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TOPOLOGICAL WAVELENGTH SHIFTS [ELECTROMAGNETIC FIELD IN LOBACHEVSKIAN GEOMETRY]

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Abstract—It is shown that in hyperbolic spaces, an electromagnetic radiation experiences shifts in spectrum as a function of curvature and distance. The equation relating distance in hyperbolic space, its curvature, and spectral shift is derived by method of horospheres. The active nature of the Lobachevskian vacuum is discussed with applications to physics.

1. INTRODUCTION

In the present paper we examine the nature and origin of spectral shifts of electromagnetic radiation in Lobachevskian (hyperbolic) geometry. Our analysis **is restricted** to purely geometrical properties of hyperbolic spaces and their relation to an electromagnetic field. It will be shown that the shift in the frequency spectrum of electromagnetic radiation **is a consequence of the non-Euclidean geometry of the space under consideration**. **This is a very subtle point of profound importance and far reaching implications**. In particular, in the case of representation of Lobachevskian space by velocities' hyperboloid, it results in the (relativistic) Doppler shift. Our method extends to any physically realizable representation of hyperbolic space, and its another value is that results are scale independent.

We will prove our statements by deriving a general type **shift formula, just from the geometry of Lobachevskian space, without any relation to physics**. We will then apply our purely geometrical equation to different physical representations. Under the assumption that ambient empty space is a 3 dimensional Lobachevskian space, we will obtain precisely the same formula for the wavelength shift as the one found in a classical analysis of the Doppler effect with a transmitter-receiver pair being in relative motion. Our exposition will be as much as possible geometrical. We believe that there is an advantage in putting things that way, as we will later show. However, other points of view are equally valid. Since Lobachevskian space is a homogeneous space with a group of motion which is the (proper) Lorentz group, there is a standard construction where the homogeneous space X with the group of motion G can be described solely in terms of the group G. In this approach, the homogeneous space is identified with a coset space of group G with respect to a stabilizer S_p of some point p in X. Since stabilizers of any two points are conjugate, it follows that any point in Lobachevskian space can be regarded as "a center". Those facts can be found in [6, 8, 12].

2. SOME FACTS ABOUT LOBACHEVSKIAN GEOMETRY

We start by introducing some facts about Lobachevskian (hyperbolic) geometry. It grew out of a frustration to prove Euclid's 5th postulate concerning parallel lines. Many professional mathematicians (and amateurs) contributed to development of hyperbolic (negatively curved) geometry, but it was Russian mathematician Nikolay Ivanovich

Lobachevsky (1792–1856) who first published it as a self contained geometrical system. On the historical development of Lobachevskian geometry we refer the reader to [10] and on its content to [1, 2, 9] where further references can be found. A lot of work on the application of Lobachevskian geometry to physics has been done by Soviet physicists [4, 5, 11].

Instead of trying to repeat what has already been written on subject of Lobachevskian spaces, we would rather like to bring to the reader's attention the most striking differences between Lobachevskian and Euclidean geometries.

The most distinctive feature of Lobachevskian geometry in our point of view is that there exists a way to establish an internal reference length [2], so words "short" and "long" have definite meaning. In Euclidean geometry those words are meaningless. To compare the length of two objects (in Euclidean space), each of them is compared against a third one, non-mathematical, arbitrarily chosen "standard".

Another difference between Lobachevskian and Euclidean geometries is that in Lobachevskian space, the volume of a ball grows as an exponential function of the radius, whereas in Euclidean space it grows as a power function of the radius. There are more distinctive features. For example, there are infinitely many parallel lines through a given point off a given line which are parallel to the line. Since sides of a triangle in Lobachevskian space depend on angles, there are no similar triangles in Lobachevskian geometry $k < 0$, (as well in elliptical geometry $k > 0$). Therefore, **self-similarity does not hold in a Lobachevskian negatively curved world,** and for example, **a Sierpinski gasket cannot be constructed there.** Finally, two Lobachevskian geometries with distinct curvature constants are not isometric [2].

Regarding physics, the most profound difference between Lobachevskian and Euclidean spaces is the way they relate to the properties of electromagnetic radiation (and perhaps other fields propagating at the speed of light), i.e., to its frequency, intensity and polarization. It will be shown that contrary to Euclidean space, a Lobachevskian vacuum is intimately interrelated to electromagnetism and it actively interacts and modifies the parameters of an electromagnetic field. In this respect, **Lobachevskian empty space** (vacuum) **can be regarded as an active medium,** while Euclidean space is inherently passive. That active behavior is the most visible on a large scale, since locally in domains whose linear size is "small enough" compared to characteristic length, Lobachevskian and Euclidean spaces look alike. In part one of this paper we deal only with the wavelength of electromagnetic radiation in a vacuum.

Intensity and polarization are discussed in the next paper [3]. Both parts are independent and can be read independently of each other.

3. GEOMETRIC ANALYSIS OF SPECTRAL SHIFTS DUE TO LOBACHEVSKIAN SPACE. METHOD OF HOROSPHERES

To relate physical entities in different points in Lobachevskian space we need to have some **machinery to measure distances in Lobachevskian space with our instruments**. By machinery we mean available instruments, since **we do not want** to give some prescriptions which would work **"in principle" only.** Analysis is carried out from the standpoint of **integral geometry**, as formulated by I. M. Gelfand [6]. In integral geometry, several dual spaces can be built in parallel to a given space using as a building blocks geometrical objects of the given space. At the same time a set of functions of interest (defined on initial space) is transformed into a set of images which are integrals over the objects of the initial space. For example, in NMR imaging information is sought about the density distribution function over some (finite) domain in R^3. This information is encoded into a Grassman manifold of $2D$ Euclidean planes in R^3. In this example the dual space to R^3 in the sense of integral geometry is Grassman space $G(2,3)$ where "points" are $2D$ planes in R^3. In our case similarly, the information we seek about a given Lobachevskian space is encoded into the objects of a dual space which are horospheres. Their physical significance is that information encoded into horospheres can be routinely extracted with our experimental gears while direct measurement in hyperbolic space can be done at most cases "in principle only". Therefore, together with Lobachevskian space, we consider a space of **geometrical objects** on a Lobachevskian space which are **horospheres**. We then use sets of **parallel horospheres** to measure distances in Lobachevskian space and to relate geometry to physics. In other words, the information we seek about the space is encoded into the objects of dual space-space of horospheres. In the next few sections we will show how information about a hyperbolic space is encoded via an electromagnetic field into geometrical objects of its dual (space of horospheres), how it is extracted, and how it is decoded.

Following Gelfand, Graev & Vilenkin [6], and Helgason [8], we introduce the notion of a horosphere (horocycle in two dimensions) in Lobachevskian space. "Horos" in Greek means infinitely remote. A horosphere in Lobachevskian space can be obtained by the following construction. First, a sphere in Lobachevskian space is defined as a

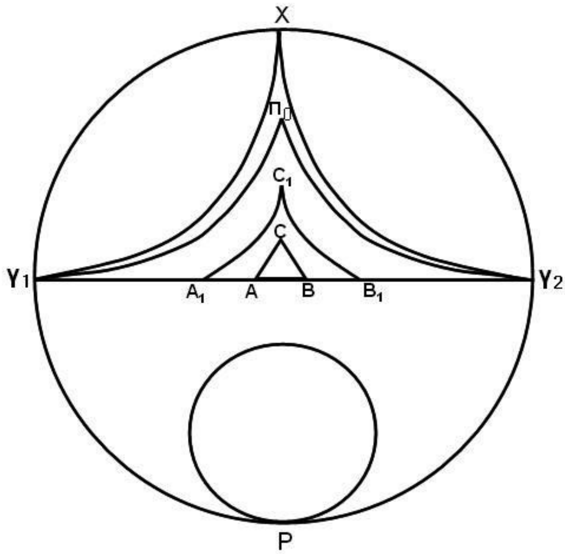

Figure 1. Lobachevskian space with application to kinematics. $\triangle ABC$ "small triangle", sum of angles equals almost π, Euclidean addition of velocities. $\triangle A_1B_1C_1$ sides comparable to radius, sum of angles less than π, Lobachevskian addition of velocities. $\triangle\gamma_1\gamma_2\pi_0$ two vertices at infinity, decay of $\pi_0 \longrightarrow 2\gamma$. $\triangle\gamma_1\gamma x$, three vertices at infinity, defect $= \pi$, hypothetical 3 massless process: particle x decays into 2 photons. Horosphere centered at point P.

locus of points to which the hyperbolic distance from a fixed point (called the center of the sphere) is constant. Then we move the center of the sphere to infinity requiring that in the process the sphere passes through the fixed point. As the center of the sphere approaches infinity, the sphere becomes a horosphere. Alternatively we can define a horosphere as a surface which is orthogonal to the family of parallel lines (geodesics) converging to one point at infinity and tangent at that point to the boundary at infinity. Two horospheres tangent to the same point at infinity are called parallel. The distance between two parallel horospheres is the distance measured along any ray in a family of parallel rays converging to the tangency point and intersecting the horospheres. Figure 1 shows some objects in the Lobachevskian space for a Poincare disc model.

The internal geometry on horospheres of Lobachevskian space is flat. This means that in the case of a $3D$ Lobachevskian space, horospheres carry the geometry of a $2D$ Euclidean plane. In other symmetric spaces that may not be true [7], but it is beyond the scope of the present paper. We observe that horospheres in Lobachevskian space form a **transitive set.** This means that any horosphere can be mapped onto any other by some **motion in Lobachevskian space.** We will use a family of **parallel horospheres** (which foliate the Lobachevskian space) as a **distance markers**. For that purpose we need **to attach a "tag" to each horosphere** which we can **read with our physical instruments.** Having such **a "tag code reader"** we need to create a one-to-one mapping between the difference in the tag readings and the distance between two points in Lobachevskian space. Since this mapping is one-to-one, we can also recover the "tag" difference from the distance which separates parallel horospheres in Lobachevskian space. In other words, we establish a **one to one relation between horospheres' parameter space and distances in Lobachevskian space.**

We will work with **homogeneous coordinates** in a 3 dimensional projective real space. By taking different normalizations of homogeneous coordinates we will get different models of Lobachevskian space and we will use them interchangeably without any special notification. Normalizing homogeneous coordinates by $[a, a] = 1 = a_0a_0 - a_1a_1 - a_2a_2 - a_3a_3$, we get **a hyperboloid model** of Lobachevskian space. In this realization, **the boundary at infinity** consists of all vectors satisfying $[b, b] = 0$, which is **a cone.** Another representation of Lobachevskian space is **on a hyperplane** $a_0 = const. > 0$. The hyperplane $a_0 = const. > 0$ intersects the cone $[b, b] = 0$ over the sphere $a_1^2 + a_2^2 + a_3^2 = const.$ In that realization, Lobachevskian space is **an interior of the** $3D$ **ball**, bounded at infinity by **a** $2D$ **sphere**.

Assuming that, $[a, a] = 1$, $[b, b] = 0$, and $a_0 > 0$, (upper sheet) $b_0 > 0$, (forward cone), we write the horosphere equation in 3D Lobachevskian space:

$$[a, b] = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 = const \tag{1}$$

We note a very simple fact which will be intensively used in further exposition: the equation of a horosphere is nothing else than a (pseudo) **scalar product of a constant value**. This is a **geometrical statement** independent of coordinate system in which this product is computed. It follows that it will have the same constant value in different coordinate systems (frames associated with different points in Lobachevskian space); however components of those two vectors will undergo change.

Therefore we can write:

$$[a, b]_T = [a, b]_D \tag{2}$$

Where subscripts T and D refer to coordinate systems at two different points in Lobachevskian space. At this point, the subscripts T and D have no other meaning than to distinguish two points. **This simple looking Equation (2) contains an invariant formulation of spectral shifts that an electromagnetic wave experiences in hyperbolic space, in abstract geometrical form.** However it reaches much further and is applicable to all physical phenomena which can be modeled on Lobachevskian geometry, i.e., geometry of constant negative curvature. **Equation (2) states the (almost) trivial truth that a real number is an invariant.** Below we will illustrate how this equation works in **different representations** as applicable to physics. At this point we would like to bring to the reader's attention that we share the belief that the notion of **space in mathematics is established by a relationship between its objects** ("points"), however **the physical representation** in each particular case **may be different**. For example, group structure doesn't depend on its representation but is established by relations among its elements. In the same way Lobachevskian geometry does not depend on its physical representation, whether it is represented by velocity space, by coordinate (configuration) space, or by the space of reflection coefficients, which also carries Lobachevskian $2D$ space geometry (Lobachevskian plane).

We observe that the easiest way is to compute the LHS of Equation (2) is in the coordinate system where $a_T = (1, 0, 0, 0)$. This is "the center" of homogeneous space. Therefore we have: $[a, b]_T = b_{0T}$. On the other hand in the coordinate system of "general position" we have:

$[a, b]_D = a_{0D}b_{0D} - a_{1D}b_{1D} - a_{2D}b_{2D} - a_{3D}b_{3D} = a_{0D}b_{0D} - (\mathbf{a}, \mathbf{b})_D = a_{0D}b_{0D} - |\mathbf{a}|_D\,|\mathbf{b}|_D \cos\alpha$

where $(\mathbf{a}, \mathbf{b})$ denotes the usual Euclidean scalar product. Since for vectors on absolute $[b, b] = 0$, $b_0 = \pm\,|\mathbf{b}|$, and because $b_0 > 0$ we have $b_0 = +\,|\mathbf{b}|$ and then: $[a, b]_D = b_{0D}\,(a_{0D} - |\mathbf{a}|_D\,|\mathbf{b}|_D \cos\alpha)$. Therefore Equation (2) takes form: $\frac{b_{0T}}{b_{0D}} = \frac{|\mathbf{b}|_T}{|\mathbf{b}|_D} = a_{0D} - |\mathbf{a}|_D \cos\alpha = a_{0D} + |\mathbf{a}|_D$, where α is an angle between Euclidean vectors $\mathbf{a}$ and $\mathbf{b}$.

Here α is set equal to π (another choice for α is $\alpha = 0$, and $\cos\alpha = \pm 1$ is related to the choice of a direction on a geodesic through two points).

Since $[a, a] = 1$, we have $1 = a_0^2 - |\mathbf{a}|^2 = (a_o - |\mathbf{a}|)^{\frac{1}{2}}\,(a_0 + |\mathbf{a}|)^{\frac{1}{2}}$, and we came to the final equation:

$$\frac{b_{0T}}{b_{0D}} = \frac{|\mathbf{b}|_T}{|\mathbf{b}|_D} = \left(\frac{a_{0D} + |\mathbf{a}|_D}{a_{0D} - |\mathbf{a}|_D}\right)^{\frac{1}{2}} \tag{3}$$

We also denote $\frac{|\mathbf{b}|_T}{|\mathbf{b}|_D} = \zeta$, $\frac{|\mathbf{a}|_D}{a_{0D}} = \xi$. In that notation and after taking the natural logarithm of both sides, Equation (3) takes form:

$$\ln \zeta^2 = \ln \frac{1+\xi}{1-\xi} \tag{4}$$

We see that RHS of Equation (4) is nothing else than **a hyperbolic distance** d_H from origin to the point ξ in the unit ball model of Lobachevskian space [1]. Hence $\xi = \tanh\left(\frac{1}{2}d_H\right)$, and finally we get:

Theorem 1.

The mapping of parameter ratio of two parallel horospheres onto Euclidean distance (which separates them) in the unit ball model is of Lobachevskian space is given by Equation (5) below.

$$\xi = \tanh\left(\ln \zeta\right) \tag{5}$$

Later we will see that the parameter ratio is nothing else than the arc length ratio, which has an immediate and clear physical significance. The equation we derived relates the parameter ratio (tag value ratio) for two parallel horospheres in Lobachevskian space to the hyperbolic distance which separates them. It has a **purely geometrical origin** and for example, it has the same validity as the statement that the circumference to diameter ratio of a circle (in Euclidean geometry) equals π. So far we have not involved any physics at all, and we have not made any assumptions on vectors a and b beyond their normalization conditions: $[a, a] = 1$, $[b, b] = 0$ and $a_0 > 0$, $b_0 > 0$. No other assumptions have been made beyond those which follow from $3D$ hyperbolic geometry.

4. REPRESENTATION OF LOBACHEVSKIAN GEOMETRY BY VELOCITY SPACE. SIGNIFICANCE OF HOROSPHERES FOR PHYSICS

Due to **an invariant and limiting value** of the speed of light (in a vacuum) we see that **velocities of all material bodies** (i.e., with the nonzero rest mass) **lie in** a $3D$ ball of radius c in Euclidean space. **Photon** (and perhaps neutrino / graviton) velocities lie on the **limiting sphere at infinity**, or on the light cone if we model Lobachevskian space on a $3D$ hyperboloid. In the hyperbolic metric inside the ball, the signed distance between two points has the meaning of uniform relative velocity. When one point approaches the limiting sphere, the hyperbolic distance between it and a second fixed point goes to infinity. For that reason we say that **photon velocities populate the boundary at infinity.** The boundary at infinity, also called **an absolute**, **does not** belong to the Lobachevskian space. To learn more about Lobachevskian geometry of velocity space and its relation to physics we refer reader to paper by Ya. A. Smorodinsky [11], N. A. Chernikov [4, 5], both in Russian and to Klein [9]. Note one **very important fact**. **Velocity space is not compact**. Now we will try to extract some physics by identifying the vectors a and b in Equation (2) with some physical entities. In order to measure distances in Lobachevskian velocity space, we employ **an electromagnetic field which labels horospheres as it propagates through free space on its way to us.** We could use Equation (5) directly. Instead, we will do the first example in every detail to give the reader some comfort and to make the bridge between **a negatively curved non-compact velocity space and physics** more visible. Later we will use Equation (5) directly.

First, in a ball of radius c in $3D$ Euclidean space we introduce **homogeneous Weierstrass coordinates** in the following way. If v denotes (Euclidean) distance from the origin $O\,(0,0,0)$ to the point $V\,(v_1,v_2,v_3)$, then homogeneous coordinates of a point V are:

$$u_0 = \frac{1}{\sqrt{1-\beta^2}}, \quad u_1 = \frac{\beta_1}{\sqrt{1-\beta^2}}, \quad u_2 = \frac{\beta_2}{\sqrt{1-\beta^2}}, \quad u_3 = \frac{\beta_3}{\sqrt{1-\beta^2}},$$

here $\beta^2 = \frac{v^2}{c^2}$, and $\beta_k = \frac{v_k}{c}$, $k = 1,2,3$.

Those coordinates, according to Buseman & Kelly [2], were introduced and used by Weierstrass (1815–1897). Please note that four homogeneous coordinates obey the normalization condition $[u,u] = 1$, and therefore only three of them are independent. This means that **velocity is a** $3D$ **geometrical object**. However, due to the invariant and limiting value of c, **velocity space is not flat**. It is a $3D$ **Lobachevskian (hyperbolic non-compact) space,** and this fact was known as early as 1909 to Felix Klein [9].

Now we identify vector b with a wave vector of an electromagnetic wave in free space having dispersion relation, $\omega^2 - \mathbf{k}^2 = 0$, which in homogeneous coordinates takes form $[k,k] = 0$, $k_0 = \omega > 0$, and $|\mathbf{k}| = \frac{2\pi}{\lambda}$. We note that k lies on the cone.

Therefore the equation of the horosphere in Lobachevskian velocity space is:

$$[u,k] = const. > 0, \qquad [u,u] = 1, \qquad [k,k] = 0, \qquad u_0 > 0, \quad k_0 > 0 \tag{6}$$

To see the physical meaning of *const.*in the RHS of Equation (6), we compute the LHS in a coordinate system in which $u = (1,0,0,0)$, i.e., in the “center” of homogeneous velocity space. We call this the **transmitter coordinate frame** (in velocity space) that and denote it with subscript T. From now on, subscript T stands for the transmitter. We **always locate the transmitter at "a center"** since that corresponds its own reference frame.

$[u, k]_T = k_{0T} = \omega_T$ is the transmitter's own frequency.

Therefore we see that a horosphere in Lobachevskian velocity space, with a boundary at infinity populated by photon velocities, is **a surface of constant frequency of an electromagnetic wave,** or **surface of constant wavelength,** or **surface of constant photon energy.** In any other **arbitrary point** (note that any two points determine unique geodesics), which we call the **detector coordinate frame** (subscript D), we have in Weierstrass coordinates:

$$[u, k]_D = \frac{k_{0D}}{\sqrt{1-\beta^2}} - \frac{k_{1D}\beta_1}{\sqrt{1-\beta^2}} - \frac{k_{2D}\beta_2}{\sqrt{1-\beta^2}} - \frac{k_{3D}\beta_3}{\sqrt{1-\beta^2}} = \frac{k_{0D}\,(1-|\beta|\cos\alpha)}{\sqrt{1-\beta^2}},$$

where $|\beta| = \frac{\sqrt{v_1^2+v_2^2+v_3^2}}{c}$ is the relative velocity between transmitter and detector.

Taking $\alpha = \pi$ which corresponds to the case of recession (**redshift**), and $k_0 = |\mathbf{k}| = \dfrac{2\pi}{\lambda}$, and repeating the calculation already done in section 1, we have :

$$\frac{v}{c} = \tanh\left(\ln\frac{\lambda_D}{\lambda_T}\right) \tag{7}$$

The approach case (**blueshift**) corresponds to $\alpha = 0$ and yields:

$$\frac{v}{c} = \tanh\left(\ln\frac{\lambda_T}{\lambda_D}\right) \tag{8}$$

This clearly shows that the roots of the Doppler effect are in Lobachevskian geometry and not in kinematics, i.e., **Doppler effect follows from the existence of topological property — the negative curvature of velocity space.**

Figure 2 shows velocity of the object (in units of c) versus wavelength ratio $\zeta = 1 + z$. Point $\beta = 0$ can be interpreted in two ways: either $v = 0$, transmitter and detector at the same point in velocity space), or $c = \infty$ (flat velocity space). We conclude that:

No spectral shift will be recorded if:

4a. **distance in velocity space is zero** (=relative velocity is 0), or

4b. **velocity space is flat i.e., Euclidean,** (= velocity is unlimited, $c = \infty$) or

4c. **both conditions hold**

We see that vanishing curvature of velocity space $k \to 0$ (or $c \longrightarrow \infty$) implies vanishing spectral shift.

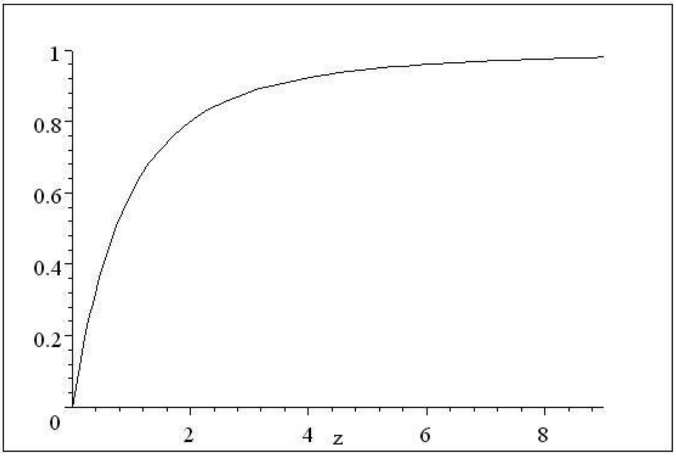

Figure 2. $\xi = \tanh(\ln(1+z))$. Vertical axis represents distance in unit ball model. Horizontal axis represents wavelength shift. The unit ball may be interpreted as a Lobachevskian universe or Lobachevskian velocity space.

5. REPRESENTATION OF LOBACHEVSKIAN GEOMETRY BY CONFIGURATION SPACE — LOBACHEVSKIAN VACUUM

Motivated by the above example we turn now to the **Lobachevskian vacuum**. Similarly to what we did before, we consider the interior of a $3D$ ball of (Euclidean) radius R in $3D$ Euclidean space. Inside the ball (**the very real space around us**) we again introduce the **homogeneous Weierstrass coordinates,** $\rho_0,\ \rho_1,\ \rho_2,\ \rho_3$**.** We use Equation (5) directly, but since the radius of the ball is now R instead of 1, we need to put $\xi = \dfrac{r}{R}$ in Equation (5).

If r denotes (Euclidean) distance from the origin to the point $\mathbf{r}$, then the homogeneous Weierstrass coordinates of point $\mathbf{r}$ are:

$$\rho_0 = \frac{1}{\sqrt{1-\xi^2}},\ \rho_1 = \frac{\xi_1}{\sqrt{1-\xi^2}},\ \rho_2 = \frac{\xi_2}{\sqrt{1-\xi^2}},\ \rho_3 = \frac{\xi_3}{\sqrt{1-\xi^2}}$$

We see that a horosphere in configuration space is a surface of constant phase (wavefront) of an electromagnetic wave $[\rho, k] = const.$

This time instead of repeating the computations as above, we apply Formula (5) directly, which yields:

$$\frac{r}{R} = \tanh\left(\ln\frac{\lambda_D}{\lambda_T}\right) \tag{9}$$

Figure 2 shows the graph of exact formula $\xi = \tanh(\ln\zeta)$, $\zeta - 1 = z \geq 0$, and radial distance is normalized to $R = 1$.

We again conclude that:

No spectral shift will be recorded if:

5a. **Distance in static hyperbolic space between transmitter and receiver is zero**, $r = 0$.or**,**

5b. **Configuration space is flat (Euclidean),** $R = \infty$, or,

5c. **Both conditions hold**

We need to emphasize that Equation (9) presents a universal, **purely geometrical statement**. By "purely geometrical" we mean that it relates **two geometrical objects** (horospheres) in Lobachevskian space via a **boundary at infinity**. It is expressed by a simple **relationship between two dimensionless ratios**, (of four lengths), which is in the very spirit of geometry. It is amazing that the "**radius-redshift-distance**" (RRD) Equation (9), shows for the first time, how **quantum physics is related to ultimate macro physics**. **The wavelength λ that we read with our "tag reader",** i.e., spectroscopic instrumentation (at fixed transmitting atom wavelength), **depends on the curvature of space and on the separation between the transmitter and receiver.** Our result can be applied to a wide range of physical phenomena whose **underlying geometry** is that of **Lobachevskian space** and it is experimentally provable.

Just one quick look at Figure 2 tells us that there are two **"linear"** regions (with different slopes), one $z \lesssim 0.8$, and the second one $z \gtrsim 4$. In those regions, **distance** is an approximate linear function of **redshift** (and vice versa). Another interesting fact about Formula (9) and its plot is that the normalized radial distance (or normalized velocity) changes with redshift z very insignificantly for values of z practically bigger than 4; our measurements in this region are simply speaking, inconclusive. One more interpretation of Formula (9) is possible. Since a wavelength can be represented only by a non-negative real number, the set of all wavelengths belongs to the upper half space (another model of the Lobachevskian space) and we can regard $\ln\frac{\lambda_D}{\lambda_T} = d_\lambda$ as a hyperbolic distance in **"a wavelength space"**. It is related to the redshift z by the Equation (10).

$$z = e^{d_\lambda} - 1 \tag{10}$$

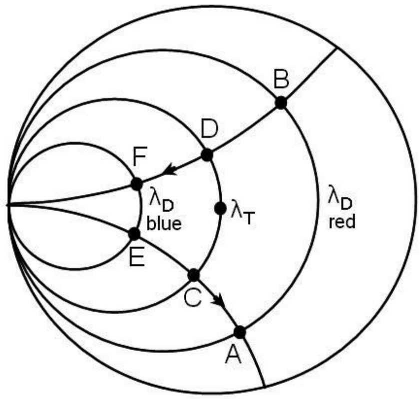

Figure 3. Lobachevskian velocity space. CD represents transmitter's own wavelength. Detector has a double image depending on direction on geodesics. Divergent direction (recession). CD $<$ AB $=$ redshifted λ. Convergent direction (approach) EF $=$ blueshifted $\lambda <$ CD. The system of coordinates is sometimes referred to in physics as a wavefront coordinate system. In mathematics, it is called a horospherical coordinate system.

Then distance r, in Equation (9) will be simply: $r = \tanh d_\lambda$, $(R = 1)$. This maps distances in λ space (spectral shifts) onto distances in Lobachevskian space. In the case of Lobachevskian velocity space, the exponent of Equation (10) can have a positive or negative sign, which corresponds to two different directions of velocity. Equation (10) is a well known fact in Lobachevskian geometry and it relates the separation of parallel horospheres to the **ratio of the arc length** along the two parallel horospheres [2].

We conclude that the segments (Euclidean segments, since horospheres are flat) cut out by parallel geodesics on horospheres are nothing else than **wavelengths of the electromagnetic field measured at detector and transmitter.**

The above diagrams clearly show that **the shift in wavelength** of electromagnetic radiation in Lobachevskian space (either kinematic or static) **is a consequence of the hyperbolicity** of space, which manifests its **presence via an exponential divergence of geodesics**. Figures 3 and 4 also have another value beyond visualizing spectral shifts. They can be regarded as a **diagrammatic technique** to compute shifts (or distances) just from the diagram itself. This is precisely what we had in mind when we mentioned that a geometrical exposition has an advantage over the group theoretical treatment.

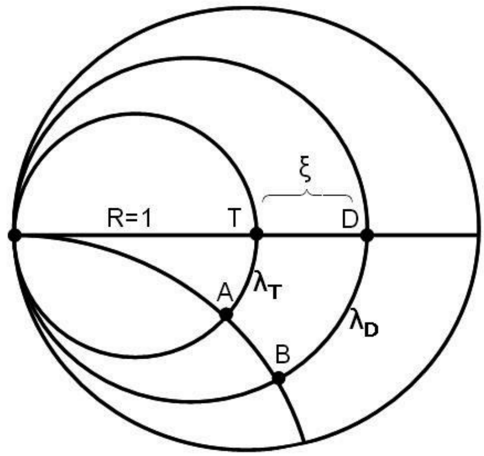

Figure 4. Lobachevskian universe. AT represents transmitter's own wavelength. BD represents wavelength at detector. Redshifts only. Separation between parallel horospheres equals ξ.

As we mentioned, the internal geometry on horospheres of $3D$ Lobachevskian space is flat. It is a geometry of a Euclidean $2D$ plane. We have seen that the phase of an electromagnetic wave in configuration space is constant on the horospheres. It follows that Lobachevskian space is **foliated by "plane waves"** in a very direct sense with a different **"horosphere color"** seen by different observers along the ray. We clearly see that there is no globally (at every point) existing monochromatic wave in Lobachevskian space. **"Monochromaticity" has only a local, pointwise meaning.** This means that (**dimensionless**) parameter on the geodesic γ in hyperbolic space e.g., geodesic ECA is simply $\zeta = \frac{\lambda_D}{\lambda_T}$. If we assume parameter being $z = \zeta - 1$,then a base point on geodesic will be $p_T = \gamma(0)$, that is the point at which transmitter is located. The set of observers distributed along the geodesic is enumerated by geodesic parameter z which can be interpreted as a spectral shift.

6. THE LOWER BOUND OF EXPERIMENTAL DETECTION OF THE CURVATURE OF HYPERBOLIC VACUUM

A question arises: **what is the estimation of the radial distance in a Lobachevskian universe within which the curvature of**

ambient space is undetectable? The answer is that it depends on the resolution power of our instruments. We will give some **scale invariant estimates** of when we can use Euclidean physics and when Lobachevskian physics comes into play. These estimates, being scale invariant, will work in other areas of physics too.

Differentiating RRD Equation (9) and keeping in mind that $\cosh(x) \geqslant 1$, and that $\triangle\zeta = \dfrac{\triangle\lambda_D}{\lambda_T}$ (since transmitter wavelength is fixed) we get :

$$\triangle\rho = \frac{\triangle r}{R} = \frac{\triangle\zeta}{\zeta}\frac{1}{\cosh^2(\ln\zeta)} \leqslant \frac{\triangle\zeta}{\zeta} = \frac{\triangle\lambda_D}{\lambda_D} = (p_s)^{-1}$$

where p_s, **as it should be expected**, is the chromatic resolving power of a spectroscope in our case. Therefore, we will see the world as **Euclidean** for:

$$\triangle r \leqslant R\frac{1}{\left(\dfrac{\lambda}{\triangle\lambda}\right)_D} = \frac{R}{p_s} \tag{11}$$

This means that to see the effect of Lobachevskian vacuum curvature we need to go **beyond distances equal to the radius of R divided by the resolving power of our spectroscopes**, $\Delta r > R\,(p_s)^{-1}$. If we assume that present day spectroscopes can deliver chromatic resolution on the order of 10^6, and if we assume that the radius of the universe is $15BLy$ (which is yet to be nailed down), we end up with $\Delta r > 15KLy$ (kilo light years) or $2.1E + 17\,\mathrm{km}$, which is about the radius of our own galaxy.

Very big numbers like 10^{17}, or very small ones like 10^{-12}, do not match our everyday experience, and do not give us "a feel" as to where we operate. For illustrative purposes, we shrink the Lobachevskian universe to a ball of Euclidean radius of 1 **kilometer** and then $\triangle r$ will be equal to or greater than 1 **millimeter**.

We can draw a few conclusions from this scale model:

1. Within a ball of 1 millimeter radius we may use formulas of Euclidean physics since we are not be able to detect any deviation from Euclidean geometry with our present spectroscopes. In such a domain physics will be based on Euclidean geometry.
2. Going 10 times further to a distance of 1 **centimeter** (this will be outside of our galaxy) **we should abandon Euclidean geometry**. Note that (in Euclidean measure) **we are still** 0.99999 **km, almost an entire** 1 **kilometer, away from the boundary** so the relative size of Euclidean domain seems to be rather quite small.

7. HOMOGENEOUS SPACE OF HOROSPHERES. APPLICATION TO COSMIC MICROWAVE BACKGROUND RADIATION (CMBR)

We would like to discuss to a greater extent the properties of the space of horospheres $\Omega(H^3)$ on a $3D$ hyperbolic space. We recall some **facts about the space horospheres** [6] in **Lobachevskian space** and we will try to draw a conclusion from there.

9a. Horospheres in Lobachevskian space are **surfaces of constant phase** of an electromagnetic wave, i.e., **wavefronts.**

9b. **Geometry on horospheres is flat.** This means **wavefronts are flat $2D$ surfaces.**

9c. Horospheres on Lobachevskian space form **a transitive set** with a Lorentz group SL(2C) as a group of motion.

9d. **Space of horospheres is a homogeneous space dual to the Lobachevskian space** [6]. It is a factor space of SL(2C) with respect to stabilizer, which is a group of unipotent matrices $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, where the real parameter z can be interpreted as a spectral shift.

It follows that in a **very direct sense** that a Lobachevskian "vacuum" is **filled with horospherical wave fronts-plane waves**. In other words, hyperbolic **space itself is decomposed into plane horospherical wavefronts.** In that setting **the relation between a hyperbolic geometry and electromagnetism seems to be very close.**

Regarding observations of Cosmic Microwave Background Radiation, it is an experimentally established fact that CMBR is homogeneous and isotropic. We can see the **presence of horospherical wavefronts at each point in each direction with our microwave antennas (CMBR)**. The described above geometry of horospherical wave fronts when applied to CMBR agrees with the observational data about homogeneity and isotropy of CMBR and is simply pictured in Figure 5. **The homogeneity and isotropy of CMBR is a geometrical property** of a homogeneous space of horospheres. Figure 5 shows CMBR in 2 dimensions for clarity reasons.

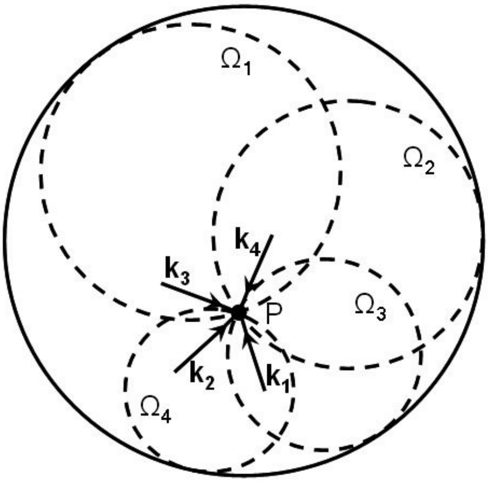

Figure 5. Plane wavefronts Ω_1-Ω_4 at point P (only 4 wave vectors shown). Point P is arbitrary, and therefore the same picture will take place at any point, which implies homogeneity. The wave vectors k are distributed evenly in all directions at any point P. This is isotropy.

8. CONCLUSIONS

In the present paper, we showed in a simple and fairly rigorous way that **Lobachevskian geometry,** represented either **static** or **kinematic** spaces, causes **shifts in spectra** of electromagnetic radiation. Shift in spectra is a measure of negative curvature and distance and is described by Equation (9). Also, it was given an analysis of the size of spatial domain over which curvature is beyond detection at present experimental power. We **conjecture** that **any field propagating at the speed of light** will be altered in **the same way** by Lobachevskian space as described above. The present paper can also be viewed as an illustration of how Lobachevskian negatively curved geometry is intimately associated with electromagnetism (electromagnetic 2 form). The behavior of the intensity and polarization of electromagnetic radiation in Lobachevskian space will be discussed in the next paper [3].

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