2 k\sqrt{1 - \frac{kd^2}{4\eta}}} \right)^2. \end{aligned} \tag{44}$$

Substituting (43) and (44) into (16) and (17) gives the beam's amplitude and phase for this model.

7. THE HELMHOLTZ EQUATION

Note, in conclusion of the previous paragraph, that with $\eta = 1$ and $d \to 0$, Eq. (37) becomes the well-known Helmholtz equation

$$\left[\Delta + k^2(z)\right] E = 0. \tag{45}$$

If in formulas (44) $d \to 0$, one obtains

$$S_0(z) = \int\limits_0^z k(z)dz, \quad S_0' = -\frac{1}{2}\int\limits_0^z \frac{dz}{k(z)}, \quad W(z) = 1 + \left[\frac{2}{w_0^2}\int\limits_0^z \frac{dz}{k(z)}\right]^2. \tag{46}$$

For an homogeneous medium, i.e., when k is not z-dependent, formulas (46) become the expressions well known from the literature:

$$S_0 = kz, \quad S_0' = -\frac{z}{2k}, \quad W(z) = 1 + \left(\frac{z}{\xi}\right)^2, \quad \xi = \frac{kw_0^2}{2}. \tag{47}$$

Consider the validity criteria for k to be z-independent in the near-zone approximation. Not going into the calculations themselves, we arrive at the next inequalities:

$$k^2 w_0^2 \gg 1, \quad z \ll w_0(kw_0)^3. \tag{48}$$

The first one is always fulfilled by virtue of beam's determination; the second one, due to the first one, is not too much strong.

Unfortunately, carrying out such a detailed investigation on the far-zone field is not simple a task, because Eq. (21) for $\vec{k}_{\perp S}$ can only be solved analitically for $r \gg w_0$. In this case the right item in the left part of (21) can be neglected, and after simple but cumbersome enough calculations we get the field's amplitude and phase in the far-zone:

$$\begin{aligned} &A = \frac{E_0 \xi z}{\sqrt{r^2 + z^2}} \exp - \left[d\sqrt{r^2 + z^2} + \frac{k\xi r^2}{2(r^2 + z^2)}\right] \quad \text{and} \\ &\Phi = k\sqrt{r^2 + z^2}. \end{aligned} \tag{49}$$

From the first of inequalities of (48) it follows that

$$k\xi = \frac{k^2 w_0^2}{2} \gg 1,$$

and from Eqs. (49), that A is not small only when r is small. Neglecting r^2 as compared to z^2 we have

$$A = \frac{E_0 \xi}{z} \exp - \left(dz + \frac{k \xi r^2}{2z^2} \right) \quad \text{and} \quad \Phi = \frac{3\pi}{2} + kz. \tag{50}$$

It is interesting to note that, if in the formula for $W(z)$ of Eqs. (48) one neglects the unity, i.e., supposes $z \gg \xi$, the expressions for the field in the near-zone and far-zone coincide. This means that formula derived for the near-zone field is valid for the far-zone as well. In spite of the fact that this statement has been proved for a special case, it apparently fits for a more general one, too. We will not dwell on the formulas for the beam reflection from two-medium's interface because of their bulkiness and because they can be readily derived from the above expressions. Since in most of media, when $z \to \infty$, k is finite-valued,

$$\lim_{z \to \infty} k(z) = k(\infty),$$

formulas (50) can be regarded as asymptotic ones for a large enough z and for an arbitrary relation k of z. The expression

$$\xi = \frac{k(\infty) w_0^2}{2} \tag{51}$$

in these asymptotic formulas is a universal one for any field described by the Helmholtz equation.

8. CONCLUSIONS

The wave beam evolution in a plane-layered medium with smoothly changing properties, at the layer's interface the properties being changed stepwise, has been considered. Unlike the usual approach based on the Helmholtz equation, a much more general equation has been used. Into such a general equation form, as it is clearly demonstrated, we can convert the difference and integral equations. Using the JWKB method, expressions for the near-zone and far-zone fields have been derived.

This general theory is applied to the wave propagation in a periodical series chain described usually by a differential-difference equation. The expression for the beam field in a medium where the propagation is described by the Helmholtz equation, can be readily derived from the results obtained for series chain by passage to the limit.

Naturally, the paper results do not exhaust the whole problem. It is of actual interest to investigate the beam propagation in the periodically and statistically inhomogeneous media, as well as the propagation through and reflection from statistically rough surfaces. These and similar problems are to be tackled in the next communications.

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