

THE TEM-MODE BANDWIDTH OF TWO-CONDUCTOR OPEN TRANSMISSION LINES

H. E. Green

Institute for Telecommunications Research
University of South Australia

Abstract—The usual aim with any waveguide is to operate it with only the fundamental mode propagating. With fully closed waveguides, finding the band over which this is possible turns on no more than knowledge of the cutoff frequencies of the fundamental and first higher order modes. With open waveguides, the question is not so simply answered. Such waveguides propagate at most a finite set of bound modes together with a continuous modal spectrum that has no counterpart with closed guides. In this paper, for several particular two-conductor transmission lines, we investigate the circumstances under which leaky wave modes, though not themselves members of any orthonormal set of basis functions, can be used to set bounds on the band over which it is to be expected that the transmitted field is substantially contained in the fundamental TEM mode. The method used relies on transverse resonance.

1 Introduction

2 Some Preliminary Physics

3 The TE-like Mode in Parallel Plate Line

4 The TE-like Leaky Mode in Two-wire Transmission Line

5 TM-like Leaky Modes in Parallel Plate and Two-wire Lines

6 Conclusion

Acknowledgment

Appendix A. Determination of G Attributable to the External Region of a Parallel Plate Line

Appendix B. Finding the Elemental Inductance and Capacitance

Appendix C. On Finding the Area of a Certain Closed Figure

Appendix D. Determination of the Capacitance Attributable to an Element of Cross-section in a Two-wire Line

Appendix E. Determination of G Attributable to the Exterior Region of a Two-wire Line

References

1. INTRODUCTION

As a first approach to the problem, it is natural to begin from what is already known about the behaviour of closed waveguides. Thus, for example, in a coaxial line first to appear is the TEM mode, which propagates from zero frequency. It remains the only propagating mode until the onset of the TE_{11} mode, the first of a denumerably infinite set of discrete, higher order modes. Normally there is little interest in operating coaxial lines with other than the TEM mode present and the bandwidth over which this is possible extends from zero frequency to the cutoff of the TE_{11} mode. Furthermore, we can find this cutoff by making use of transverse resonance, which serves to show that it corresponds closely to the frequency which makes the mean circumference of the line a wavelength in the dielectric medium with which it is filled [1].

Many two-conductor open waveguides are configured to propagate a TEM mode from zero frequency and one might look to finding the cutoff of the first higher order mode as an upper bound on the TEM-mode bandwidth, possibly to be obtained by a suitably devised version of the transverse resonance technique [2]. However, when this is attempted, complications appear at once. Open waveguides do not possess a denumerably infinite set of discrete modes as do closed waveguides. In general it is found that the normal modes consist of a finite set of bound modes together with a continuous modal spectrum bounded at infinity that has no counterpart with closed waveguides [3].

This follows from the fact that the Helmholtz equation together with its boundary conditions, which must include one at infinity for the now unbounded field, is no longer self-adjoint [4]. Moreover, when transverse resonance is used as a means of finding the eigenvalues, complex solutions appear corresponding to fields that are not bounded

at infinity and therefore not part of any proper mode set. It would, however, be wrong to regard these simply as nuisances to be discarded. They have a physical interpretation as leaky wave modes per medium of which power is conveyed away from the open waveguide and have found practical application in a class of leaky wave antennas in which the open waveguide is used as the radiating element [5].

For bound modes transporting energy in the direction of the longitudinal axis of the guide (which we will take as the z -axis of a cartesian frame in which x, y are the transverse plane coordinates) without loss by leakage to the external surroundings, it has to be the case that $k_z^2 \geq k^2$, where k is the free space wave number. For the two-conductor transmission lines to be considered in this paper, only the fundamental TEM mode is capable of fulfilling this condition and so is the only member of the finite bound mode set, all that stands in addition being the continuous mode spectrum.

The classical method for determining the electromagnetic field excited by a source in a closed waveguide is to use the Lorentz reciprocity theorem to evaluate the modal amplitudes. As Rozzi and Mongiardo note [3], the same process can be followed with open waveguides, with the field distributing itself between the bound mode and the continuous orthonormal spectrum. By appropriate deformation of the contour of integration in the complex wavenumber plane, the total field in the line can be decomposed into a sum of the residue from the bound mode pole on the real axis and a continuous spectrum contained in a branch cut integral. If in turn the path of this branch cut integral is deformed into a path enclosing all the branch cuts along the negative imaginary axis and a steepest descent path around the branch point corresponding to the wave number with which energy is radiated from the structure, the continuous spectrum current becomes further decomposable into a sum of leaky wave modes and a residual wave [6, 7].

When the line is operated at low frequencies, the leaky mode component is not present and the field consists only of the bound and residual waves. Under this condition the former strongly dominates the latter and the line can be considered to operate essentially as the TEM mode transmission line that it is commonly supposed to be. As the frequency is increased, leaky wave modes contained within the non-physical spectral gap region are able to make their presence felt indirectly by influencing a now much stronger residual wave. At higher frequencies still, the leaky waves become physical, directly influencing the behaviour of the line by contributing to continuous energy loss as the wave moves away from the source that produced it. Under these circumstances the line no longer operates as a TEM mode transmission

line.

Knowledge of the position in the complex plane of the leaky wave mode pole which, as frequency is increased, first has an impact thereby defines a kind of pseudo-cutoff or critical frequency, an upper frequency limit on essentially TEM mode operation, which sets the bandwidth over which it is reasonable to treat the waveguide as essentially a TEM mode transmission line. Since the intent in practice is normally to stay clear of mode boundaries by a margin, even its fuzzy nature or no more than an approximate knowledge of where it is will be quite useful.

In the light of the above and considering the ease with which the equation for transverse resonance can be set up and solved for many open waveguides, including many with non-separable cross-sectional geometries, it is natural to want to use it in answering the question of their essentially TEM-mode bandwidth. As noted in [3], while with fully closed waveguides transverse resonance is exact, in open waveguides it is an approximation which to be accurate requires the leaky mode poles to have a dominant influence in the complex wavenumber plane integration process. Desirably this implies leaky modes with a small rate of leakage, which in turn implies that transverse resonance occurs with a reasonably high Q factor. Finding the critical frequency then rests on a transverse resonance analysis of the open waveguide in which a suitable lumped equivalent circuit is used to connect between the interior of the guide and the space external to it.

Even before much of what has been pointed out above had been elucidated, a method which early found favour as a tool for leaky mode antenna analysis was to treat the open waveguide as a perturbation of a closely similar but closed waveguide in which ideally the internal fields could be found by separation. This technique was used by Rotman in 1949 [8] to study the channel guide antenna, an ordinary rectangular waveguide having one of its narrow faces removed to provide an opening between its interior and the surrounding space. Rotman used a transverse resonance formulation in which the open face is represented by an equivalent circuit given by Marcuvitz [9] for the case of a parallel plate line radiating into a half-space through an infinite baffle. The conductive and susceptive parts of this circuit take into account respectively leakage by radiation from the open edge and energy storage in its immediate neighbourhood. Rotman tested his theoretical results against some published experimental data and found useful agreement.

Almost certainly independently and in the same year, based on the same principles, Cullen [10] published a solution to a very similar problem. Cullen's problem differs from Rotman's only in

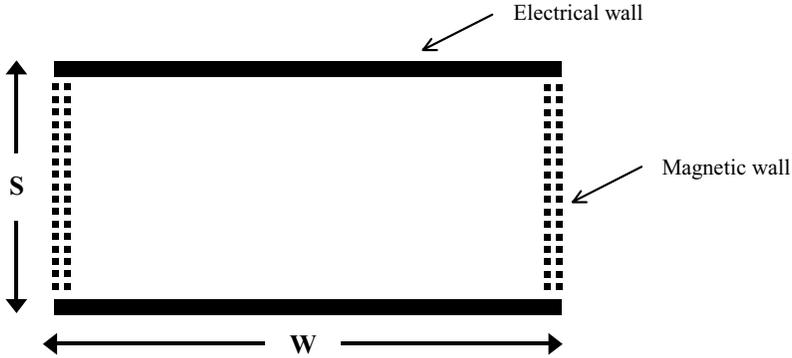


Figure 1. Prototype waveguide for perturbation.

having an unflanged transition between its interior and exterior regions and in treating the effects of the exterior region through quasi-static approximations. Cullen compared the predictions of his theory with the results of experiments which he also carried out himself and reported satisfactory agreement between the two. Both solutions rely on the assumption that only a single mode is present in the perturbed waveguide and that it is essentially the same as that in the unperturbed guide, which in this instance is the TE_{01} mode of a rectangular waveguide. The restricted aims of this paper, where we shall be interested only in parallel plate lines or others that for the most part are conformal distortions of them, allow a this simple starting point to continue to be used.

2. SOME PRELIMINARY PHYSICS

Consider the waveguide shown in Fig. 1 which is bounded on opposite pairs of faces by electric and magnetic conductors. This is our choice for the closed waveguide that is to be perturbed to provide solutions to several related open waveguide problems. It possesses a denumerably infinite set of modes that may be found by separation of the wave equation subject to the conditions imposed by the boundaries. The fundamental of these modes is the TEM mode which propagates from zero frequency, but at higher frequencies higher order TE and TM modes will begin to appear. The lowest order members of these two families are the TE_{01} and TM_{01} modes and which is first to appear depends on the aspect ratio of the cross-section. For $w > s$ it will be the TE_{01} and for the converse, the TM_{01} ; when $w = s$, both appear together as a degenerate pair of modes.

Separation of the reduced wave equation to find these modes is a standard textbook problem [11] that does not require repetition here. However it is interesting to examine the fields associated with them. Both can be resolved into pairs of crossing plane waves. Within each pair, each wavefront, while maintaining the same angle with the longitudinal axis of the guide, propagates by reflection in the walls in a manner analogous to passage of a ray of light along a hall of mirrors. For the TE_{01} mode the electric field is normal to the electrically conducting walls which therefore provide the boundaries to contain the waves in the manner of a TEM parallel plate line while they are reflected between the magnetic walls. For the TM_{01} case it is the dual.

The angles which the normals to these wavefronts make with the longitudinal axis of the guide are frequency dependent. At high frequencies, they are very oblique, but with decreasing frequency eventually reach the condition of being normal, when each plane wave no longer has any axial component of velocity. At this frequency propagation ceases and the mode has reached its cutoff condition. Even so, if the waves are to continue to exist under this circumstance, conditions must be such that in the round trip between the two reflecting walls, each wave returns with a phase to reinforce itself. Mode cutoff therefore corresponds to the transverse resonance condition.

The perturbation to be introduced is to replace the non-physical magnetically conducting walls with physical open circuits when, although much of the foregoing argument retains its validity, these edges are no longer perfect and some leakage of energy will occur from them. For waveguides with no losses, the Q of transverse resonance is infinite and the cutoff discrete. However, when as here the case is otherwise, the resonant Q will be finite and it becomes more sensible to speak of a leaky mode with a critical rather than a cutoff frequency [12]. It is with these critical frequencies and the Q 's associated with them, a measure of the degree to which the leaky mode is dominant, that we will be concerned in the remainder of this paper. We will begin by considering critical frequency in parallel plate lines as a means of advancing on the final objective of treating the two-wire line.

3. THE TE-LIKE MODE IN PARALLEL PLATE LINE

From a theoretical point of view, the simplest form of parallel plate line is that shown in Fig. 2, made by cutting a slit through a finite thickness but otherwise infinite conducting slab, the presence of the slab serving to prevent coupling of the fringing fields at the two open faces. Determining the critical frequency for this configuration is

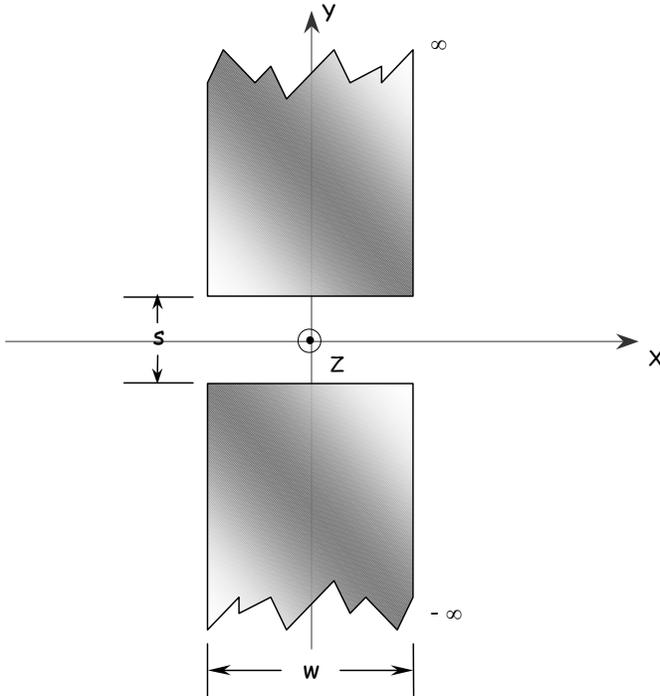


Figure 2. Slit in infinite slab.

simplified if use is made of symmetry, i.e., if the problem geometry is bifurcated by introducing an electrically conducting plane along $x = 0$. Each half of what remains is a Rotman's problem for which a solution already exists in the literature [8]. We will therefore be brief in developing it, adding no more detail than is needed to take us on to the next stage.

The equivalent circuit for determination of resonance is shown in Fig. 3a. It consists of a section of transmission line short circuited at one end and terminated at the other in a parallel combination of G and B representing the open face of the slot. Expressions for these quantities normalised in terms of the characteristic impedance of the transmission line are available from an earlier consideration by Marcuvitz [9] of a parallel plate line radiating through a conducting baffle into a half space.

In anticipation of later simplifications, a useful way to begin is to replace the transmission line part of the resonator to the left of the reference plane TT' with its π -equivalent, lumped circuit which,

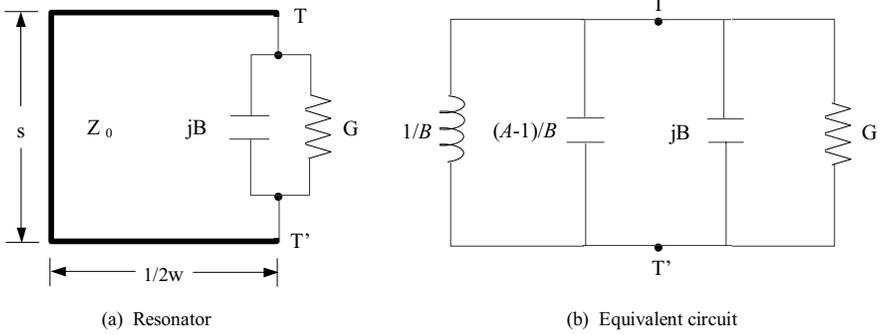


Figure 3. Equivalent resonator for the bifurcated parallel plate line.

because of the short circuit, immediately degenerates into the pair of parallel susceptances shown in Fig. 3b. What remains is simply a lumped, parallel resonator for which resonant frequency and Q are readily found. It is to be noted that all the inductance (magnetic energy storage) is supplied by the transmission line while capacitance (electric energy storage) is shared between the transmission line and the fringing field. Radiation losses in the guise of G determine the Q .

Again looking ahead, it will be found useful to write the susceptances contributed by the transmission line in terms of its transmission ($\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$) parameters, when the general equations for resonance and Q are easily shown to be

$$\mathcal{A} + j\mathcal{B}G = 0 \quad (1a)$$

$$Q = \frac{1}{|\mathcal{B}|G} \quad (1b)$$

For a uniform transmission line of length $\frac{1}{2}w$ and characteristic impedance Z_0

$$\mathcal{A} = \mathcal{D} = \cos \frac{1}{2}kw \quad (2a)$$

$$\mathcal{B} = jZ_0 \sin \frac{1}{2}kw \quad (2b)$$

$$\mathcal{C} = \frac{j}{Z_0} \sin \frac{1}{2}kw \quad (2c)$$

Substituted into eqns. (1a,b), these lead to as the equation for resonance

$$\cot \frac{1}{2}kw - b = 0 \quad (3)$$

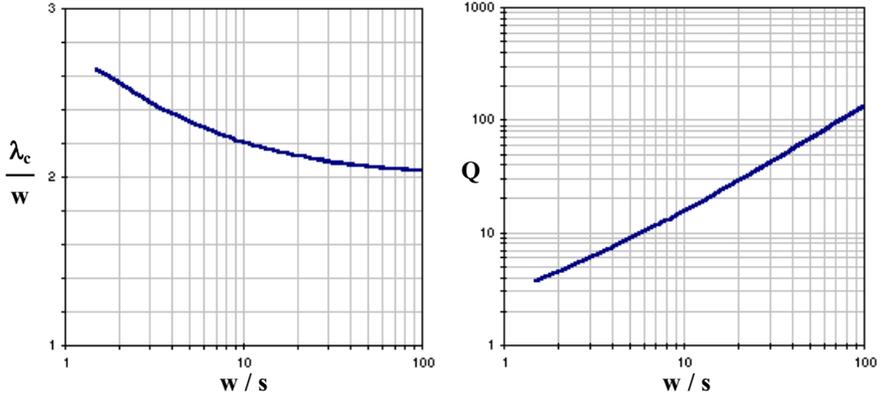


Figure 4. Normalised critical wavelength and Q in idealised parallel plate line.

and for Q

$$Q = \frac{1}{g \sin \frac{1}{2}kw} \quad (4)$$

where $g = GZ_0$ is the normalised conductance, and $b = BZ_0$ is the normalised susceptance.

An expression similar to eqn. (3) is to be found in Rotman's paper [8].

Interestingly Marcuvitz [9] provides two formulas for each of G and B , an accurate formula based on a modal expansion and a simplified approximation obtained quasi-statically[†] Fig. 4 shows the critical wavelength normalised to the line width w , i.e., the ratio $2\pi/k_cw$, where k_cw is the smallest solution of eqn. (3), as a function of w/s . The results given in this figure were derived using the accurate formulas but substitution of the approximate forms leads to differences indistinguishable within the line width. At least so far as resonant wavelength is concerned, the reason lies in the fact that it is the transmission line that contributes most of the resonator capacitance, so that even a relatively large error in fringing field susceptance tends to be washed out in the final result. Overall the contribution of the open edge is to effectively widen the resonator and to produce a sharpness of resonance which increases monotonically with w/s , indicative of the growing dominance of the leaky mode.

[†] In the case of the susceptance, there is a sign error in the accurate formula (although not in the accompanying graph) which is corrected in [13]

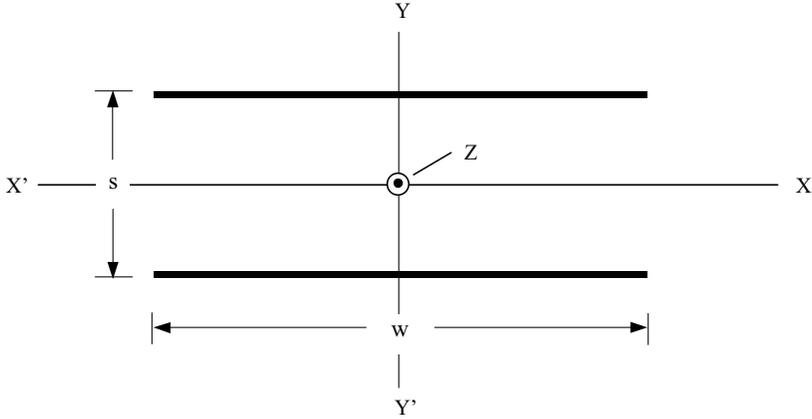


Figure 5. Physically realisable parallel plate line.

A more realistic form of parallel plate line consists of the pair of thin conducting strips shown in Fig. 5. Here the fringing fields at each open edge are no longer kept separate but interact. However, given new values for edge admittance, a result can be obtained directly by their substitution into the previous equations. Formulas for G and B of comparable quality to Marcuvitz's do not seem to have been published for this case but the success in the previous example of approximate results obtained quasi-statically does suggest a way out. This problem also has strong similarity to Cullen's [10], from which it differs only in the need to account for interaction of the two fringing fields. The fact that Cullen obtains satisfactory results from quasi-static approximations lends added weight to their use here.

B can be estimated from electrostatic considerations. A formula for static capacitance attributable to the fringing field in a parallel plate capacitor is to be found in [14][‡]. It is a lower bound on the correct result, the accuracy of which improves with increasing w/s . Adapted to supply a normalised susceptance, it is

$$b = \frac{ks}{2\pi} \left(1 + \ln \frac{2\pi w}{s} \right) \quad (5)$$

The formula for static capacitance that underlies this result is known to be accurate to four percent or better for $w/s > 2$.

Finding G requires further work and is addressed in Appendix A

[‡] This problem is also solved in Morse and Feshbach [15] but there is an error which leads to the fringing capacitance given there being out by a factor of 2.

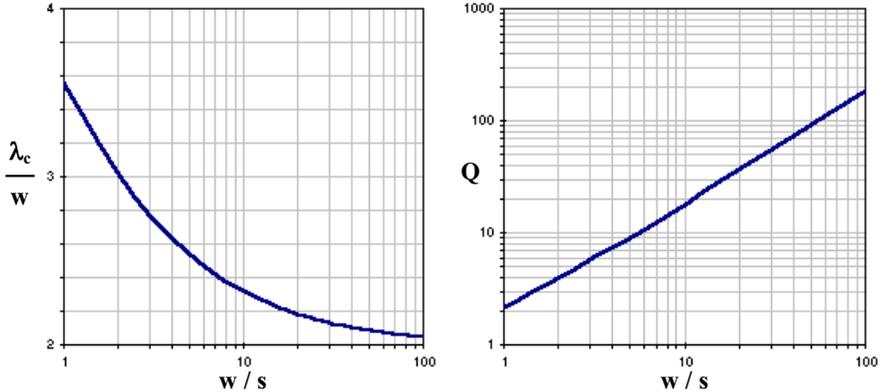


Figure 6. Normalised critical wavelength and Q in physical parallel plate line.

where it is shown that for $w/s \gg 1$

$$g = \frac{ks}{4} \{1 + J_0(kw)\} \quad (6)$$

It is shown in the Appendix that for $w/s > 3$, about five percent or better is a reasonable expectation for the accuracy of this result.

The results for critical wavelength normalised in terms of w and Q are shown as a function of w/s in Fig. 6. They exhibit similar trends to those appearing in Fig. 4, although for w/s at the smaller end of its range, they are probably more indicative than highly accurate.

4. THE TE-LIKE LEAKY MODE IN TWO-WIRE TRANSMISSION LINE

It is interesting to extend the foregoing ideas to determining the critical frequency of the lowest order TE-like leaky mode in two-wire transmission line. This is essentially a conformal distortion of the previous case in which the parallel conducting surfaces, parts of constant coordinate surfaces in a cartesian frame, now become parts of constant coordinate surfaces in a bipolar coordinate system [16].

Consider the cross-section shown in Fig. 7, the condition for transverse resonance of which is to be found. If for a moment we think about the electrostatic field which surrounds such a pair of conductors when they are at equal and opposite potentials, it is clear that on each side of the diagram there is a line of force joining the points of intersection of the circles with a common tangent drawn parallel to

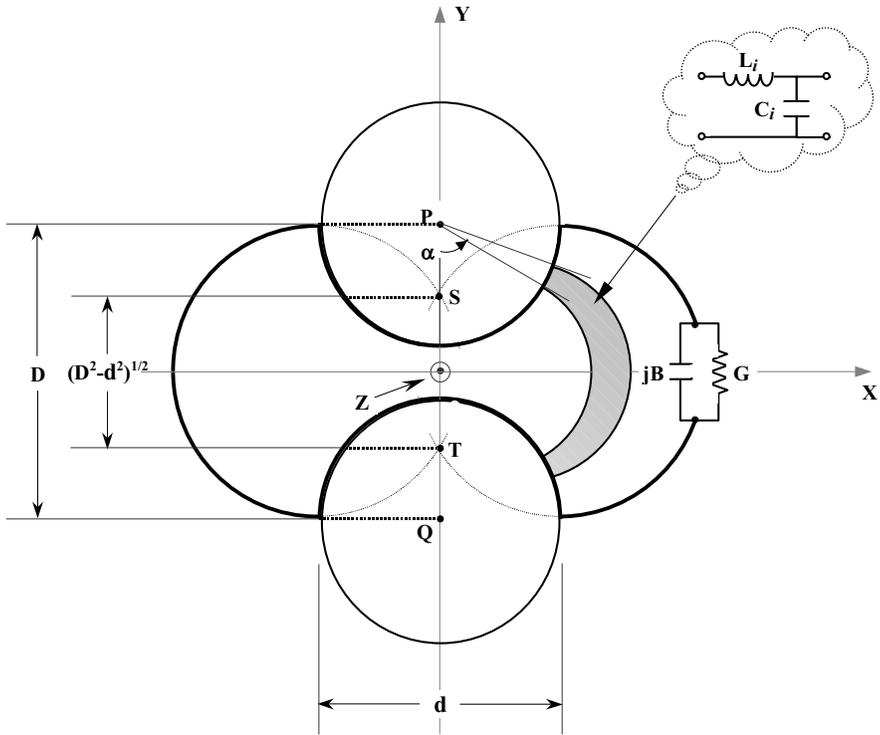


Figure 7. Resonator and its circuit representation.

the vertical axis of symmetry. These lines of force, which will appear to have their origin in line charges located at points S and T , are semicircles of diameter equal to the centre-to-centre spacing of the circular conductors [17]. They will be taken as the boundaries between the interior and exterior of the resonator whose resonant wavelength is to be determined.

The method used follows closely that employed previously. Symmetry about the plane $x = 0$ is again used to allow bisection of the resonator and the interior region is represented by lumped susceptances derived from the transmission parameters. The external region is modelled as a parallel combination of G and B . The immediate problem is to find all these quantities.

When a quasi-TEM wave is propagating transversely in the cross-section, we can expect that its electric lines of force will follow the same paths as those of the electrostatic field instanced above, i.e., they will be segments of circles which pass through S and T . We may therefore segment the interior region into n elementary sub-regions between

successive lines of force drawn at equal increments in the angle α joining the centre of each circular conductor with the takeoff or reception point of the line force on the conductor surface. Our strategy is to represent each of these elementary sub-regions circuitally as an inductance-capacitance L-section and to find the transmission parameters of the entire interior region as the cascade of their individual transmission matrices.

The question of actually finding these elemental inductances and capacitances is one better deferred to Appendix B. Here we will proceed on the assumption that they are known. It is shown in any number of texts that the transmission matrix of an LC L-section is

$$T_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} = \begin{bmatrix} 1 - L_i C_i \omega^2 & jL_i \omega \\ jC_i \omega & 1 \end{bmatrix} \quad (7)$$

from which it follows that that for the entire interior region is

$$T = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \prod_{i=1}^n T_i \quad (8)$$

A program to find the elements of this matrix has been written in MATLAB. Some numerical experimentation has shown that using 2,000 sub-regions guarantees four figure consistency of the transmission parameters of the interior region. However, since it is not excessively costly in machine time to use more, 3,000 have been used in this work to provide an extra comfort margin.

At the chosen boundary between the interior and exterior regions, the exterior portion is to be represented by a parallel combination of G and B . As was the case with the parallel plate line, in the expectation that because capacitance is shared between interior and exterior regions an approximation for the exterior capacitance will suffice, B has been determined electrostatically as the residuum of the total capacitance of the cross-section not attributable to the interior region. The method given in Appendix D allows this to be done conveniently. Finding an acceptable formula for G is much more complex and in a sense only minimally rewarding in that the critical wavelength, the more interesting of the two resonance parameters, does not depend on it. However the question is considered in Appendix E.

Fig. 8 shows the data for the TE-like mode over the range of diameter ratios $1.1 \leq D/d \leq 1.6$ (corresponding to TEM mode characteristic impedances in the range 50–125 ohms). It has been prepared in a form that affords easiest comparison with the earlier Figs. 4 and 6 where similar data for the parallel plate line is presented. Here normalised critical wavelength and Q form the ordinates in a

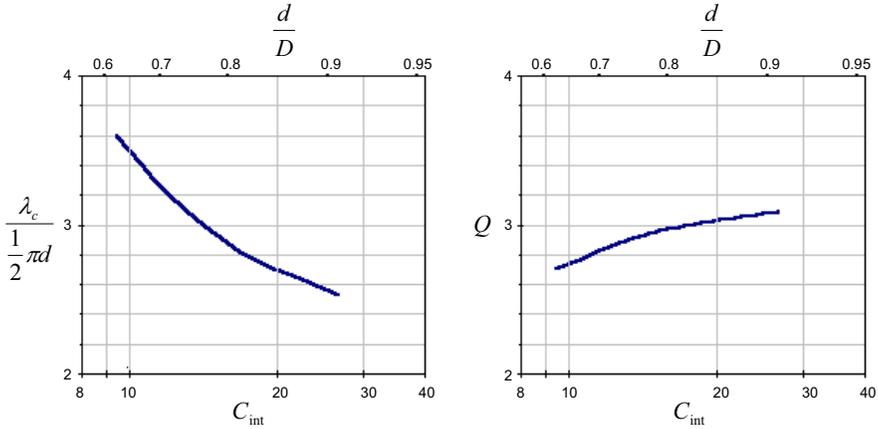


Figure 8. Normalised critical wavelength and Q for the TE-like mode in two-wire line.

plot with the interior capacitance of the resonator as abscissa. (Recall that in the parallel plate case w/s , the chosen abscissa, is directly proportional to the interior capacitance of the resonator.) Wavelength has been normalised to $\frac{1}{2}\pi d$, the length of the conducting surface interior to the resonator and chosen as having some equivalence with w in the parallel plate case, i.e., normalised critical wavelength is the ratio $4c/\omega_c d$, where ω_c is the critical angular frequency determined by solving eqn. (1a) with appropriate insertions for the quantities involved and c is the velocity of light. All three figures show similar trends save that in this instance Q remains small, indicating a rapidly attenuating leaky mode. It is only for these very low impedance lines, not infrequently used as components of baluns, that the TE-like leaky mode is of any importance. For the wider spaced, hundreds of ohms TEM mode characteristic impedance, two-wire lines normally deployed, it is the TM-like higher mode that becomes more interesting. This is the problem that we will now address.

5. TM-LIKE LEAKY MODES IN PARALLEL PLATE AND TWO-WIRE LINES

In the case of parallel plate line, application of the result argued for in Section 2 leads to an immediate outcome. This suggests that the TM-like leaky mode critical wavelength corresponds to putting the plates half a wavelength apart, as in the unperturbed case. Credence is lent to this by observing that TM resonance within a pair of parallel plates

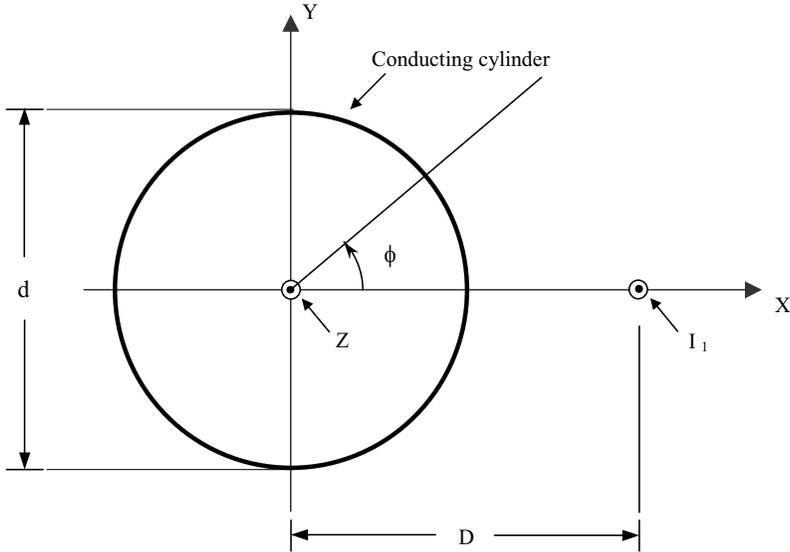


Figure 9. Wire parallel to cylinder-cross-sectional view.

is exactly the same phenomenon as resonance in a Fabry-Perot cavity operating in its lowest order mode, for which the plates must be a half wavelength apart. In fact this is precisely the simplified model on which elementary analyses of it are based [11]. To be sure, in any real Fabry-Perot cavity, corrective action is taken to limit losses by diffraction at the edges, a remedy that we are not proposing. However, the result will be to have produced a “badly designed” cavity in which resonance can be expected to occur at much the same wavelength, even if with a much diminished Q .

It strains permissiveness only a little more to apply this idea to a two-wire transmission line, even if it strains the imagination a great deal more to agree that what now we are dealing with much resembles a Fabry-Perot cavity. As most often deployed, two-wire lines are made with an inter-wire spacing of several times the conductor diameter. The canonical problem that holds the key to further advance is that of an electrical line source parallel to a conducting cylinder.

Consider the configuration depicted in cross-section in Fig. 9 in which the axis of the cylinder is used to define the z -axis, any point on which can be chosen as origin. The perpendicular from this origin through the line source then becomes the initial line for angular measurement in a cylindrical coordinate system. It is then easy to use some results presented by Harrington [18] to show that the surface

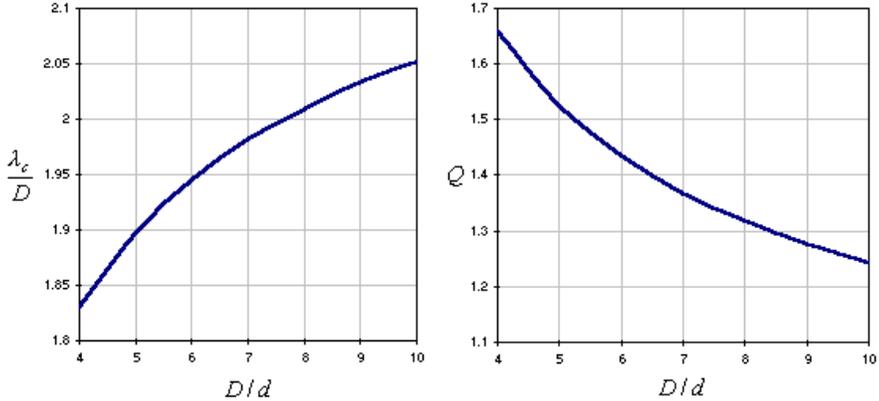


Figure 10. Normalised critical wavelength and Q in for TM-like mode in two-wire line.

current density induced on the cylinder by the line current I_1 is

$$J_z = \frac{-I_1}{\pi d} \sum_{n=-\infty}^{\infty} \frac{H_n^{(2)}(kD)e^{jn\phi}}{H_n^{(2)}\left(\frac{1}{2}kd\right)} \quad (9)$$

For small kd , only the $n = 0$ term is significant, when the cylinder is essentially equivalent to another line current given by

$$I_2 = \frac{1}{2}d \int_{-\pi}^{\pi} J_z d\phi \simeq -I_1 \frac{H_O^{(2)}(kD)}{H_O^{(2)}\left(\frac{1}{2}kd\right)} \quad (10)$$

It was, of course, in the expectation of obtaining a result of this kind that we made the implicit assumption that the first of the pair of conductors could be represented by a line current and hence chose our canonical problem.

The induced line current which replaces the second conductor now becomes the source for induction of a further line current on the first, and for resonance to occur, it must do so such that the effect is one of reinforcement, i.e., with a round trip phase lag of 2π . Hence the equation for resonance is

$$\arg \left\{ \frac{H_O^{(2)}(kD)}{H_O^{(2)}\left(\frac{1}{2}kd\right)} \right\} = -\pi \quad (11)$$

The solution of this equation is presented in Fig. 10 which shows critical wavelength normalised to conductor spacing as a function of the

spacing to diameter ratio. It is clear that a good first approximation is just to assume that the critical wavelength corresponds to having the centres of the wires a half wavelength apart, a result arrived at earlier by Marin [12] but in a much more elaborate analysis.

It is also possible to work out a Q factor for the resonance, a result not readily available from Marin's analysis. In that each wire can intercept only a small fraction of the radiation from the other, the expectation has to be for a small Q and a mode with high leakage losses. The basis of the calculation is to determine the ratio in which, after each reflection in the second cylinder, the magnitude of the current re-induced in the first is decreased and noting that the time required for this to happen is one period of the oscillation. Resonant Q then follows as [19]

$$Q = \frac{-\pi}{2 \ln \left| \frac{H_O^{(2)}(kD)}{H_O^{(2)}\left(\frac{1}{2}kd\right)} \right|_{res}} \quad (12)$$

Q is shown as a function of D/d also in Fig. 10. For $D/d = 5$ for example, corresponding to a TEM-mode characteristic impedance of 276 ohms, the Q is found to be about 1.5, corresponding to an extremely leaky guide and probably too low to assert any strong dominance of the leaky wave mode poles on the continuous mode spectrum.

6. CONCLUSION

In this paper, using a transverse resonance technique originally used for determination of the eigenvalues in leaky wave antennas, we have found the critical wavelengths of the lowest order TE and TM-like leaky modes in the parallel plate and two-wire transmission lines. These can be used to set a bound on the bandwidth over which the fundamental TEM mode is substantially the only propagating field. Particularly in a context where the aim is to avoid by a margin any conditions that would allow significant energy to be coupled into other than the TEM mode, this is a satisfactory approximation.

ACKNOWLEDGMENT

The author would like to thank Rocco Melino, Hooi Sing Choy and Chun Ouyang who assisted him at various stages during preparation of this paper.

**APPENDIX A. DETERMINATION OF G
ATTRIBUTABLE TO THE EXTERNAL REGION OF A
PARALLEL PLATE LINE**

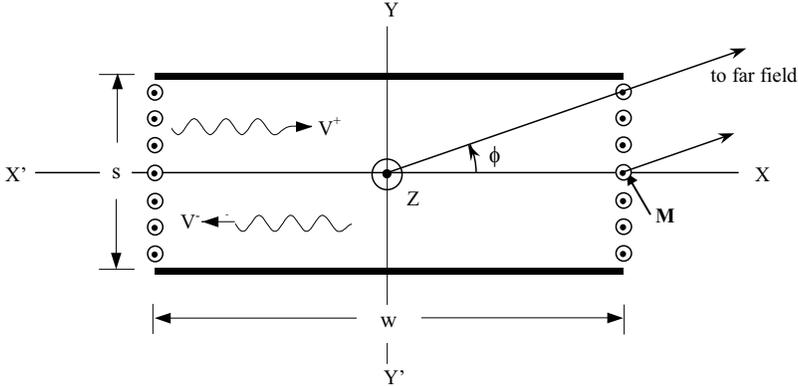


Figure A1. Geometry for determining edge conductance in parallel plate line.

A formula for the edge conductance for the case where $w \gg s$ is easily obtained in the following way. Consider the cross-section shown in Fig. A1 and let the voltage travelling waves traversing it be

$$V^+ = V_0 e^{-jkx} \quad (\text{A1a})$$

$$V^- = -V_0 e^{jkx} \quad (\text{A1b})$$

Note that this choice ensures that $V^+ + V^- = 0$ at $x = 0$. Then, at the right hand end ($x = \frac{1}{2}w$),

$$V = V^+ + V^- = -j2V_0 \sin\left(\frac{1}{2}kw\right) \quad (\text{A2})$$

Assuming that the electric field at the open edge can be approximated [18] as that in the interior of a parallel plate capacitor, we then have

$$\mathbf{E} = -\hat{y}j \frac{2V_0}{s} \sin\left(\frac{1}{2}kw\right) \quad (\text{A3})$$

Using the equivalence principle [18], the open edge is now to be closed with a conductor supporting on its surface a magnetic current density \mathbf{M} , given by

$$\mathbf{M} = \mathbf{E} \times \hat{x} = \hat{z}j \frac{2V_0}{s} \sin\left(\frac{1}{2}kw\right) \quad (\text{A4})$$

If we go through the same steps for the left hand end, exactly the same result is obtained. This follows because the reversal of the direction of the normal compensates for the change in sign of the aperture field. Hence we are left to solve the problem of radiation from a conducting bar of rectangular cross-section, on opposite ends of which there are equal, in-phase, magnetic strip currents.

Since we expect resonance of the interior cross-section to occur when $kw \approx \pi$, the restriction $w \gg s$ implies that $ks \ll \pi$, in which case diffraction effects due to the bar can be neglected and the radiation field computed assuming the strip currents to be located in free space. Moreover it also will involve only small error to concentrate these strip currents as equivalent line currents \mathbf{K} at their centres, i.e.,

$$\mathbf{K} = \int_{-\frac{1}{2}s}^{\frac{1}{2}s} \mathbf{M} dy = \hat{z} j 2V_0 \sin\left(\frac{1}{2}kw\right) \quad (\text{A5})$$

Two dimensional radiation from an electric line current is a problem solved in [18] and the result for a magnetic line current can be inferred directly from it by duality. Taking into account the array factor due to the two in phase magnetic line sources, the electric far field ($\rho \gg w$) will be

$$\mathbf{E} = \hat{\phi} j 4kV_0 \sin\left(\frac{1}{2}kw\right) \cos\left(\frac{1}{2}kw \cos \phi\right) \sqrt{\frac{j}{8\pi k\rho}} e^{-jk\rho} \quad (\text{A6})$$

From this the Poynting vector can be integrated over unit axial length of the far field cylinder to determine the total radiated power. The result is

$$P = \frac{2kV_0^2 \sin^2\left(\frac{1}{2}kw\right)}{\eta} \{1 + J_0(kw)\} \quad (\text{A7})$$

where $\eta = 120\pi$ is the impedance of space. The power out of each aperture separately will be half this, making the aperture conductance

$$G = \frac{\frac{1}{2}P}{VV^*} = \frac{k}{4\eta} \{1 + J_0(kw)\} \quad (\text{A8})$$

When this is normalised in terms of the characteristic admittance per unit length of the cross-section ($1/\eta s$), the result is that shown in eqn. (6).

It is noteworthy that as $kw \rightarrow \infty$ and there is no interaction between the edges, eqn. (6) reduces to the approximate formula given by Marcuvitz in [9] for a single open (but flanged) edge. This approximate formula is known to be accurate to better than five

percent for $ks < 1$. Since, as we have noted, transverse resonance occurs when $kw \approx \pi$, it is reasonable then to expect that the value obtained for Q by inserting eqn. (6) into eqn. (4) will be accurate to about this order for $w/s > 3$.

APPENDIX B. FINDING THE ELEMENTAL INDUCTANCE AND CAPACITANCE

The problem taken up in this appendix is that of finding the inductance and capacitance that are to be used in an L-section representation of the elemental region contained between the line conductors and near neighbouring, electric lines of force.

Consider Fig. 7 and let it be required to determine the inductance and capacitance for the i th such region which is contained between the $(i - 1)$ th and the i th lines of force, $i = 1, n$. This is the shaded region shown in the figure. All these lines of force are arcs of circles that leave or enter the conductors normally and appear to have originated or terminated on virtual line charges at S and T . To set up an identification system for them, let the quadrant on each conductor between the axis joining the centres of the conductors, the Y axis in the chosen coordinate system, and the end of the resonator be divided into n equal sub-arcs, with neighbouring lines of force coming away or entering from the end points of each given sub-arc. Let radius vectors be drawn from the centres of the conductors to the end points of these sub-arcs. A parameter which then serves to define the i th line of force is the angle contained between the Y axis and the radius vector drawn to its point of exit or entry. This angle will be $\alpha_i = \pi i/2n$.

The associated set of magnetic lines of force will all be parallel to OZ , i.e., normal to the cross-sectional plane shown in Fig. 7, and might be thought of as being supported by fictional static magnetic charges contained on magnetically conducting planes lying in cross-sections unit distance apart. Since the region that they occupy contains no electric currents, it follows from the magnetostatic form of Ampere's Law that this magnetic field must be uniform over the cross-section. Hence the inductance attributable to the shaded region in Fig. 7 will be

$$L_i = \mu_0 \Delta A_i \quad (\text{B1})$$

where ΔA_i is the area of the shaded region.

In Appendix C a formula is derived which gives the area bounded by the i th electric line of force, the arcs of the conductor cross-section and the Y axis in terms of the parameter α . By subtracting the areas so determined for consecutive lines of force the areas of successive repetitions of the shaded region are found, i.e., $\Delta A_i = A(\alpha_i) - A(\alpha_{i-1})$.

From this the inductance of each elemental L-section follows through eqn. (B1).

The capacitance is determined electrostatically from a knowledge of the charge distribution on the conducting surfaces when a static potential difference exists between them. The capacitance attributable to each shaded region bears the same ratio to the total capacitance per unit length of the cross-section as does the charge on either of the sub-arcs that help define it to the total charge on the conductor of which the sub-arc is part. In Appendix D it is shown how the surface charge between the Y axis and the exit or entry point of the i th line of force can be found as a function of the parameter α . As with the inductance, the capacitance to be inserted into the L-section equivalent is then determined by differencing the capacitances attributable to regions bounded at their outer ends by consecutive lines of force.

Once these L 's and C 's are known, it is a simple matter to use eqn. (8) to determine the transmission parameters of the resonator for insertion into eqns. (1a,b).

APPENDIX C. ON FINDING THE AREA OF A CERTAIN CLOSED FIGURE

In this appendix, we provide a method for determining the area of the interior region shown in Fig. 7 as bounded by the vertical axis of symmetry, a line of force and the two conductors. This is most easily done by subtracting from the circular segment cut off by the line of force and the vertical axis of symmetry the two identical sub-areas contained within the conductor cross-sections and themselves bounded by two arcs and the axis.

As a prelude, it is useful first to define a special function $seg(r, s)$, the area of the minor segment cut from a circle of radius r by a chord of length s . It requires only simple geometry to show that

$$seg(r, s) = r^2 \sin^{-1} \frac{s}{2r} - \frac{s}{2} \sqrt{r^2 - \left(\frac{s}{2}\right)^2} \quad (C1)$$

All lines of force are circles with their centres on the horizontal axis of symmetry and passing through the points S and T . It requires only some tedious geometry to show that the radius of curvature of the line which enters or leaves the conductors at positions defined by the angle α is

$$\rho(\alpha) = \frac{D - d \cos \alpha}{2 \sin \alpha} \quad (C2)$$

However, of those lines which are interior to the right hand half of the resonator, some only will have their centres on the positive X axis. It

is necessary therefore to determine the critical angle α_c beyond which the segment cut off by the line of force and the vertical axis passes from being the minor to the major segment. This can be done by determining where a circle of radius $\frac{1}{2}\sqrt{D^2 - d^2}$ and centre the origin cuts the conductors, when it is found that

$$\alpha_c = \cos^{-1} \frac{d}{D} \quad (\text{C3})$$

The area of the segment bounded by the vertical axis of symmetry and the interior line of force is then

$$A_1(\alpha) = \begin{cases} \text{seg} \left\{ \rho(\alpha), \frac{1}{2}\sqrt{D^2 - d^2} \right\}, & \alpha < \alpha_c \\ \pi\rho^2(\alpha) - \text{seg} \left\{ \rho(\alpha), \frac{1}{2}\sqrt{D^2 - d^2} \right\}, & \alpha > \alpha_c \end{cases} \quad (\text{C4})$$

The subareas to be subtracted and contained within the conductor cross-sections are next found as the sum of an interior triangle and two residual segments. After some amount of trigonometry, this can be shown to be

$$A_2(\alpha) = \frac{1}{8}d \sin \alpha \left\{ \sqrt{D^2 - d^2} - (D - d) \right\} + \text{seg} \left\{ \frac{1}{2}d, d \sin \frac{1}{2}\alpha \right\} \\ + \text{seg} \left\{ \rho(\alpha), \sqrt{\frac{1}{2}(D - d \cos \alpha) (D - \sqrt{D^2 - d^2})} \right\} \quad (\text{C5})$$

The required area which we set out to find is therefore

$$A(\alpha_i) = A_1(\alpha_i) - 2A_2(\alpha_i) \quad (\text{C6})$$

and from this the element of inductance is found by the method given in Appendix B.

APPENDIX D. DETERMINATION OF THE CAPACITANCE ATTRIBUTABLE TO AN ELEMENT OF CROSS-SECTION IN A TWO-WIRE LINE

The root problem, from which the result sought readily follows, is to determine under static conditions the charge contained on an arc of either conductor between the vertical axis of symmetry and a radius vector making an angle α with it. The capacity per unit axial length of the cross-section attributable to the arc then bears the same ratio to the total capacitance as does the charge on the arc to the total

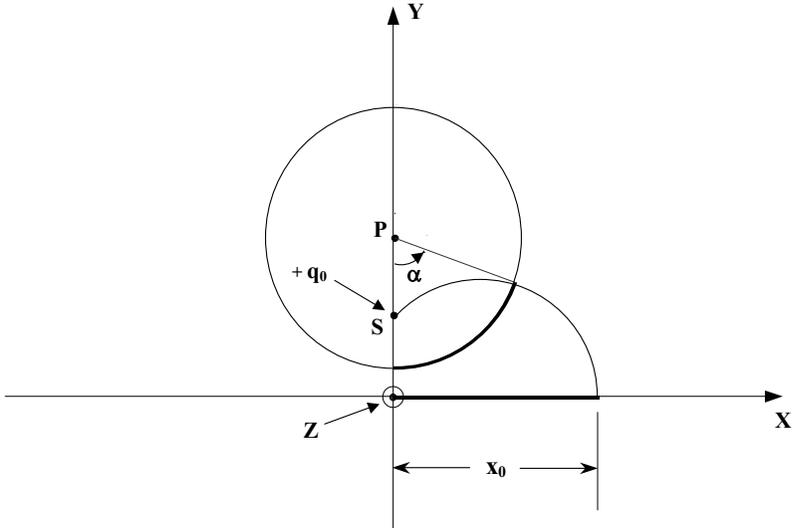


Figure D1. Geometry for determining resonator capacitance.

charge on the conductor. From this the element shunt capacitance follows by differencing in a manner similar to that used to find the series inductance in the equivalent ladder network.

Consider Fig. D1 which shows a virtual line charge q_0 per unit length located on the vertical axis of symmetry at S , i.e., $y = \frac{1}{2}\sqrt{D^2 - d^2}$, and let the plane $y = 0$ be replaced with an infinite, perfectly conducting sheet. Then by image theory, in the upper half space this is equivalent to the original problem. Now trace a line of force emanating at the line charge and passing through the conducting cylinder at a position defined by the angle α . Then it is not hard to show that this will terminate on the conducting plane at $x_0 = \frac{1}{2}(D + d) \tan \frac{1}{2}\alpha$, when the charge on the arc will be the negative of that contained on the sheet in $0 < x < x_0$. We do this because it is easier to integrate across the sheet than around the arc.

With no great effort, it can be shown that the charge density on the sheet is

$$\rho(x) = \frac{-\frac{1}{2}q_0\sqrt{D^2 - d^2}}{\pi\left(x^2 + \frac{1}{4}(D^2 - d^2)\right)} \quad (\text{D1})$$

and hence the charge on the arc is

$$q(\alpha) = \int_0^{x_0} \rho(x)dx = \frac{q_0}{\pi} \tan^{-1} \left(\sqrt{\frac{D+d}{D-d}} \tan \frac{1}{2}\alpha \right) \quad (\text{D2})$$

In [17] C_0 , the total capacitance per unit length of the line, is found. Hence the capacity contributed by the arc is

$$C(\alpha) = \frac{C_0 q(\alpha)}{q_0} = \frac{C_0}{\pi} \tan^{-1} \left(\sqrt{\frac{D+d}{D-d}} \tan \frac{1}{2} \alpha \right) \quad (\text{D3})$$

In this paper, we have assumed that the susceptance of the region exterior to the resonator is contributed by the exterior static capacitance, when

$$B = \frac{1}{2} C_0 \omega \left(1 - \frac{2C \left(\frac{1}{2} \pi \right)}{C_0} \right) \quad (\text{D4})$$

APPENDIX E. DETERMINATION OF G ATTRIBUTABLE TO THE EXTERIOR REGION OF A TWO-WIRE LINE

We compute G using the equivalence principle [18]. This requires placing magnetic sheet currents on each end face of the resonator which is then closed with a perfect conductor. Energy loss, which manifests itself circuitally as a conductance, occurs by radiation of a cylindrical wave from the magnetic current into the two dimensional space exterior to the resonator. To determine G , two things are required; one must either know or be able to make plausible assumptions about the nature of the sheet currents and then be able to calculate the radiation pattern which they produce in the presence of a conducting object made up by juxtaposition of two pairs of hemicylinders, the ends of the resonator and the top and bottom halves of the original conductors, the latter banished to the exterior region by assumption.

As was done in Appendix A where we considered the parallel plate line case, here we will also assume that the form of the electric field along an open end face of the resonator is the same as along a similarly placed line of force in the static field produced by a voltage V between the conductors, though without the further assumption that this is the same as along an interior line of force. This may be found in terms of the geometric parameters shown in Fig. E1 when, after a lot of routine algebra which is omitted here, it is possible to show that

$$\mathbf{E}(\phi') = \begin{cases} \hat{\phi}' \frac{-V \sqrt{D^2 - d^2}}{D(D \cos \phi' + d) \cosh^{-1} \frac{D}{d}} & -\frac{\pi}{2} < \phi' < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases} \quad (\text{E1})$$

Application of the equivalence principle then gives for the magnetic sheet current $\mathbf{M} = -\hat{\rho}' \times \mathbf{E}$.

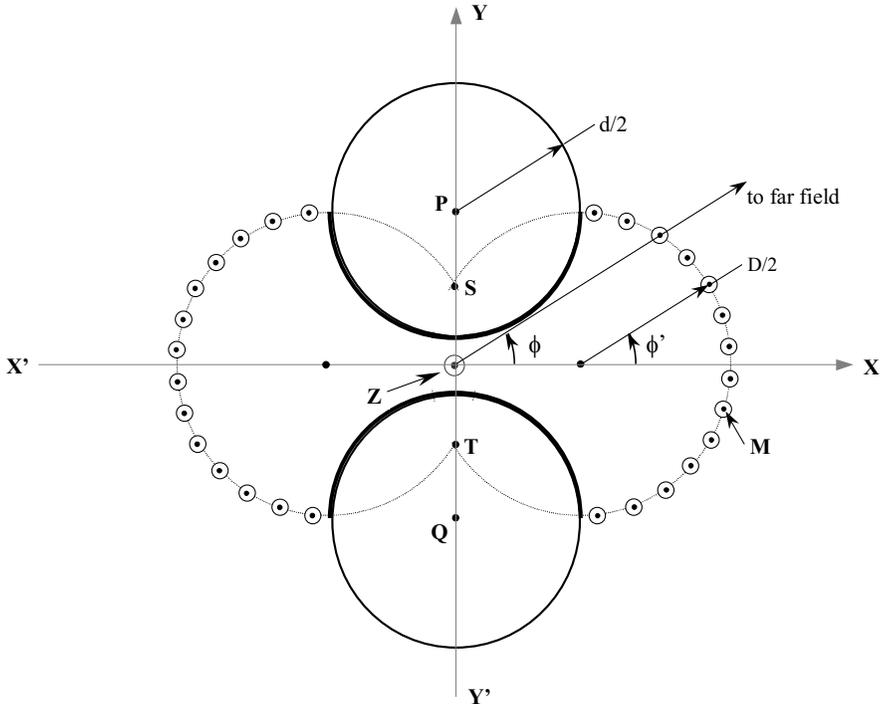


Figure E1. Geometry for determining G .

There is no simple way of computing the radiation pattern that this will produce in the presence of the quadrihemicylindrical, conducting structure which supports it. As an approximation we propose to replace this structure with a pair of cylinders which are simply extensions of the ends of the resonator (i.e., cylinders of diameter D) and, even though they intersect (because their centres are spaced $d < D$ apart), to treat them as isolated conducting cylinders. This can be expected to lead to greatest error in the weakest parts of the field and is justified by the fact that determining conductance involves integration of the field, an error suppressing process. Finally, of course, there is the consideration that G itself plays no part in determining the critical frequency, although it does have a first order impact on resonant Q .

The radiation field produced by an isolated conducting cylinder on the surface of which there is an axially directed magnetic sheet current harmonically distributed in ϕ' is easily found taking as a point of departure the field produced by an axial magnetic line source, a

problem to which the solution is given in [18]. This suggests as a way forward expanding the magnetic current obtained from eqn. (E1) as a Fourier series and combining this with the previous result to obtain an expression for the field. These steps involve considerable detail that we will choose to omit here in favour of simply stating the end result, most easily done in terms of the magnetic field which contains only the single, z -directed component

$$H_z(\phi) = \frac{V \sqrt{\left(\frac{D}{d}\right)^2 - 1}}{\pi D \cosh^{-1} \frac{D}{d}} \frac{j2}{\eta} \sqrt{\frac{j2}{\pi k \rho}} e^{-jk\rho} f(\phi) \quad (\text{E2})$$

where

$$f(\phi) = \sum_{m=0}^{\infty} {}_l j^m c_m \cos m\phi \quad (\text{E3})$$

$$H_m^{(2)'} \left(\frac{1}{2} kD \right)$$

and

$$c_m = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos m\phi'}{\frac{D}{d} \cos \phi' + 1} d\phi' \quad (\text{E4})$$

Note the priming of the summation in eqn. (E3), intended to indicate that the first term is to be halved; note also that the series substantially converges in around kD terms, meaning that in practice only a few of the c_m need to be determined.

In our problem, this is the radiation field produced by the cylindrical aperture having its centre on the X -axis at $x = \frac{1}{2}d$. That due to the other, the cylinder with its centre at $x = -\frac{1}{2}d$, will be the lateral inversion of this, obtainable from the previous result under the transformation $\phi \rightarrow \pi - \phi$. Combining both aperture fields with the correct phasing gives as the total radiation pattern

$$H_z = \frac{V \sqrt{\left(\frac{D}{d}\right)^2 - 1}}{\pi D \cosh^{-1} \frac{D}{d}} \frac{j2}{\eta} \sqrt{\frac{j2}{\pi k \rho}} e^{-jk\rho} \left\{ f(\phi) e^{j\frac{1}{2}kd \cos \phi} + f(\pi - \phi) e^{-j\frac{1}{2}kd \cos \phi} \right\} \quad (\text{E5})$$

From this the Poynting vector is easily found. When, similarly to the procedure used in Appendix A, this is integrated over unit axial length of the far field circumscribing cylinder, the conductance per unit length of each aperture can be shown to be

$$G_z = \frac{16}{\pi^3 \eta k D^2} \frac{\left(\left(\frac{D}{d} \right)^2 - 1 \right)}{\left(\cosh^{-1} \frac{D}{d} \right)^2} \int_0^{\frac{\pi}{2}} \left| f(\phi) e^{j\frac{1}{2}kd \cos \phi} + f(\pi - \phi) e^{-j\frac{1}{2}kd \cos \phi} \right|^2 d\phi \quad (\text{E6})$$

Note that in doing this, the range of integration has been minimised by making use of the fourfold symmetry of the radiation field. However, it does not appear possible to reduce this formula further to any simple, explicit kind of result and values of G would seem best computed from it directly numerically.

REFERENCES

1. Green, H. E., "Determination of the cutoff of the first higher order mode in coaxial line by the transverse resonance technique," *IEEE Trans. Mic. Th. Tech.*, Vol. MTT-37, 1652–1653, 1989.
2. Sorrentino, R., "Transverse resonance technique," *Numerical Techniques for Microwave and Millimetre-wave Passive Structures*, T. Itoh (ed.), John Wiley & Sons, 1989.
3. Rozzi, T. and M. Mongiardo, *Open Electromagnetic Waveguides*, Institution of Electrical Engineers, London, First Edition, 1997.
4. Friedrichs, K. O., "Criteria for discrete spectra," *The Theory of Electromagnetic Waves*, First Edition, M. Kline (ed.), Dover Publications Inc., 1951.
5. Tamir, T., "Leaky wave antennas," *Antenna Theory Pt 2*, R. E. Collin and F. J. Zucker (eds.), McGraw-Hill Book Coy., First Edition, 1969.
6. Langston, W. L., J. T. Williams, D. R. Jackson, and F. Mesa, "Spurious radiation from a practical source on a covered microstrip line," *IEEE Trans. Mic. Th. Tech.*, Vol. MTT-49, 2216–2226, 2001.
7. Shigesawa, H., M. Tsuji, and A. A. Oliner, "The nature of the spectral-gap between bound and leaky solution when dielectric loss is present in printed-circuit lines," *Radio Sci.*, Vol. 28, 1235–1243, 1993.
8. Rotman, W., "The channel guide antenna," *Proc. Natl. Electron. Conf.*, Vol. 5, 190–202, 1949.
9. Marcuvitz, N. (ed.), *Waveguide Handbook*, First Edition, Dover Publications, 1965.
10. Cullen, A. L., "On the channel section waveguide radiator," *Phil. Mag.*, Vol. 40, 417–428, 1949.
11. Collin, R. E., *Foundations for Microwave Engineering*, First Edition, McGraw-Hill Book Coy., 1966.
12. Marin, L., "Transient electromagnetic properties of two, infinite parallel wires," *J. App. Phys.*, Vol. 5, 335–345, 1975.
13. Melino, R. and H. E. Green, "An error in the 'Waveguide

- Handbook',” *IEEE Trans. Mic. Th. Tech.*, Vol. MTT-48, 1972, 2000.
14. Love, A. E. H., “Some electrostatic distributions in two dimensions,” *Proc. London Math. Soc.*, Vol. 22, 337, 1923.
 15. Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics*, First Edition, McGraw-Hill Book Coy., 1953.
 16. Moon, P. and D. Spencer, *Field Theory Handbook*, First Edition, Springer-Verlag, 1961.
 17. Green, H. E., “A simplified derivation of the capacitance of a two-wire transmission line,” *IEEE Trans. Mic. Th. Tech.*, Vol. MTT-47, 365–366, 1999.
 18. Harrington, R. F., *Time Harmonic Electromagnetic Waves*, First Edition, McGraw-Hill Book Coy., 1961.
 19. Altman, J. L., *Microwave Circuits*, First Edition, Van Nostrand Coy., 1964.