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GREEN'S FUNCTION EXPANSIONS IN DYADIC ROOT FUNCTIONS FOR SHIELDED LAYERED WAVEGUIDES

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Abstract—Dyadic Green's functions for inhomogeneous parallel-plate waveguides are considered. The usual residue series form of the Green's function is examined in the case of modal degeneracies, where second-order poles are encountered. The corresponding second-order residue contributions are properly interpreted as representing "associated functions" of the structure by constructing a new dyadic root function representation of the Hertzian potential Green's dyadic. The dyadic root functions include both eigenfunctions (corresponding to first-order residues) and associated functions, analogous to the idea of Jordan chains in finite-dimensional spaces. Numerical results are presented for the case of a two-layer parallel-plate waveguide.

1. INTRODUCTION

In waveguiding structures the field due to an impressed source is often represented in terms of a Green's function. In general three-dimensional parallel-plate waveguides the Green's function can be obtained in a number of ways. One method (method i) is to expand the Green's function in the vertical-coordinate (perpendicular to the guiding plates) eigenfunctions, obtaining a discrete series of one-dimensional eigenfunctions multiplied by a closed-form two-dimensional Green's function in the longitudinal infinite coordinates. Another method (method ii) is to expand the Green's function as a two-dimensional continuous eigenfunction expansion in the infinite longitudinal coordinates (i.e., a Fourier transform), obtaining a two-dimensional integral representation having as kernel a one-dimensional closed-form Green's function in the vertical coordinate. In another method (method iii) the Green's function is represented as a three-dimensional expansion in terms of a two-dimensional continuous eigenfunction expansion in the infinite longitudinal coordinates (Fourier transform) and a one-dimensional discrete eigenfunction expansion in the vertical coordinate. This results in a purely spectral form that is not convenient computationally since it involves both integrals and series, but is useful for theoretical manipulations.

The ability to obtain the Green's function by usual eigenfunction methods depends on the eigenfunctions forming a basis in the underlying function space. This question, in turn, depends on self-adjointness of the governing differential operator. In general, the eigenfunctions associated with a self-adjoint boundary value problem form a basis in the underlying Hilbert space [1]. If the boundary value problem is not self-adjoint then the eigenfunctions may not form a basis, although the root functions, which include the eigenfunctions as well as functions associated to them (associated functions), will often form the desired basis [2].

If the medium filling the waveguide is lossless and a radiation condition is imposed in the longitudinal infinite coordinates, then the governing three-dimensional differential operator is formally self-adjoint, but not self-adjoint. If the medium filling the waveguide is allowed to admit dielectric loss, however small (which will be assumed in the following), and is homogeneous, then the problem is self-adjoint in the usual $\mathbf{L}^2$ space; the resulting eigenfunctions form a basis, and ordinary eigenfunction expansions are sufficient to represent the Green's function. Any of the methods (i), (ii), or (iii) are valid.

For parallel-plate waveguides filled with inhomogeneous lossy media the situation becomes more complex, as one obtains a self-adjoint operator in the longitudinal (infinite) coordinates, but not generally in the vertical coordinate. For certain combinations of structural and electrical parameters resulting in nontrivial modal degeneracies, the vertical-coordinate eigenfunctions fail to form a basis in the underlying Hilbert space. In this case the discrete series of eigenfunctions will not represent the Green's function, and methods (i) and (iii) will fail at these points. In most cases of physical interest, however, the root functions will form a basis in the desired space. In this case root function expansion solutions can be used, providing a correct representation using methods (i) or (iii), although the resulting expansions are somewhat complicated. In electromagnetic theory this problem has been pursued primarily in the Soviet literature [3–8]. A related discussion concerning associated waves is provided in [9, pp. 50–59], where the example of a hollow parallel-plate waveguide with an impedance wall is explicitly considered.

It can be appreciated that, in general, method (ii), which combines Fourier transforms in the infinite longitudinal coordinates, and the solution (using nonspectral methods) of a one-dimensional boundary-value problem in the vertical coordinate, is the "safest" method to obtain the Green's function. It is easily shown that the integrand of the resulting inverse Fourier transform representation of the Green's function is meromorphic in the transform plane, and complex-plane analysis can be used to obtain a discrete residue series form for the Green's function. The resulting series is obviously related to an eigenfunction expansion; poles of unity order correspond to eigenfunctions, whereas higher-order poles correspond to associated functions. The benefits of this procedure, compared to starting directly with a root function expansion, are that the root functions are automatically obtained (both eigenfunctions and associated functions) and normalized. Thus, knowledge of the dispersion behavior of the modes (necessary to find the poles and ascertain their multiplicity) leads to a simple, rigorous method of obtaining a discrete series form for the Green's function, without the need to explicitly study root functions and their complicated orthogonality relationships.

In this paper we present dyadic Green's functions for inhomogeneous parallel plate waveguides, developed using method (ii), and obtain a discrete dyadic residue series form using complex-plane analysis. The purpose of the paper is fourfold. First, we discuss the possibility of nontrivial modal degeneracies which render eigenfunction expansion methods (the usual method) invalid at certain points, necessitating the consideration of root functions. Second, in order to interpret the dyadic residue series form of the Green's dyadic, we introduce the idea of dyadic root functions (dyadic eigenfunctions and dyadic associated functions), rather than the usual vector or scalar eigenfunctions. These dyadic functions capture the physics of the problem in a compact form. Third, for first-order poles we show that the resulting dyadic residues are dyadic eigenfunctions of the governing differential operator, and that for higher-order poles the dyadic residues are related to dyadic associated functions. Although it is not our intent to develop the dyadic root function expansion approach thoroughly, for completeness we discuss dyadic root function expansions and properties of the dyadic root functions. Finally, we examine numerical consequences of nontrivial modal degeneracies and the failure of the eigenfunctions to form a basis in such cases.

2. FORMULATION

2.1. Nontrivial Modal Degeneracies

For any waveguiding structure source-free Maxwell's equations, field continuity conditions, and boundary conditions can be converted into a functional equation for the discrete modes of the structure,

$$A(k_\rho, \omega, \boldsymbol{\varsigma})X = 0. \tag{1}$$

In (1) k_ρ is the longitudinal spatial Fourier-transform variable representing the modal propagation constant, ω is the temporal Fourier-transform variable representing angular frequency, $\boldsymbol{\varsigma}$ is a vector of n electrical (e.g., permittivity, permeability) and structural (e.g., dielectric thickness, waveguide dimensions) parameters, and X represents a modal field distribution. We consider each of the variables (k_ρ, ω) in the complex plane $\mathbf{C}$, and assume that $\varsigma \in \mathbf{C}^n$ is specified. Nontrivial solutions of (1) are obtained from the implicit dispersion equation

$$H(k_\rho, \omega, \varsigma) = \det\big(A(k_\rho, \omega, \varsigma)\big) = 0 \tag{2}$$

leading to modal propagation constants $k_\rho = k_{\rho n}(\omega, \varsigma)$. If

$$\frac{\partial}{\partial k_\rho} H(k_\rho, \omega, \varsigma) = 0 \tag{3}$$

in addition to (2), then $k = k_{\rho n}$ is a double root of $H = 0$ signifying a modal degeneracy (i.e., a point at which two modes have equal propagation constants). If the corresponding modal field patterns are different, in particular, orthogonal in some sense, then the degeneracy is trivial, otherwise the degeneracy is called nontrivial [8]. Obviously, higher-order roots can be obtained in an analogous manner.

At a nontrivial modal degeneracy the complex pair $(k_{\rho n}, \omega_n)$ obtained from the solution of (2) and (3) can be classified in several different ways. If, in addition to (2) and (3),

$$\left(\frac{\partial}{\partial \omega} H(k_\rho, \omega, \varsigma)\right)\left(\frac{\partial^2}{\partial k_\rho^2} H(k_\rho, \omega, \varsigma)\right) \neq 0, \tag{4}$$

then $(k_{\rho n}, \omega_n)$ is called a fold-point, which has been shown to occur at modal cutoff points as well as at certain modal interaction points [10–12, 15]. Alternately, if, in addition to (2) and (3),

$$\frac{\partial}{\partial \omega} H(k_\rho, \omega, \varsigma) = 0, \tag{5}$$

then $(k_{\rho n}, \omega_n)$ is called a critical-point of the mapping $H : \mathbf{C}^{n+2} \to \mathbf{C}$; certain critical points have been shown to occur in the vicinity of traditional mode-coupling regions [10, 13–15]. In the following we will be interested in fold-type points, which are analogous to those explicitly considered in [12] for an open grounded dielectric waveguide. It has been shown [11] that the frequency ω_n arising from the solution of (2) and (3) is a branch point in the complex frequency plane, which impacts transient modal analysis [16].

As an example, consider the parallel-plate waveguide depicted in Figure 1. Dispersion curves ($\beta = k_\rho$, $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$) for the first eight TM^ρ modes are shown in Figure 2 for $d_1 = d_2 = d$, $\varepsilon_3 = \varepsilon_0$, and $\varepsilon_2 = (2.25 - i0)\varepsilon_0$ (lossless case). It is obvious that modes do not couple or interact. The case of strong dielectric loss, $\varepsilon_2 = (2.25 - i1.0)\varepsilon_0$, is

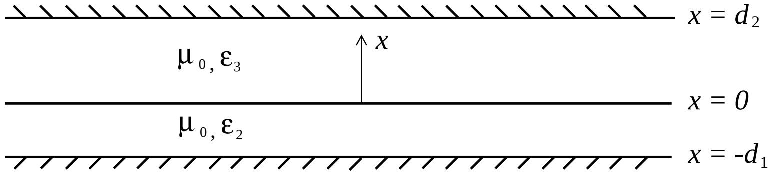

Figure 1. Two layer parallel-plate waveguide.

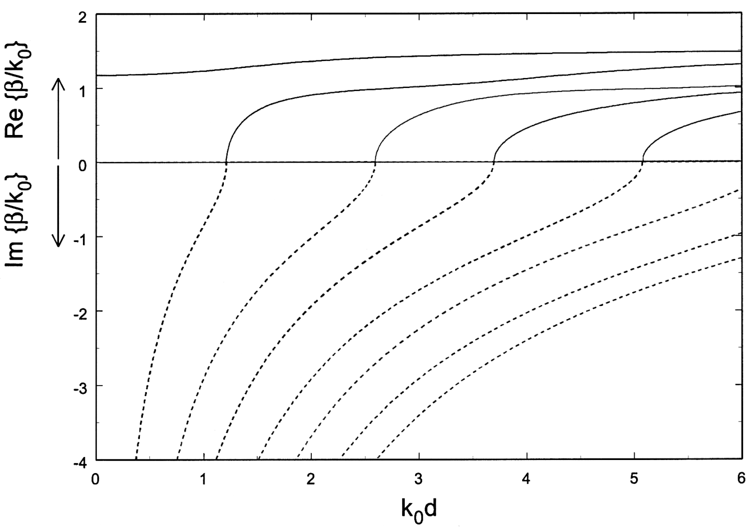

Figure 2. Dispersion curves for the first eight TM$^\rho$ modes of the two-layer waveguide shown in Fig. 1, with $\varepsilon_3 = \varepsilon_0$, $\varepsilon_2 = (2.25 - i0.0)\varepsilon_0$, and $d_1 = d_2 = d$ (lossless case).

shown in Figure 3 (the first six TM$^\rho$ modes are shown). The modes significantly differ from those in the lossless case.

At a critical value of loss, $\varepsilon_2 = (2.25 - i1.735522)\varepsilon_0$, the first two modes intersect at $k_0 d = 1.303347$ forming a nontrivial modal degeneracy, as shown in Figure 4. Although this critical value of loss is quite high, it is considered here merely to demonstrate the desired phenomena; the resulting degeneracy is convenient since it involves the two lowest-order modes. Exactly the same phenomena occurs for lower values of loss, with the corresponding modal degeneracies involving higher-order modes. Furthermore, although such large loss values are not relevant for most electronics applications, material properties similar to the considered values may occur in geophysical applications. Nontrivial modal degeneracies also occur for anisotropic media even in

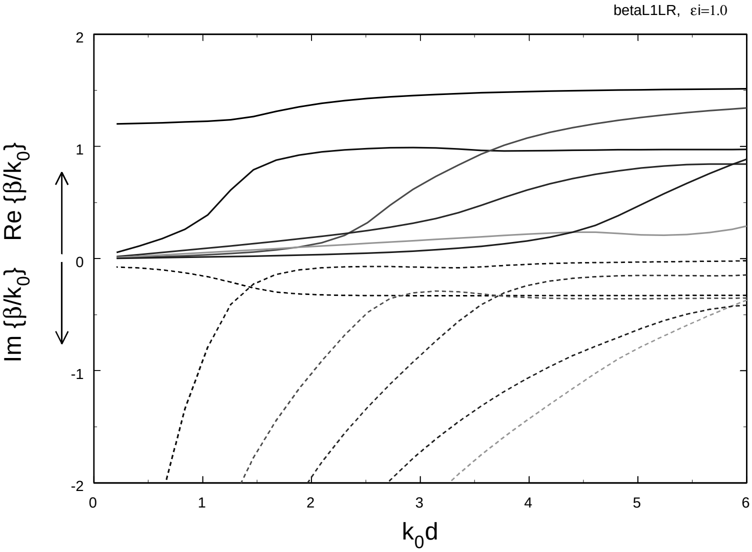

Figure 3. Dispersion curves for the first six TM^{ρ} modes of the two-layer waveguide shown in Fig. 1, with $\varepsilon_3 = \varepsilon_0$, $\varepsilon_2 = (2.25 - i1.0)\varepsilon_0$, and $d_1 = d_2 = d$ (lossy case).

the lossless case.

For loss below the critical value, i.e., for $\mathrm{Im}\{\varepsilon_2\} > -1.7355219$, these two modes do not intersect (see, e.g., Figure 3), yet for $\mathrm{Im}\{\varepsilon_2\}$ near the critical loss value these two modes approach each other near $k_0d = 1.3$. In this case a fold point that interconnects these two modes resides in the fourth quadrant of the complex k_0d (analogously, frequency) plane, which is the quadrant in which the fold point resides in the lossless case. As loss increases this fold point moves towards the real axis (see, e.g., Figure 6(c) in [12] for the analogous case of an open dielectric slab), and at the critical loss value the fold point resides on the real axis at $k_0d = 1.303347$. As loss is further increased beyond the critical value the fold point moves into the first-quadrant of the complex plane. For the critical value of loss, and at the corresponding critical value of k_0d (frequency) corresponding to a modal degeneracy, double poles of the Green's function exist, which are the subject of this paper.

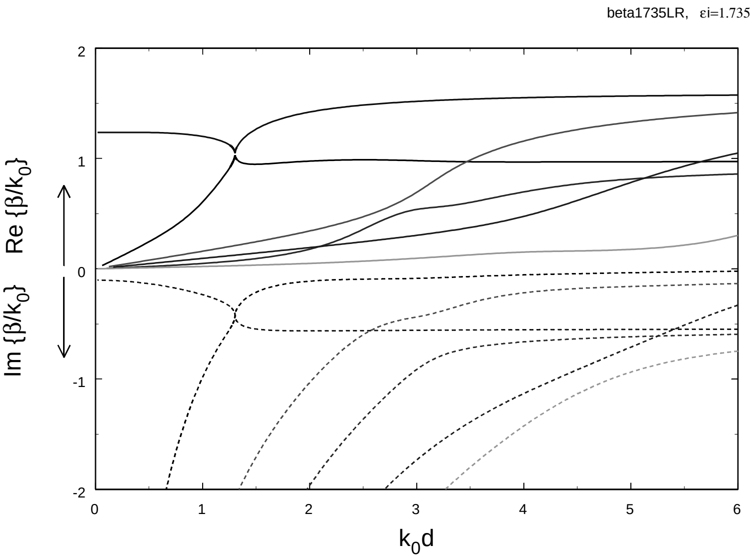

Figure 4. Dispersion curves for the first six TM^{ρ} modes of the two-layer waveguide shown in Fig. 1, with $\varepsilon_3 = \varepsilon_0$, $\varepsilon_2 = (2.25 - i1.735523)\varepsilon_0$, and $d_1 = d_2 = d$ (first two modes are degenerate).

2.2. Dyadic Green's Functions and Dyadic Residues

Consider a planarly inhomogeneous medium inside a parallel-plate waveguide. Maxwell's equations are

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}) &= -i\omega\mu(x)\mathbf{H}(\mathbf{r}) \\ \boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r}) &= i\omega\varepsilon(x)\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \end{aligned} \tag{6}$$

where $\mathbf{J}$ represents an impressed source. Assume that $\varepsilon(x)$ and $\mu(x)$ are piecewise constant, complex-valued functions, forming M material layers, with the j^{th} layer being

$$\Omega_j = \{(x, y, z) : x \in (a_{j-1}, a_j), -\infty < y, z < \infty\}, \tag{7}$$

where

$$\left.\begin{aligned} \varepsilon(x) &= \varepsilon_j \\ \mu(x) &= \mu_j \end{aligned}\right\} \quad \text{for} \quad x \in (a_{j-1}, a_j) \tag{8}$$

with $\mathrm{Im}\{\varepsilon_j\} < 0$, $\mathrm{Im}\{\mu_j\} \leq 0$. As an example, a three-layer parallel-plate waveguide is shown in Figure 5. For simplicity, assume that the source current has compact support in one region, Ω_i.

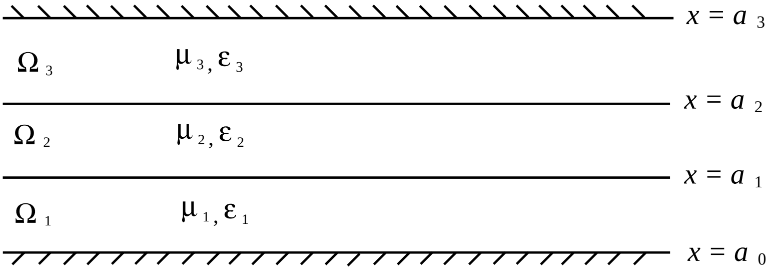

Figure 5. General, three layer parallel-plate waveguide.

In addition to satisfying Maxwell's equations (6), **E** and **H** satisfy the continuity conditions at each dielectric interface a_i, $i = 1, \ldots, M-1$,

$$\begin{aligned} \widehat{\mathbf{x}} \times \left(\mathbf{E}^{+}(\mathbf{r}) - \mathbf{E}^{-}(\mathbf{r})\right) &= \mathbf{0}, \quad \mathbf{r} \in a_i, \\ \widehat{\mathbf{x}} \times \left(\mathbf{H}^{+}(\mathbf{r}) - \mathbf{H}^{-}(\mathbf{r})\right) &= \mathbf{0}, \quad \mathbf{r} \in a_i, \end{aligned} \tag{9}$$

where $+/-$ indicates a position infinitesimally above/below the interface, the boundary conditions at the surface of the perfect conductors, interfaces a_0 and a_M,

$$\widehat{\mathbf{x}} \times \mathbf{E}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in a_0, a_M, \tag{10}$$

and the condition at infinity

$$\lim_{r\to\infty} |\mathbf{E}(\mathbf{r})|, |\mathbf{H}(\mathbf{r})| = O(r^{-1-\delta}), \qquad \begin{array}{l} \delta > 0, \ \mathbf{r} \in \Omega_j, \\ j = 1, 2, \ldots, M \end{array} . \tag{11}$$

The condition at infinity replaces the usual radiation condition for the case of dissipative media, and leads to a unique solution of the excitation problem [6]. It also renders the fields Fourier transformable in the longitudinal coordinates in the classical sense, since they decay sufficiently fast as $|y|, |z| \to \infty$.

The electric field in the j^{th} layer is

$$\begin{aligned} \mathbf{E}^{(j)}(\mathbf{r}) &= (k_j^2 + \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot)\boldsymbol{\pi}^{(j)}(\mathbf{r}), \\ \mathbf{H}^{(j)}(\mathbf{r}) &= i\omega\varepsilon_j \boldsymbol{\nabla} \times \boldsymbol{\pi}^{(j)}(\mathbf{r}) \end{aligned} \tag{12}$$

where $k_j = \omega\sqrt{\mu_j \varepsilon_j}$, and where the potential $\boldsymbol{\pi}^{(j)}$ is obtained by solving the set of M Helmholtz equations (one for each region)

$$\begin{aligned}
&(\nabla^2 + k_i^2)\boldsymbol{\pi}^{(i)}(\mathbf{r}) = -\frac{\mathbf{J}(\mathbf{r})}{i\omega\varepsilon_i}, && \mathbf{r}\in\Omega_i, \\
&(\nabla^2 + k_j^2)\boldsymbol{\pi}^{(j)}(\mathbf{r}) = \mathbf{0}, && \begin{array}{c}\mathbf{r}\in\Omega_j,\\ j=1,2,\ldots,M,\\ j\neq i,\end{array}
\end{aligned} \tag{13}$$

subject to the condition at infinity

$$\lim_{r\to\infty}\left|\boldsymbol{\pi}^{(j)}(\mathbf{r})\right| = O(r^{-1-\delta}), \qquad \begin{array}{l}\delta>0,\ \mathbf{r}\in\Omega_j,\\ j=1,2,\ldots,M,\end{array} \tag{14}$$

and appropriate boundary and continuity conditions on $\boldsymbol{\pi}$ at the interfaces a_i, $i = 0, 1, \ldots, M$, arising from (9)–(10). Specifically, at each dielectric interface a_i, $i = 1, \ldots, M-1$, the potential must satisfy [17]

$$\begin{aligned}
\pi_\beta^+ &= N_\mp^2 \pi_\beta^-, \quad \beta = x, y, z, \\
\frac{\partial \pi_\alpha^+}{\partial x} &= N_\mp^2 \frac{\partial \pi_\alpha^-}{\partial x}, \quad \alpha = y, z, \\
\left(\frac{\partial \pi_x^+}{\partial x} - \frac{\partial \pi_x^-}{\partial x}\right) &= -(N_\mp^2 - 1)\left(\frac{\partial \pi_y^-}{\partial y} + \frac{\partial \pi_z^-}{\partial z}\right),
\end{aligned} \tag{15}$$

where $N_\mp^2 = \varepsilon^-/\varepsilon^+$, and at each surface of the perfect conductors, interfaces a_0 and a_M,

$$\begin{aligned}
\pi_\alpha &= 0, \quad \alpha = y, z, \\
\frac{\partial \pi_x}{\partial x} &= 0.
\end{aligned} \tag{16}$$

To facilitate spectral analysis one can consider the M Helmholtz equations (13) as one equation,

$$-\big(\nabla^2 + k^2(x)\big)\boldsymbol{\pi}(\mathbf{r}) = \frac{\mathbf{J}(\mathbf{r})}{i\omega\varepsilon(x)}, \quad \mathbf{r} \in \bigcup_{j=1}^{M} \Omega_j, \tag{17}$$

where $k(x) = \omega\sqrt{\mu(x)\varepsilon(x)}$ subject to the above described continuity and boundary conditions imposed at the various interfaces, and the condition at infinity.

The equation for the potential (17) can be solved subject to the continuity and boundary conditions (15)–(16) to yield [17]

$$\boldsymbol{\pi}^{(j)}(\mathbf{r}) = \int_\Omega \underline{\mathbf{G}}^{(j,i)}(\mathbf{r}, \mathbf{r}') \cdot \frac{\mathbf{J}(\mathbf{r}')}{i\omega\varepsilon_i} d\Omega', \tag{18}$$

where $\underline{\mathbf{G}}^{(j,i)}(\mathbf{r}, \mathbf{r}')$ provides the field at $\mathbf{r}$ in region j due to an elemental current source at $\mathbf{r}'$ in region i, and is not, in general, a diagonal dyadic. The Green's dyadic is a solution of

$$(\nabla^2 + k_i^2)\underline{\mathbf{G}}^{(i,i)}(\mathbf{r}, \mathbf{r}') = -\underline{\mathbf{I}}\boldsymbol{\delta}(\mathbf{r} - \mathbf{r}'), \qquad \mathbf{r}, \mathbf{r}' \in \Omega_i, \tag{19}$$

$$(\nabla^2 + k_j^2)\underline{\mathbf{G}}^{(j,i)}(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{0}}, \qquad \begin{array}{c} \mathbf{r} \in \Omega_j, \ \mathbf{r}' \in \Omega_i, \\ j = 1, 2, \ldots, M, \ j \neq i, \end{array} \tag{20}$$

which can be combined into the single equation

$$-\big(\nabla^2 + k^2(x)\big)\underline{\mathbf{G}}^{(j,i)}(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{I}}\boldsymbol{\delta}(\mathbf{r} - \mathbf{r}') \tag{21}$$

where $\mathbf{r}' \in \Omega_i$. The Green's dyadic is subject to the condition at infinity

$$\lim_{r \to \infty} \left|\underline{\mathbf{G}}^{(j,i)}\right| = O(r^{-1-\delta}), \qquad \begin{array}{l} \delta > 0, \ \mathbf{r} \in \Omega_j, \\ j = 1, 2, \ldots, M \end{array} \tag{22}$$

and to the boundary and continuity conditions imposed on π_α elevated to dyadic level; at each dielectric interface a_i, $i = 1, \ldots, M-1$,

$$G^+_{x\beta} = N^2_\mp G^-_{x\beta}, \quad G^+_{\alpha\alpha} = N^2_\mp G^-_{\alpha\alpha}, \tag{23}$$

$$\left(\frac{\partial G^+_{xx}}{\partial x} - \frac{\partial G^-_{xx}}{\partial x}\right) = 0, \quad \frac{\partial G^+_{\alpha\alpha}}{\partial x} = N^2_\mp \frac{\partial G^-_{\alpha\alpha}}{\partial x}, \tag{24}$$

$$\left(\frac{\partial G^+_{x\alpha}}{\partial x} - \frac{\partial G^-_{x\alpha}}{\partial x}\right) = -(N^2_\mp - 1)\frac{\partial G^-_{\alpha\alpha}}{\partial \alpha}, \tag{25}$$

$$\beta = x, y, z, \quad \alpha = y, z, \tag{26}$$

and at the perfectly conducting walls (a_0 and a_M) boundary conditions are

$$G_{\alpha\alpha} = 0, \qquad \alpha = y, z \tag{27}$$

$$\frac{\partial G_{x\beta}}{\partial x} = 0, \qquad \beta = x, y, z. \tag{28}$$

Due to the longitudinal invariance in the y and z coordinates and the condition at infinity, one can write (21) in the spatial Fourier transform domain as

$$-\left(\frac{d^2}{dx^2} - p^2(x)\right)\underline{\mathbf{G}}^{(j,i)}(x, x', k_y, k_z) = \underline{\mathbf{I}}\boldsymbol{\delta}(x - x') \tag{29}$$

where $p(x) = \sqrt{k_y^2 + k_z^2 - k^2(x)}$, leading to

$$\underline{\mathbf{G}}^{(j,i)}(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\mathbf{G}}^{(j,i)}(x, x', k_y, k_z)e^{ik_y(y-y')}e^{ik_z(z-z')}dk_y dk_z \tag{30}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\underline{\mathbf{G}}^{(j,i)}(x, x', k_\rho)H_0^{(2)}(k_\rho \rho)k\rho dk_\rho \tag{31}$$

where $k_\rho^2 = k_y^2 + k_z^2$ and $\rho = \sqrt{(y-y')^2 + (z-z')^2}$. The first form (30) comes from a double Fourier transform on the infinite coordinates y and z, and the second form (31) can be obtained from the first form. Therefore, both forms are "safe" in that they don't rely on spectral properties of the x-coordinate part of the operator $-\nabla^2$. Instead, they make use of completeness of the y- and z-coordinate continuous eigenfunctions in $\mathbf{L}^2(-\infty, \infty) \times \mathbf{L}^2(-\infty, \infty)$ (i.e., the Fourier transform technique). It can be shown that the form

$$\begin{aligned}\underline{\mathbf{G}}^{(j,i)}(x, x', k_\rho) = \,&\widehat{\mathbf{x}}\widehat{\mathbf{x}} r_n^{(j,i)}(x, x', k_\rho) + (\widehat{\mathbf{y}}\widehat{\mathbf{y}} + \widehat{\mathbf{z}}\widehat{\mathbf{z}})r_t^{(j,i)}(x, x', k_\rho)\\ &+ \left(\widehat{\mathbf{x}}\widehat{\mathbf{y}}\frac{\partial}{\partial y} + \widehat{\mathbf{x}}\widehat{\mathbf{z}}\frac{\partial}{\partial z}\right) r_c^{(j,i)}(x, x', k_\rho)\end{aligned} \tag{32}$$

holds in general for any planarly-layered medium (i.e., for any number of layers). The coefficients $r_\nu^{(j,i)}$ are determined by the specific structure of the waveguide and have the general form

$$r_n^{(j,i)}(x, x', k_\rho) = \frac{n_n(x, x', k_\rho)}{z^{tm}(k_\rho)} = \frac{n_1^n(x_<, k_\rho)n_2^n(x_>, k_\rho)}{z^{tm}(k_\rho)}, \tag{33}$$

$$r_t^{(j,i)}(x, x', k_\rho) = \frac{n_t(x, x', k_\rho)}{z^{te}(k_\rho)} = \frac{n_1^t(x_<, k_\rho)n_2^t(x_>, k_\rho)}{z^{te}(k_\rho)}, \tag{34}$$

$$r_c^{(j,i)}(x, x', k_\rho) = \frac{n_c(x, x', k_\rho)}{z^{tm}(k_\rho)z^{te}(k_\rho)} = K_c(k_\rho)\frac{n_2^n(x, k_\rho)n_2^t(x', k_\rho)}{z^{te}(k_\rho)z^{tm}(k_\rho)}, \tag{35}$$

where $x_<$ ($x_>$) indicates the lesser (greater) of the pair x and x', $z^{tm(te)}(k_\rho) = 0$ are the dispersion equations for TM (TE) surface-wave modes of the inhomogeneous parallel-plate waveguide, and n_ν, $\nu = n, t, c$, are coefficients that depend on (i, j) and on the structure of the waveguide (see, e.g., (118)–(124)).

Due to the presence of the top and bottom perfectly conducting plates, the dyadic $\underline{\mathbf{G}}^{(j,i)}(x, x', k_\rho)$ is meromorphic in the lower-half complex k_ρ-plane, leading to

$$\begin{aligned}&\underline{\mathbf{G}}^{(j,i)}(\mathbf{r},\mathbf{r}') = \\ &-\frac{i}{2}\sum_{n_e=1}^{\infty}\left(\widehat{\mathbf{x}}\widehat{\mathbf{x}}R_n^{(j,i)}(\mathbf{r},\mathbf{r}',k_{\rho n_e}) + \left(\widehat{\mathbf{x}}\widehat{\mathbf{y}}\frac{\partial}{\partial y} + \widehat{\mathbf{x}}\widehat{\mathbf{z}}\frac{\partial}{\partial z}\right)R_c^{(j,i)}(\mathbf{r},\mathbf{r}',k_{\rho n_e})\right) \\ &-\frac{i}{2}\sum_{n_h=1}^{\infty}\left((\widehat{\mathbf{y}}\widehat{\mathbf{y}}+\widehat{\mathbf{z}}\widehat{\mathbf{z}})R_t^{(j,i)}(\mathbf{r},\mathbf{r}',k_{\rho n_h})+\left(\widehat{\mathbf{x}}\widehat{\mathbf{y}}\frac{\partial}{\partial y}+\widehat{\mathbf{x}}\widehat{\mathbf{z}}\frac{\partial}{\partial z}\right)R_c^{(j,i)}(\mathbf{r},\mathbf{r}',k_{\rho n_h})\right)\end{aligned} \tag{36}$$

where $R_{n,t,c}^{(j,i)}$ are residues of the product $\left(r_{n,t,c}^{(j,i)}(x,x',k_\rho)H_0^{(2)}(k_\rho\rho)k_\rho\right)$, respectively, with the index n_e used for TM (E) modes and n_h for TE (H) modes.

Using the formula for the residue of a function f at an m^{th}-order pole $k_\rho = k_{\rho n}$,

$$\mathrm{Res}_n(f) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dk_\rho^{m-1}}(k_\rho - k_{\rho n})^m f(k_\rho)\bigg|_{k_\rho = k_{\rho n}}, \tag{37}$$

we have for first-order poles (the usual case)

$$R_\nu^{(i,j)}(\mathbf{r},\mathbf{r}',k_{\rho n}) = \frac{n_\nu(x,x',k_{\rho n})H_0^{(2)}(k_{\rho n}\rho)k_{\rho n}}{z'(k_{\rho n})}. \tag{38}$$

It will be shown that n_ν are proportional to the product of eigenfunctions $u_n(x)$ and conjugate adjoint eigenfunctions $\bar{v}_n(x')$ (see (82)–(85)). Therefore, the first-order residues have the form of the product of eigenfunctions and conjugate adjoint eigenfunctions, multiplied by the radial Green's functions $H_0^{(2)}(k_{\rho n}\rho)$,

$$H_0^{(2)}(k_{\rho n}\rho) \sim \sqrt{\frac{2}{\pi k_{\rho n}\rho}}e^{-i(k_{\rho n}\rho - \pi/4)} \sim \frac{e^{-ik_{\rho n}\rho}}{\sqrt{\rho}} \tag{39}$$

for $|k_{\rho n}\rho| \gg 1$, which provides the usual cylindrical wave propagation behavior.

For second-order poles,

$$R_{\nu}^{(i,j)}(\mathbf{r},\mathbf{r}',k_{\rho n}) = \frac{d}{dk_\rho}\left[(k_\rho - k_{\rho n})^2 \frac{n_\nu(x,x',k_\rho)H_0^{(2)}(k_\rho\rho)k_\rho}{z(k_\rho)}\right]\Bigg|_{k_\rho=k_{\rho n}} \tag{40}$$

$$= \frac{2}{z''(k_{\rho n})}\left(\frac{d}{dk_\rho}\left[n_\nu(x,x',k_\rho)H_0^{(2)}(k_\rho\rho)k_\rho\right]\Bigg|_{k_\rho=k_{\rho n}} - \frac{\left[n_\nu(x,x',k_{\rho n})H_0^{(2)}(k_{\rho n}\rho)k_{\rho n}\right]z'''(k_{\rho n})}{3z''(k_{\rho n})}\right). \tag{41}$$

It is clear from (41) that the second-order residues can be written as

$$R_{\nu}^{(i,j)}(\mathbf{r},\mathbf{r}',k_{\rho n}) = c_0 n_\nu(x,x',k_{\rho n})H_0^{(2)}(k_{\rho n}\rho) + c_1 n'_\nu(x,x',k_{\rho n})H_0^{(2)}(k_{\rho n}\rho) + c_2 n_\nu(x,x',k_{\rho n})\rho H_1^{(2)}(k_{\rho n}\rho) \tag{42}$$

where $c_{0,1,2}$ are constants and $n'_\nu = dn_\nu/dk_\rho$. The term

$$n_\nu(x,x',k_{\rho n})H_0^{(2)}(k_{\rho n}\rho) \tag{43}$$

is the same as that encountered in the case of first-order residues (38), and has the usual propagation behavior of eigenfunctions propagating as cylindrical waves. Since derivatives of eigenfunctions lead to associated functions as shown later (see, e.g., (97)), then the term

$$n'_\nu(x,x',k_{\rho n})H_0^{(2)}(k_{\rho n}\rho) \tag{44}$$

is related to associated functions propagating as cylindrical waves. The last term in (42),

$$n_\nu(x,x',k_{\rho n})\rho H_1^{(2)}(k_{\rho n}\rho) \tag{45}$$

involves the eigenfunctions via n_ν. However, the propagation factor is $\rho H_1^{(2)}$, which has the asymptotic form

$$\rho H_1^{(2)}(k_{\rho n}\rho) \sim \rho\left(\sqrt{\frac{2}{\pi k_{\rho n}\rho}}e^{-i(k_{\rho n}\rho - 3\pi/4)}\right) \sim \sqrt{\rho}e^{-ik_{\rho n}\rho} \tag{46}$$

for $|k_{\rho n}\rho| \gg 1$. Therefore, this term has unusual propagation characteristics, and would grow according to the factor $\sqrt{\rho}$ if not for the exponential decay provided by the complex-valued poles (due to the assumed dielectric loss, which itself is often the reason for the occurrence of nontrivial modal degeneracies). Thus, the second-order poles lead to residue contributions which are very different than for the first-order poles.

In the case of lossless media the conditions at infinity (11), (14), and (22) are no longer valid. In this case waveguide radiation conditions must be used, and the corresponding problem is non self-adjoint even for the case of homogeneous media. As a result, complex modes may still occur in the lossless case, although they carry zero total power flux. A principle of radiation that leads to a radiation condition is formulated in [18, 19].

2.3. Dyadic Eigenfunctions

To interpret the dyadic residue series (36) we introduce dyadic eigenfunctions satisfying (analogous to (29))

$$-\left(\frac{d^2}{dx^2} - p^2(x)\right)\underline{\mathbf{u}}_n(x) = \lambda_n \underline{\mathbf{u}}_n(x), \tag{47}$$

where $\underline{\mathbf{u}}_n$ has scalar components $u_{n,i,j}$, $i,j = x,y,z$, and where $p^2(x) = k_\rho^2 - k^2(x)$. We rewrite (47) as

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right)\underline{\mathbf{u}}_n(x) = \gamma_n \underline{\mathbf{u}}_n(x), \quad x \in (a_0, a_M), \tag{48}$$

where $\gamma_n = \lambda_n - k_\rho^2$, subject to the continuity conditions at the dielectric interfaces a_i, $i = 1, \ldots, M-1$,

$$u^+_{n,x\beta} = N^2_\mp u^-_{n,x\beta}, \quad u^+_{n,\alpha\alpha} = N^2_\mp u^-_{n,\alpha\alpha}, \tag{49}$$

$$\frac{\partial u^+_{n,xx}}{\partial x} = \frac{\partial u^-_{n,xx}}{\partial x}, \quad \frac{\partial u^+_{n,\alpha\alpha}}{\partial x} = N^2_\mp \frac{\partial u^-_{n,\alpha\alpha}}{\partial x}, \tag{50}$$

$$\left(\frac{\partial u^+_{n,x\alpha}}{\partial x} - \frac{\partial u^-_{n,x\alpha}}{\partial x}\right) = -(N^2_\mp - 1)(ik_\alpha)u^-_{n,\alpha\alpha}, \tag{51}$$

$$\beta = x,y,z, \quad \alpha = y,z, \tag{52}$$

and boundary conditions at the perfect conducting walls (a_0 and a_M)

$$u_{n,\alpha\alpha} = 0, \qquad \alpha = y,z, \tag{53}$$

$$\frac{\partial u_{n,x\beta}}{\partial x} = 0, \qquad \beta = x,y,z. \tag{54}$$

Note that $u_{n,\beta\alpha}(x) = u_{n,\beta\alpha}(x, \gamma_n)$.

From the conditions (49)–(54) it is clear that only dyadic components $u_{n,x\beta}$, $\beta = x, y, z$, and $u_{n,\alpha\alpha}$, $\alpha = y, z$, are nonzero, and, moreover, that the equations for $u_{n,xx}$ and for the pairs $\left(u_{n,\alpha\alpha}(x), u_{n,x\alpha}(x)\right)$, $\alpha = y, z$, all decouple from each other. Also, it can be shown that the continuity and boundary conditions for $u_{n,\alpha\alpha}, \alpha = y, z$, result in TE modes, those for $u_{n,xx}$ lead to TM modes, and those for $u_{n,x\alpha}$, $\alpha = y, z$, can result in either type of mode (depending on the presence or absence of $u^{-}_{n,\alpha\alpha}$ on the right-side of (51)). Therefore we have five decoupled (independent) dyadic problems,

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right) \underline{\mathbf{u}}^{xx}_{n_e}(x) = \gamma_{n_e} \underline{\mathbf{u}}^{xx}_{n_e}(x), \tag{55}$$

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right) \underline{\mathbf{u}}^{\alpha\alpha}_{n_e}(x) = \gamma_{n_e} \underline{\mathbf{u}}^{\alpha\alpha}_{n_e}(x), \qquad \alpha = y, z, \tag{56}$$

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right) \underline{\mathbf{u}}^{\alpha\alpha}_{n_h}(x) = \gamma_{n_h} \underline{\mathbf{u}}^{\alpha\alpha}_{n_h}(x), \qquad \alpha = y, z, \tag{57}$$

with n_e indicating TM (E) modes and n_h indicating TE (H) modes, where

$$\underline{\mathbf{u}}^{xx}_{n_e}(x) = \begin{bmatrix} u_{n_e,xx}(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \underline{\mathbf{u}}^{yy}_{n_e}(x) = \begin{bmatrix} 0 & u_{n_e,xy}(x) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\underline{\mathbf{u}}^{yy}_{n_h}(x) = \begin{bmatrix} 0 & u_{n_h,xy}(x) & 0 \\ 0 & u_{n_h,yy}(x) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \underline{\mathbf{u}}^{zz}_{n_e}(x) = \begin{bmatrix} 0 & 0 & u_{n_e,xz}(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\underline{\mathbf{u}}^{zz}_{n_h}(x) = \begin{bmatrix} 0 & 0 & u_{n_h,xz}(x) \\ 0 & 0 & 0 \\ 0 & 0 & u_{n_h,zz}(x) \end{bmatrix}. \tag{58}$$

Note that $\underline{\mathbf{u}}^{\alpha\alpha}_{n}$ only needs to be obtained for one α, either $\alpha = y$ or $\alpha = z$; up to an arbitrary constant the corresponding non-zero entries have the same form.

Since the operator is nonself-adjoint we must consider dyadic eigenfunctions adjoint to the ones considered above. Let $\Gamma = (a_0, a_M)$ and consider the inner-product

$$\langle \underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle = \int_\Gamma \underline{\mathbf{u}}(x) : \overline{\underline{\mathbf{v}}}(x) dx \tag{59}$$

$$= \int_\Gamma \sum_{i,j=1}^{3} u_{ij}(x) \bar{v}_{ij}(x) dx \tag{60}$$

utilizing the double-dot product notation [20], where the overbar indicates complex conjugation. The dyadic functions of interest belong to the Hilbert space $H = \mathbf{L}^2(\Gamma)$ consisting of dyadics such that

$$\|\underline{\mathbf{u}}\|^2 = \langle \underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle = \int_\Gamma \underline{\mathbf{u}}(x) : \overline{\underline{\mathbf{u}}}(x) dx \tag{61}$$

$$= \int_\Gamma \sum_{i,j=1}^{3} |u_{ij}(x)|^2 dx < \infty. \tag{62}$$

Defining the operator $A : H \to H$ as

$$\begin{aligned} A &= -\left(\frac{d^2}{dx^2} + k^2(x) \right), \\ D_A &= \{ \underline{\mathbf{u}} : \|\underline{\mathbf{u}}\| < \infty \} \end{aligned} \tag{63}$$

where D_A is the domain of A, then from

$$\langle A\underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle = \langle \underline{\mathbf{u}}, A^* \underline{\mathbf{v}} \rangle \tag{64}$$

the operator adjoint to (63) is

$$\begin{aligned} A^* &= -\left(\frac{d^2}{dx^2} + \bar{k}^2(x) \right), \\ D_{A^*} &= \{ \underline{\mathbf{v}} : \|\underline{\mathbf{v}}\| < \infty \} = D_A \end{aligned} \tag{65}$$

leading to the adjoint eigenvalue problems

$$-\left(\frac{d^2}{dx^2} + \bar{k}^2(x) \right) \underline{\mathbf{v}}_{n_e}^{xx}(x) = \gamma_{n_e}^* \underline{\mathbf{v}}_{n_e}^{xx}(x), \tag{66}$$

$$-\left(\frac{d^2}{dx^2} + \bar{k}^2(x) \right) \underline{\mathbf{v}}_{n_e}^{\alpha\alpha}(x) = \gamma_{n_e}^* \underline{\mathbf{v}}_{n_e}^{\alpha\alpha}(x), \qquad \alpha = y, z, \tag{67}$$

$$-\left(\frac{d^2}{dx^2} + \bar{k}^2(x) \right) \underline{\mathbf{v}}_{n_h}^{\alpha\alpha}(x) = \gamma_{n_h}^* \underline{\mathbf{v}}_{n_h}^{\alpha\alpha}(x), \qquad \alpha = y, z, \tag{68}$$

where $\gamma_n^* = \overline{\gamma}_n$. Adjoint continuity conditions are found to be quite different than (49)–(52), and are obtained as

$$v_{n,x\beta}^- = v_{n,x\beta}^+, \quad v_{n,\alpha\alpha}^- = \overline{N}_\mp^2 v_{n,\alpha\alpha}^+, \tag{69}$$

$$\frac{\partial v_{n,x\beta}^-}{\partial x} = \overline{N}_\mp^2 \frac{\partial v_{n,x\beta}^+}{\partial x}, \tag{70}$$

$$\frac{\partial v_{n,\alpha\alpha}^-}{\partial x} - \overline{N}_\mp^2 \frac{\partial v_{n,\alpha\alpha}^+}{\partial x} = -(\overline{N}_\mp^2 - 1)(ik_\alpha) v_{n,x\alpha}^+, \tag{71}$$

$$\beta = x, y, z, \quad \alpha = y, z, \tag{72}$$

at the dielectric interfaces a_i, $i = 1, \ldots, M-1$. Adjoint boundary conditions at the perfectly conducting walls $(x = a_0, a_M)$ are the same as (53)–(54),

$$v_{n,\alpha\alpha} = 0, \qquad \alpha = y, z, \tag{73}$$

$$\frac{\partial v_{n,x\beta}}{\partial x} = 0, \qquad \beta = x, y, z. \tag{74}$$

From the boundary and continuity conditions it is found that the adjoint eigenfunctions have the form

$$\underline{\mathbf{v}}^{xx}_{n_e}(x) = \begin{bmatrix} v_{n_e,xx}(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \underline{\mathbf{v}}^{yy}_{n_e}(x) = \begin{bmatrix} 0 & v_{n_e,xy}(x) & 0 \\ 0 & v_{n_e,yy}(x) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\underline{\mathbf{v}}^{yy}_{n_h}(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v_{n_h,yy}(x) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \underline{\mathbf{v}}^{zz}_{n_e}(x) = \begin{bmatrix} 0 & 0 & v_{n_e,xz}(x) \\ 0 & 0 & 0 \\ 0 & 0 & v_{n_e,zz}(x) \end{bmatrix},$$

$$\underline{\mathbf{v}}^{zz}_{n_h}(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v_{n_h,zz}(x) \end{bmatrix}. \tag{75}$$

It is a simple matter to obtain the orthogonality relationship

$$(\gamma_n - \gamma_m)\langle \underline{\mathbf{u}}^{\beta\beta}_n, \underline{\mathbf{v}}^{\beta\beta}_m \rangle = 0, \quad \beta = x, y, z. \tag{76}$$

Furthermore, by properties of the dyadic double-dot product,

$$\langle \underline{\mathbf{u}}^{\alpha\alpha}_n, \underline{\mathbf{v}}^{\beta\beta}_m \rangle = 0, \quad \alpha, \beta = x, y, z, \quad \alpha \neq \beta \tag{77}$$

If the dyadic eigenfunctions form an orthonormal basis of H then making the expansion

$$\begin{aligned} \underline{\mathbf{G}}(x, x', k_\rho) = &\sum_{n_e} \left(a_{n_e} \underline{\mathbf{u}}^{xx}_{n_e}(x) + b_{n_e} \underline{\mathbf{u}}^{yy}_{n_e} + c_{n_e} \underline{\mathbf{u}}^{zz}_{n_e} \right) \\ &+ \sum_{n_h} \left(d_{n_h} \underline{\mathbf{u}}^{yy}_{n_h}(x) + e_{n_h} \underline{\mathbf{u}}^{zz}_{n_h}(x) \right) \end{aligned} \tag{78}$$

and exploiting orthonormality leads to

$$\begin{aligned} &\underline{\mathbf{G}}(x, x', k_\rho) \\ &\quad = \sum_{n_e} \left(\frac{\underline{\mathbf{u}}^{xx}_{n_e}(x)\bar{v}_{n_e,xx}(x')}{k_\rho^2 + \gamma_{n_e}} + \frac{\underline{\mathbf{u}}^{yy}_{n_e}(x)\bar{v}_{n_e,yy}(x')}{k_\rho^2 + \gamma_{n_e}} + \frac{\underline{\mathbf{u}}^{zz}_{n_e}(x)\bar{v}_{n_e,zz}(x')}{k_\rho^2 + \gamma_{n_e}} \right) \\ &\qquad + \sum_{n_h} \left(\frac{\underline{\mathbf{u}}^{yy}_{n_h}(x)\bar{v}_{n_h,yy}(x')}{k_\rho^2 + \gamma_{n_h}} + \frac{\underline{\mathbf{u}}^{zz}_{n_h}(x)\bar{v}_{n_h,zz}(x')}{k_\rho^2 + \gamma_{n_h}} \right) \end{aligned} \tag{79}$$

such that

$$\underline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk_\rho k_\rho H_0^{(2)}(k_\rho \rho)\Bigg\{\sum_{n_e}\Bigg(\frac{\underline{\mathbf{u}}_{n_e}^{xx}(x)\bar{v}_{n_e,xx}(x')}{k_\rho^2+\gamma_{n_e}} + \frac{\underline{\mathbf{u}}_{n_e}^{yy}(x)\bar{v}_{n_e,yy}(x')}{k_\rho^2+\gamma_{n_e}} + \frac{\underline{\mathbf{u}}_{n_e}^{zz}(x)\bar{v}_{n_e,zz}(x')}{k_\rho^2+\gamma_{n_e}}\Bigg) + \sum_{n_h}\Bigg(\frac{\underline{\mathbf{u}}_{n_h}^{yy}(x)\bar{v}_{n_h,yy}(x')}{k_\rho^2+\gamma_{n_h}} + \frac{\underline{\mathbf{u}}_{n_h}^{zz}(x)\bar{v}_{n_h,zz}(x')}{k_\rho^2+\gamma_{n_h}}\Bigg)\Bigg\}. \tag{80}$$

Due to the factor (ik_α) in the continuity conditions (51) and (71), the $x\alpha$ entries of the dyadics $\underline{\mathbf{u}}_n^{\alpha\alpha}$ have a multiplicative factor (ik_α) such that complex-plane analysis leads to the Green's dyadic as a partial (in the x-coordinate) eigenfunction expansion, $(k_{\rho n} = i\sqrt{\gamma_n})$

$$\underline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{4i}\sum_{n_e}\big(\underline{\mathbf{u}}_{n_e}^{xx}(x)\bar{v}_{n_e,xx}(x') + \underline{\mathbf{u}}_{n_e}^{yy}(x)\bar{v}_{n_e,yy}(x') + \underline{\mathbf{u}}_{n_e}^{zz}(x)\bar{v}_{n_e,zz}(x')\big)H_0^{(2)}(k_{\rho n_e}\rho) + \frac{1}{4i}\sum_{n_h}\big(\underline{\mathbf{u}}_{n_h}^{yy}(x)\bar{v}_{n_h,yy}(x') + \underline{\mathbf{u}}_{n_h}^{zz}(x)\bar{v}_{n_h,zz}(x')\big)H_0^{(2)}(k_{\rho n_h}\rho) \tag{81}$$

where the $x\alpha$ entries of the dyadics $\underline{\mathbf{u}}_n^{\alpha\alpha}$ have a multiplicative factor $\partial/\partial\alpha$.

Comparing (36) and (38) for first-order poles with (81) we see that

$$\frac{2n_n(x,x',k_{\rho n_e})k_{\rho n_e}}{\big(z^{tm}(k_{\rho n_e})\big)'} = u_{n_e,xx}(x)\bar{v}_{n_e,xx}(x'), \tag{82}$$

$$\frac{2n_t(x,x',k_{\rho n_h})k_{\rho n_h}}{\big(z^{te}(k_{\rho n_h})\big)'} = u_{n_h,\alpha\alpha}(x)\bar{v}_{n_h,\alpha\alpha}(x'), \tag{83}$$

$$\frac{2n_c(x,x',k_{\rho n_h})k_{\rho n_h}}{z^{tm}(k_{\rho n_h})\big(z^{te}(k_{\rho n_h})\big)'} = u_{n_h,x\alpha}(x)\bar{v}_{n_h,\alpha\alpha}(x'), \tag{84}$$

$$\frac{2n_c(x,x',k_{\rho n_e})k_{\rho n_e}}{\big(z^{tm}(k_{\rho n_e})\big)' z^{te}(k_{\rho n_e})} = u_{n_e,x\alpha}(x)\bar{v}_{n_e,\alpha\alpha}(x') \tag{85}$$

which provides an interpretation of the first-order dyadic residues in (36) in terms of dyadic eigenfunctions associated with eigenvalues of unit multiplicity.

2.4. Dyadic Root Functions

Since the operator is not self-adjoint the Green's dyadic (32) may have poles of order greater than one. In this case, in addition to eigenfunctions it is necessary to consider root functions [21–23], similar to the idea of a Jordan chain in matrix theory.

Consider A as defined by (63). An element $\underline{\mathbf{0}} \neq \underline{\mathbf{u}}_{n,m-1} \in H$ is a root function of rank m of the operator $A : H \to H$ corresponding to an eigenvalue γ_n if

$$\begin{aligned}(A - \gamma_n I)^m \underline{\mathbf{u}}_{n,m-1} &= \underline{\mathbf{0}},\\ (A - \gamma_n I)^{m-1} \underline{\mathbf{u}}_{n,m-1} &\neq \underline{\mathbf{0}},\end{aligned} \tag{86}$$

where m is a positive integer. Every eigenfunction of A is a root function of rank 1 ($\underline{\mathbf{u}}_{n,0} \equiv \underline{\mathbf{u}}_n$), and the root functions having rank $m > 1$ are called associated functions (functions associated with the eigenfunction). The root system of A is defined as the union of the eigenfunctions and the associated functions.

In practice, to determine the associated functions, one starts with an eigenfunction $\underline{\mathbf{u}}_n \equiv \underline{\mathbf{u}}_{n,0}$ satisfying $(A-\gamma_n I)\underline{\mathbf{u}}_n = 0$. If the equation

$$(A - \gamma_n I)\underline{\mathbf{u}}_{n,1} = \underline{\mathbf{u}}_n \tag{87}$$

has a solution $\underline{\mathbf{u}}_{n,1}$, then $\underline{\mathbf{u}}_{n,1}$ is a root function of rank 2; more specifically, an associated function associated with the eigenvalue γ_n and eigenfunction $\underline{\mathbf{u}}_n$. Continuing, if

$$(A - \gamma_n I)\underline{\mathbf{u}}_{n,2} = \underline{\mathbf{u}}_{n,1} \tag{88}$$

is solvable, then $\underline{\mathbf{u}}_{n,2}$ is another root function (rank 3), associated with the eigenvalue γ_n and eigenfunction $\underline{\mathbf{u}}_n$. In general, we consider

$$(A - \gamma_n I)\underline{\mathbf{u}}_{n,k} = \underline{\mathbf{u}}_{n,k-1} \tag{89}$$

such that the chain $\{\underline{\mathbf{u}}_n, \underline{\mathbf{u}}_{n,1}, \underline{\mathbf{u}}_{n,2}, \ldots, \underline{\mathbf{u}}_{n,j}\}$ consisting of the eigenfunctions and associated functions is called a Jordan or Keldysh chain of length $j+1$ corresponding to the eigenvalue γ_n. The same ideas apply to the adjoint eigenfunctions.

It is straightforward to obtain the orthogonality relationships

$$(\gamma_n - \gamma_m)\langle \underline{\mathbf{u}}^{\beta\beta}_{n,p}, \underline{\mathbf{v}}^{\beta\beta}_{m,q}\rangle + \langle \underline{\mathbf{u}}^{\beta\beta}_{n,p-1}, \underline{\mathbf{v}}^{\beta\beta}_{m,q}\rangle - \langle \underline{\mathbf{u}}^{\beta\beta}_{n,p}, \underline{\mathbf{v}}^{\beta\beta}_{m,q-1}\rangle = 0, \tag{90}$$

$\beta = x, y, z$, and $p, q = 0, 1, 2, \ldots$, where for notational convenience we define $\underline{\mathbf{u}}^{\beta\beta}_{n,p} \equiv 0$ for $p < 0$, and that

$$\langle \underline{\mathbf{u}}^{\alpha\alpha}_{n,p}, \underline{\mathbf{v}}^{\beta\beta}_{m,q}\rangle = 0, \quad \alpha, \beta = x, y, z, \quad \alpha \neq \beta, \quad p, q = 0, 1, 2, \ldots \tag{91}$$

Note that if $\gamma_n \neq \gamma_m$, then recursively we see that

$$\langle \underline{\mathbf{u}}_{n,p}^{\beta\beta}, \underline{\mathbf{v}}_{m,q}^{\beta\beta} \rangle = 0, \quad p, q = 0, 1, 2, \ldots \tag{92}$$

In particular, if only the k^{th} eigenvalue has multiplicity two (all others having unit multiplicity) we consider the set $\{\underline{\mathbf{u}}_k^{\beta\beta}, \underline{\mathbf{u}}_{k,1}^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta}, \underline{\mathbf{v}}_{k,1}^{\beta\beta}\}$ corresponding to the double eigenvalue γ_k, and $\{\underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_n^{\beta\beta}\}$ corresponding to the other eigenvalues $n \neq k$. Then

$$\langle \underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_m^{\beta\beta} \rangle = 0, \quad n \neq m, \quad n, m \neq k, \tag{93}$$

$$\langle \underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta} \rangle = \langle \underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_{k,1}^{\beta\beta} \rangle = 0, \quad n \neq k, \tag{94}$$

$$\langle \underline{\mathbf{u}}_k^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta} \rangle = 0, \tag{95}$$

$$\langle \underline{\mathbf{u}}_k^{\beta\beta}, \underline{\mathbf{v}}_{k,1}^{\beta\beta} \rangle = \langle \underline{\mathbf{u}}_{k,1}^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta} \rangle. \tag{96}$$

The third condition, $\langle \underline{\mathbf{u}}_k^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta} \rangle = 0$, is quite different than in the case of rank 1 root functions (eigenfunctions, in which case the usual normalization is $\langle \underline{\mathbf{u}}_k^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta} \rangle = 1$, and, regardless of normalization, $\langle \underline{\mathbf{u}}_k^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta} \rangle \neq 0$).

By taking derivatives of (89) it can be seen that

$$\underline{\mathbf{u}}_{n,p}^{\beta\beta}(x, \gamma_n) = \left(\frac{1}{p!} \frac{\partial^p}{\partial \gamma_n^p} - c \right) \underline{\mathbf{u}}_n^{\beta\beta}(x, \gamma_n) \tag{97}$$

where c is an arbitrary constant. Therefore, associated functions are related to derivatives of eigenfunctions, as discussed after (46).

For nonself-adjoint operators the eigenfunctions do not generally constitute a basis in the desired function space. However, the root system often does, and in this case the Green's dyadic may be expanded in the root functions. For simplicity we will assume all eigenvalues have multiplicity one except the k^{th} TM eigenvalue, which has multiplicity two, and we concentrate on the xx-component of the Green's dyadic (the other components follow similarly). Then

$$G_{xx}(x, x', k_\rho) = \sum_{n_e \neq k} a_{n_e} u_{n_e,xx}(x) + b_k u_{k,xx}(x) + c_k u_{k,1,xx}(x). \tag{98}$$

Exploiting orthogonality (93)–(96) assuming the root functions have been normalized as

$$\langle \underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_n^{\beta\beta} \rangle = 1, \quad n \neq k, \tag{99}$$

$$\langle \underline{\mathbf{u}}_k^{\beta\beta}, \underline{\mathbf{v}}_{k,1}^{\beta\beta} \rangle = \langle \underline{\mathbf{u}}_{k,1}^{\beta\beta}, \underline{\mathbf{v}}_k^{\beta\beta} \rangle = 1, \tag{100}$$

we obtain

$$a_{n_e} = \frac{\bar{v}_{n_e,xx}(x')}{\gamma_{n_e} + k_\rho^2}, \quad c_k = \frac{\bar{v}_{k,xx}(x')}{\gamma_k + k_\rho^2}, \tag{101}$$

$$b_k = \frac{\bar{v}_{k,1,xx}(x')}{\gamma_k + k_\rho^2} - \frac{\bar{v}_{k,xx}(x')q}{\gamma_k + k_\rho^2} - \frac{k_{\rho k}^2 \bar{v}_{k,xx}(x')}{(\gamma_k + k_\rho^2)^2} \tag{102}$$

where $q = \langle u_{k,1,xx}, u_{k,1,xx} \rangle$. Therefore

$$\begin{aligned} G_{xx}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk_\rho k_\rho H_0^{(2)}(k_\rho \rho) \Bigg\{ \sum_{n_e \neq k} \frac{u_{n_e,xx}(x) \bar{v}_{n_e,xx}(x')}{\gamma_{n_e} + k_\rho^2} \\ + u_{k,xx}(x) \left(\frac{\bar{v}_{k,1,xx}(x')}{\gamma_k + k_\rho^2} - \frac{\bar{v}_{k,xx}(x')q}{\gamma_k + k_\rho^2} - \frac{k_{\rho k}^2 \bar{v}_{k,xx}(x')}{(\gamma_k + k_\rho^2)^2} \right) \\ + \frac{u_{k,1,xx}(x) \bar{v}_{k,xx}(x')}{\gamma_k + k_\rho^2} \Bigg\} \end{aligned} \tag{103}$$

leading to

$$\begin{aligned} G_{xx}(\mathbf{r}, \mathbf{r}') = \frac{1}{4i} \Bigg(\sum_{n_e \neq k} u_{n_e,xx}(x) \bar{v}_{n_e,xx}(x') H_0^{(2)}(k_{\rho n_e} \rho) \\ + \big(u_{k,xx}(x) \bar{v}_{k,1,xx}(x') + u_{k,1,xx}(x) \bar{v}_{k,xx}(x') \\ - u_{k,xx}(x) \bar{v}_{k,xx}(x') q \big) H_0^{(2)}(k_{\rho k} \rho) \\ - \frac{\partial}{\partial k_\rho} \left(\frac{u_{k,xx}(x) \bar{v}_{k,xx}(x') k_\rho^3 H_0^{(2)}(k_\rho \rho)}{(k_\rho + i\sqrt{\gamma_k})^2} \right) \Bigg|_{k_\rho = k_{\rho k}} \Bigg) \end{aligned} \tag{104}$$

The summation corresponds to rank one root functions (eigenfunctions), and represents the residues due to first-order poles (38) in (36). The other terms in (104), arising from the k^{th} (multiplicity two) eigenvalue ($b_k u_{k,xx}(x) + c_k u_{k,1,xx}(x)$ in (98)), provide the contribution

$$\begin{aligned} e_0 u_{k,xx}(x) \bar{v}_{k,xx}(x') H_0^{(2)}(k_{\rho n} \rho) \\ + e_1 \big(u_{k,xx}(x) \bar{v}_{k,xx}(x') \big)' H_0^{(2)}(k_{\rho n} \rho) \\ + e_2 u_{k,xx}(x) \bar{v}_{k,xx}(x') \rho H_1^{(2)}(k_{\rho n} \rho) \end{aligned} \tag{105}$$

to the Green's function, where $e_{0,1,2}$ are constants. Comparing (105) and (42) we see that the contribution to the Green's function from the root functions associated with multiplicity two eigenvalues corresponding to nontrivial degeneracies, (105), has the same form as the Green's function contribution due to second-order residues, (42).

2.4.1. Example 1 — Homogeneously Filled Parallel Plates

Consider a parallel-plate structure, homogeneously filled with a medium characterized by ε and μ_0, with plate separation d. This can be obtained from the structure depicted in Figure 1 if $\varepsilon_3 = \varepsilon$, $d_2 = d$, and $d_1 = 0$. It can be shown that the Green's dyadic (32) has the form

$$\underline{\mathbf{G}}(x, x', k_\rho) = \widehat{\mathbf{x}}\widehat{\mathbf{x}} r_n(x, x', k_\rho) + (\widehat{\mathbf{y}}\widehat{\mathbf{y}} + \widehat{\mathbf{z}}\widehat{\mathbf{z}}) r_t(x, x', k_\rho). \tag{106}$$

where

$$r_t(x, x', k_\rho) = \frac{\cosh[p(x - x' \mp d)] - \cosh[p(x + x' - d)]}{2p \sinh pd} \tag{107}$$

$$= \sinh px_< \frac{\sinh[p(d - x_>)]}{p \sinh pd} \tag{108}$$

$$r_n(x, x', k_\rho) = \frac{\cosh[p(x - x' \mp d)] + \cosh[p(x + x' - d)]}{2p \sinh pd} \tag{109}$$

$$= \cosh px_< \frac{\cosh[p(d - x_>)]}{p \sinh pd} \tag{110}$$

for $x \gtrless x'$, with $p = \sqrt{k_\rho^2 - k^2}$, $k^2 = \omega^2 \mu_0 \varepsilon$, and $k_\rho^2 = k_y^2 + k_z^2$. It is clear from the form of (106) that vertical currents excite vertical potentials, whereas horizontal currents excite parallel, horizontal potentials (i.e., the dyadic is diagonal in this simple case; in (32) $r_c = 0$).

The integrands are meromorphic in the lower-half complex k_ρ-plane, with simple pole singularities at

$$\sinh p_n d = 0. \tag{111}$$

Therefore, poles occur at $p_n = \pm in\pi/d$, $n = 0, 2, 3, \ldots$, such that

$$k_{\rho n} = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2}. \tag{112}$$

Closing the integration contour in (31) with a semicircle of infinite radius in the lower-half k_ρ-plane and invoking Cauchy's theorem we obtain the Green's components as a sum of residues,

$$\underline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{4i}\sum_{n=0}^{\infty}\left\{\widehat{\mathbf{x}}\widehat{\mathbf{x}}\frac{\varepsilon_n}{d}\cos\left(\frac{n\pi}{d}x\right)\cos\left(\frac{n\pi}{d}x'\right)\right.$$
$$\left. + (\widehat{\mathbf{y}}\widehat{\mathbf{y}} + \widehat{\mathbf{z}}\widehat{\mathbf{z}})\frac{2}{d}\sin\left(\frac{n\pi}{d}x\right)\sin\left(\frac{n\pi}{d}x'\right)\right\}H_0^{(2)}(k_{\rho n}\rho) \tag{113}$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$ for $n > 0$. This discrete summation form represents an expansion over the eigenfunctions in the vertical coordinate, multiplied by a Green's function for the radial direction.

The dyadic eigenfunctions are obtained in the form (58) and (75) with $n_e = n_h = n$,

$$u_{n,xx}(x) = v_{n,xx}(x) = \sqrt{\frac{\varepsilon_n}{d}}\cos\frac{n\pi}{d}x, \tag{114}$$

$$u_{n,\alpha\alpha}(x) = v_{n,\alpha\alpha}(x) = \sqrt{\frac{2}{d}}\sin\frac{n\pi}{d}x, \tag{115}$$

$$u_{n,x\alpha}(x) = v_{n,x\alpha}(x) = 0, \quad \alpha = y, z, \tag{116}$$

$$\gamma_n = \left(\frac{n\pi}{d}\right)^2 - k^2, \quad n = 0, 1, 2, \ldots. \tag{117}$$

In this case the operator $A = -(d^2/dx^2 + k^2)$ is nonself-adjoint since k is complex-valued. However, since k is constant with respect to x the natural operator to consider for the homogeneous parallel-plate waveguide is $A = -d^2/dx^2$, leading to a self-adjoint problem. By self-adjointness, poles of multiplicity greater than one cannot occur. Thus, the homogeneously-filled parallel-plate waveguide is manifestly self-adjoint, and only residues corresponding to first-order poles (related to eigenfunctions) are implicated in the Green's function.

2.4.2. Example 2 — Two-Layer Medium

Consider the two-layer parallel-plate waveguide depicted in Figure 1. The Hertzian potential is given as (18) where $\underline{\mathbf{G}}_\pi^{(j,i)}(\mathbf{r},\mathbf{r}')$ has the form (30) or (31) with $\underline{\mathbf{G}}^{(j,i)}(x,x',k_\rho)$ given by (32). The coefficients in (32) are given by (33)-(35), where for $(j,i) = (3,3)$ the coefficients are

$$n_1^t(x,k_\rho) = (\sinh p_2 d_1 \cosh p_3 x + (p_2/p_3)\cosh p_2 d_1 \sinh p_3 x), \tag{118}$$

$$n_2^t(x,k_\rho) = \sinh p_3(d_2 - x)$$
$$= (\sinh p_3 d_2 \cosh p_3 x - \cosh p_3 d_2 \sinh p_3 x), \tag{119}$$

$$n_1^n(x,k_\rho) = (N_{23}^2 \cosh p_2 d_1 \cosh p_3 x + (p_2/p_3) \sinh p_2 d_1 \sinh p_3 x), \tag{120}$$

$$\begin{aligned} n_2^n(x,k_\rho) &= \cosh p_3(d_2 - x) \\ &= (\cosh p_3 d_2 \cosh p_3 x - \sinh p_3 d_2 \sinh p_3 x), \end{aligned} \tag{121}$$

$$K_c(k_\rho) = (N_{23}^2 - 1) \sinh p_2 d_1 \cosh p_2 d_1 \tag{122}$$

$$z^{te}(k_\rho) = p_3 \sinh p_2 d_1 \cosh p_3 d_2 + p_2 \cosh p_2 d_1 \sinh p_3 d_2, \tag{123}$$

$$z^{tm}(k_\rho) = p_3 N_{23}^2 \cosh p_2 d_1 \sinh p_3 d_2 + p_2 \sinh p_2 d_1 \cosh p_3 d_2, \tag{124}$$

where $p_j = \sqrt{k_\rho^2 - k_j^2}$.

The dyadic integrand (32) is meromorphic in the complex k_ρ-plane, leading to (36) in terms of residues $R_{n,t,c}^{(j,3)}$.

For eigenvalues having unit multiplicity the eigenfunctions in region 3 have the form (58) and are, from (55)–(57) and (66)–(68),

$$u_{n_e,xx}^{(3)}(x) = C_{n_e,xx} \frac{1}{\sinh \xi_2 d_1} n_1^n(x, \kappa_{n_e}) \tag{125}$$

$$= C_{n_e,xx} \frac{-\xi_2}{\xi_3 \sinh \xi_3 d_2} n_2^n(x, \kappa_{n_e}), \tag{126}$$

$$v_{n_e,xx}^{(3)}(x) = C_{n_e,xx}^* \frac{1}{\cosh \overline{\xi}_3 d_2} \bar{n}_2^n(x, \kappa_{n_e}) \tag{127}$$

$$= C_{n_e,xx}^* \frac{1}{\overline{N}^2 \cosh \overline{\xi}_2 d_1} \bar{n}_1^n(x, \kappa_{n_e}), \tag{128}$$

$$u_{n_h,\alpha\alpha}^{(3)}(x) = C_{n_h,\alpha\alpha} \frac{G_\alpha}{\xi_3 \cosh \xi_3 d_2} n_2^t(x, \kappa_{n_h}) \tag{129}$$

$$= C_{n_h,\alpha\alpha} \frac{-G_\alpha}{\xi_2 \cosh \xi_2 d_1} n_1^t(x, \kappa_{n_h}), \tag{130}$$

$$v_{n,\alpha\alpha}^{(3)}(x) = C_{n,\alpha\alpha}^* \frac{-1}{\cosh \overline{\xi}_3 d_2} \bar{n}_2^t(x, \kappa_n) \tag{131}$$

$$= C_{n,\alpha\alpha}^* \frac{\overline{\xi}_3}{\overline{\xi}_2 \cosh \overline{\xi}_2 d_1} \bar{n}_1^t(x, \kappa_n), \tag{132}$$

$$u_{n,x\alpha}^{(3)}(x) = C_{n,\alpha\alpha} \frac{-1}{\sinh \xi_3 d_2} n_2^n(x, \kappa_n), \tag{133}$$

$$v_{n,x\alpha}^{(3)}(x) = C_{n,x\alpha}^* \frac{-1}{\sinh \overline{\xi}_3 d_2} \bar{n}_2^n(x, \kappa_n), \tag{134}$$

where n_e indicates TM modes, n_h indicates TE modes, and n can represent either mode type. In the above, $\xi_j^2 = \kappa_n - k_j^2$ where $\kappa_{n_h} = -\gamma_{n_h}$ satisfies

$$z^{te}(\kappa_{n_h}) = 0, \tag{135}$$

$\kappa_{n_e} = -\gamma_{n_e}$ satisfies

$$z^{tm}(\kappa_{n_e}) = 0, \tag{136}$$

$$G_\alpha = \frac{\xi_2 z^{tm}}{(N_2 - 1)(ik_\alpha)\sinh\xi_2 d_1 \sinh\xi_3 d_2}, \tag{137}$$

$C_{n_e,xx}$, $C^*_{n_e,xx}$, $C_{n_h,\alpha\alpha}$, and $C^*_{n,x\alpha}$ are independent constants, and $C^*_{n,\alpha\alpha}$ is

$$C^*_{n,\alpha\alpha} = \frac{C^*_{n,x\alpha}(ik_\alpha)(\overline{N}^2 - 1)\overline{\xi}_3 \cosh\overline{\xi}_2 d_1 \cosh\overline{\xi}_3 d_2}{\overline{\xi}_2 \bar{z}^{te}}. \tag{138}$$

Comparing (118)–(121) and (125)–(134), it is clear that the numerators of the Greens function coefficients (33)–(35) are related to the eigenfunctions (corresponding to unit multiplicity eigenvalues).

As shown in the numerical results, multiplicity two eigenvalues exist for certain combinations of electrical and structural parameters. Associated functions corresponding to rank two root functions can be obtained from (125)–(134) via (97),

$$\underline{\mathbf{u}}_{n,1}^{\beta\beta}(x,\gamma_n) = \left(\frac{\partial}{\partial\gamma_n} - c\right)\underline{\mathbf{u}}_n^{\beta\beta}(x,\gamma_n), \tag{139}$$

and normalized as (100). In this case the operator $A = -\big(d^2/dx^2 + k(x)\big)$ is manifestly nonself-adjoint, having multiplicity two eigenvalues when

$$z(k_\rho)\big|_{k_{\rho n}} = \frac{d}{dk_\rho} z(k_\rho)\bigg|_{k_{\rho n}} = 0$$

with z being either z^{te} or z^{tm} in (123) or (124), respectively.

3. NUMERICAL RESULTS

The following numerical results are for the two-layer parallel-plate waveguide shown in Figure 1, where $d_1 = d_2 = d$, $\varepsilon_3 = \varepsilon_0$, and $\varepsilon_2 = (2.25 - i\varepsilon_i)\varepsilon_0$. For simplicity the component G_{xx} is examined;

Table 1. $\mathrm{Im}(\varepsilon_2) = 0.0$, $k_0 d = 1.303347$.

ρ/d	$G_{xx}^{\mathrm{int.}}$	$G_{xx}^{\mathrm{residue}}$	$\left\lvert \frac{G_{xx}^{\mathrm{int.}} - G_{xx}^{\mathrm{residue}}}{G_{xx}^{\mathrm{int.}}} \right\rvert \times 100\%$
0.5	$(0.101453, -0.213712)$	$(0.102225, -0.213712)$	0.326%
1.0	$(1.297863, -0.173037)$	$(1.298841, -0.173036)$	0.075%
5.0	$(-0.082335, 0.026255)$	$(-0.082334, 0.026256)$	0.002%
10.0	$(0.049561, 0.001304)$	$(0.049561, 0.001303)$	0.002%

Table 2. $\mathrm{Im}(\varepsilon_2) = -1.0$, $k_0 d = 1.303347$.

ρ/d	$G_{xx}^{\mathrm{int.}}$	$G_{xx}^{\mathrm{residue}}$	$\left\lvert \frac{G_{xx}^{\mathrm{int.}} - G_{xx}^{\mathrm{residue}}}{G_{xx}^{\mathrm{int.}}} \right\rvert \times 100\%$
0.5	$(0.063880, -0.176013)$	$(0.064240, -0.176105)$	0.198%
1.0	$(-0.018109, -0.126584)$	$(-0.018105, -0.126584)$	0.003%
5.0	$(0.002171, -0.001183)$	$(0.002171, -0.001183)$	0.000%
10.0	$(-0.001426, 0.000508)$	$(-0.001426, 0.000508)$	0.000%

other components are similarly obtained. In all of the results the source is a vertical electric dipole located at (x', y', z') with $x'/d = 0.25$, $y'/d = z'/d = 0.0$, and the observation point is (x, y, z) with $x/d = 0.75\,\mathrm{cm}$ and $\rho = \sqrt{y^2 + z^2}$ varying.

The dyadic Green's function as a sum-of-residues, (36), has been computed for the case of both single and multiple poles corresponding to the plots shown in Figures 2–4. In Tables 1–3 the real-line integration form of the Greens function, (31), is compared to the sum-of-residues form, (36).

The lossless case is considered in Table 1 and the case of $\mathrm{Im}(\varepsilon_2) = -1.0$ is shown in Table 2. In both cases only first-order poles are encountered, and the first five modes are used in the residue series. Residues are evaluated via (38), with numerical derivatives computed using Ridders' method of polynomial extrapolation [24, p. 182]. The modal degeneracy case, $\mathrm{Im}(\varepsilon_2) = -1.735522$, is shown in Table 3, where the first two modes form the modal degeneracy (see Figure 4) and the next three modes yield first-order residues. The second-order residue is computed from (40) using numerical derivatives. In the

Table 3. $\mathrm{Im}(\varepsilon_2) = -1.7355219$, $k_0 d = 1.303347$.

ρ/d	$G_{xx}^{\text{int.}}$	G_{xx}^{residue}	$\left\lvert \frac{G_{xx}^{\text{int.}} - G_{xx}^{\text{residue}}}{G_{xx}^{\text{int.}}} \right\rvert \times 100\%$
0.5	$(0.046295, -0.174910)$	$(0.046885, -0.175013)$	0.331%
1.0	$(-0.033862, -0.120206)$	$(-0.033995, -0.120254)$	0.113%
5.0	$(0.009484, -0.008429)$	$(0.009491, -0.008445)$	0.138%
10.0	$(0.000393, -0.000841)$	$(0.000393, -0.000842)$	0.108%

Table 4. $\mathrm{Im}(\varepsilon_2) = -1.735522$, $k_0 d = 1.303347$.

ρ/d	$\sum(G_{xx}^{\text{residue}})_{\text{1st}-\text{order}}$	$(G_{xx}^{\text{residue}})_{\text{2nd}-\text{order}}$
0.5	$(-0.032638, 0.004325)$	$(0.079523, -0.179338)$
1.0	$(-0.005760, 0.001105)$	$(-0.028234, -0.121359)$
5.0	$(0.0, 0.0)$	$(0.009491, -0.008445)$
10.0	$(0.0, 0.0)$	$(0.000393, -0.000842)$

event of a modal degeneracy the second-order residues provide the main contribution to the Green's function, as discussed later.

In all three tables the agreement between the Green's functions is quite good. In general, one observes that as ρ increases the agreement between the two methods improves. Note that any disagreement between the two methods is due either to numerical errors in evaluating the integral form (31), or to errors evaluating the Hankel function having complex argument in (36). In principle, the two Green's functions should agree exactly. No particular emphasis was placed on refining the numerical methods utilized, the point of the paper being analysis and interpretation of the residues in the case of modal degeneracies.

For the case of a modal degeneracy the contributions to the Green's function component G_{xx} of the first order residues, corresponding to eigenfunctions, and the second order residue, corresponding to an associated function, is shown in Table 4. It is clear that the second-order residue provides the dominant contribution to the Green's function.

Finally, in Figure 6 we show $|G_{xx}|$ versus ρ/d for $\mathrm{Im}(\varepsilon_2) = \varepsilon_i = 0.0$, $\varepsilon_i = -1.0$, and for the critical value $\varepsilon_i = \varepsilon_{ic} = -1.735522$. It is

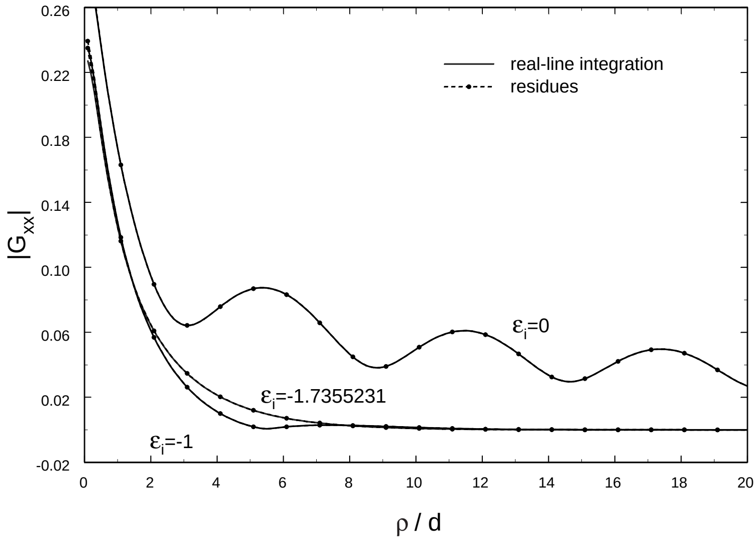

Figure 6. $|G_{xx}|$ versus distance ρ/d for the two-layer waveguide shown in Fig. 1, $\varepsilon_3 = \varepsilon_0$, $\varepsilon_2 = (2.25 - i\varepsilon_i)\varepsilon_0$, and $d_1 = d_2 = d$, with $\varepsilon_i = 0.0$ (lossless case), $\varepsilon_i = -1.0$ (lossy case), and $\varepsilon_i = -1.735523$ (critical value of loss resulting in modal degeneracy and associated double pole). Results from the real-line integration (31) and the sum of residues (36) (using the first five modes) are shown.

interesting to note that the mode attenuation for $\varepsilon_i = \varepsilon_{ic}$ is less than for $\varepsilon_i = -1.0$, even though the dielectric loss is greater. This is due to the fact that as loss increases the dominant mode is "pushed" into the air-region (upper region) of the waveguide.

4. CONCLUSIONS

The dyadic Green's function for inhomogeneous parallel-plate waveguides has been developed as a residue series in the presence of nontrivial modal degeneracies. It was shown that first-order dyadic residues correspond to eigenfunctions of the structure, whereas second-order dyadic residues lead to both eigenfunctions and associated functions that occur in the case of modal degeneracies. A new dyadic root function representation of the Hertzian potential Green's dyadic was developed as an aid to interpreting the dyadic residue series in the case of modal degeneracies, and numerical results for a two-layer parallel-plate waveguide were presented.

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