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A NOTE ON THE BACKWARD SCATTERING THEOREM

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Abstract—Recent efforts by C.-T. Tai to emphasize backward scattering within the makeup of the optical theorem are examined here from first principles. The present work exploits spectral field representations and a common asymptotic procedure so as to build up both the scattered fields and their contribution to the extinction integral. The result of all this is to reaffirm the strictly forward scattering nature of the optical theorem as commonly understood, while at the same time reconciling it with a backward scattering interpretation. The backward scattering, it so turns out, is backward in *reciprocal space*, wherein it affects the Fourier transform of the currents induced throughout the scattering object. The standard forward scattering attribute of the optical theorem, forward in the context of actual space, remains unimpaired. In truth, however, the backward spectral attribute is a mere technical formality, made available for only one of the two signature options which one can exercise when making specific the details of transformation. The alternate signature option leads to a forward appearance in spectral space also, with the actual value of the current transform appearing in the optical theorem quite intact. We develop these results in detail and then, for completeness, summarize the special form which they adopt for scattering obstacles with axial symmetry.

1. OVERVIEW

In a recent publication [1][1], C.-T. Tai has sought to steer the physical interpretation of the optical theorem away from its traditional, forward scattering emphasis, toward a complete, backward scattering reversal (*cf.* (I-48), (I-53), and (I-A11)). In this note we propose to reassert the unshakably forward scattering nature of the optical theorem while reconciling it, in a fairly general context, with a backward scattering attribute. Such a claim may, at first blush, appear logically contradictory, but the looming paradox dissolves when it is recognized that the backward direction is here assigned in *reciprocal space* to the Fourier transform of the induced current which is the source of scattered field response.[2]

The backward interpretation in Fourier space will however be seen as a thin formality attached to just one of the two signature options which one can exercise in the course of transform definition. The alternate signature choice leads to a forward scattering picture in spectral space also, without in any way affecting the value of the current transform which figures as an ingredient in the optical theorem grouping.

To reach our goal, we begin by representing scattered fields $\mathbf{E}_s(\mathbf{r})$ and $\mathbf{H}_s(\mathbf{r})$ as Fourier transforms over their current source distribution $\mathbf{J}_s(\mathbf{r})$. With such spectral representation in hand, and with a view in particular to its far-field behavior, attention is naturally directed to a certain elementary lemma (*q.v.,* Eq. (11) below) providing asymptotic

[1] Since Reference 1 will serve as our main point of contact, we shall simply allude to its results with a prefix I-. For instance, (I-A11), p. I-608, points to Equation (A11) on p. 608 of [1].

[2] Of course, in any given problem, such as scattering by a wedge [Sommerfeld *et al.*] or else a sphere [Mie *et al.*], we are most anxious to ascertain the detailed features of this current distribution. In particular, for a scattering object which is perfectly conducting, the current distribution adheres to the surface alone. Surface current determination for a perfectly conducting sphere, and its use in computing excitation power, receive a most thorough treatment in [1], (I-16)–(I-43). Our present arguments will however remain quite general, and they will not in any way depend upon fixing the current, be it through imposition of boundary conditions or else through some sort of a volume-distributed demand for total field self-consistency.

estimates of angular integrals. We recapitulate its simple derivation on the basis of repeated integration by parts, with due reference to antecedent work, and then reinforce this with an alternative demonstration more in keeping with a stationary phase argument, albeit one involving a slight departure from the normal pattern of stationary phase mechanics. We proceed to exploit this lemma both for the far-field spectral estimates themselves and then, in ordinary space, for the excitation power so as to yield a forward scattering optical theorem. And, when all computational details have been duly sorted out, there will remain in this forward scattering structure a current transform which does offer the option of being interpreted as referring to a *backward* direction. It is in this manner that opposed propagation directions are permitted to reside in harmony within a single master expression.

In our opinion, the theoretical/physical import of this forward/backward duality is strictly nil, an empty formality, inasmuch as it is overridden at will through a mere signature choice, and since, as will be seen in Eqs. (26) and (27) below, it serves as little more than a stepping stone between two physically and mathematically unique expressions unburdened by any signature ambiguity.

For completeness, we indicate also how the underlying computations must be modified so as to bring out an optical theorem analogue in the context of scatterers endowed with axial symmetry.[3]

Spectral field representation is well known [2], but, for the sake of logical and stylistic continuity, we summarize it briefly below. So far as is possible, we utilize a standard SI symbolism for time harmonic fields evolving as $\exp(i\omega t)$, with the latter factor suppressed. For definiteness, angular frequency ω is taken as positive. No systematic effort is made to provide verbal definitions for symbols whose use and meaning are part of the electromagnetic *lingua franca.*

2. SPECTRAL FIELD REPRESENTATION

We write

$$\tilde{\mathbf{E}}_{\pm}(\mathbf{K}) = \int \mathrm{e}^{\pm i\mathbf{K}\cdot\mathbf{r}} \mathbf{E}_s(\mathbf{r}) d^3\mathbf{r} \tag{1}$$

for the Fourier transform of $\mathbf{E}_s(\mathbf{r})$, and similarly for $\tilde{\mathbf{J}}_{\pm}(\mathbf{K})$. And, as field $\mathbf{E}_s(\mathbf{r})$ is related to its current source $\mathbf{J}_s(\mathbf{r})$ via

$$\nabla \times (\nabla \times \mathbf{E}_s) - k^2 \mathbf{E}_s = -i\omega\mu_o \mathbf{J}_s, \tag{2}$$

[3] These are the so-called two-dimensional obstacles.

it follows that

$$\left[(K^2 - k^2)\mathbf{I} - \mathbf{KK}\right] \cdot \tilde{\mathbf{E}}_{\pm} = -i\omega\mu_o\tilde{\mathbf{J}}_{\pm}, \tag{3}$$

with $\mathbf{KK}$ being the dyadic product of vectors $\mathbf{K}$, while $\mathbf{I}$ is the unit tensor.

Since, for any symmetric tensor $\mathbf{T}$ of the form

$$\mathbf{T} = \alpha\mathbf{I} + \beta\mathbf{KK} \tag{4}$$

encountered in (3), there exists a symmetric inverse

$$\mathbf{T}^{-1} = \frac{1}{\alpha(\alpha + \beta K^2)}\left[(\alpha + \beta K^2)\mathbf{I} - \beta\mathbf{KK}\right], \tag{5}$$

we see at once that (3) can be inverted so as to read

$$\tilde{\mathbf{E}}_{\pm}(\mathbf{K}) = -i\omega\mu_o k^{-2}(K^2 - k^2)^{-1}\left[k^2\mathbf{I} - \mathbf{KK}\right] \cdot \tilde{\mathbf{J}}_{\pm}(\mathbf{K}), \tag{6}$$

whereupon Fourier inversion yields

$$\mathbf{E}_s(\mathbf{r}) = -i\omega\mu_o k^{-2}(2\pi)^{-3}\int \mathrm{e}^{\mp i\mathbf{K}\cdot\mathbf{r}}(K^2 - k^2)^{-1}\left[k^2\mathbf{I} - \mathbf{KK}\right]\cdot\tilde{\mathbf{J}}_{\pm}(\mathbf{K})d^3\mathbf{K}. \tag{7}$$

For the time being we dispense with an entirely similar expression[4] for the complementary magnetic field $\mathbf{H}_s(\mathbf{r})$ simply because, futher along, in Eq. (20), we shall obtain its asymptotic version directly from a far-field electric counterpart on the basis of Faraday's equation. As regards Eq. (7), it is important to note that, in consequence of a vestigial, ambient conductivity, wave number k must be understood as the limit of $k - i\varepsilon$ when $\varepsilon \to 0+$. This proviso displaces the simple poles away from the real axis in precisely the sense required to produce a physically acceptable asymptotic structure, as we shall soon see in Eq. (17).

Current $\mathbf{J}_s(\mathbf{r})$, and the field $\{\mathbf{E}_s(\mathbf{r}), \mathbf{H}_s(\mathbf{r})\}$ which it radiates, exist only as a response to some external field excitation, which we take here to be that of a plane wave, with an electric portion

$$\mathbf{E}_i(\mathbf{r}) = \mathbf{E}_o\mathrm{e}^{-ik\hat{\mathbf{n}}\cdot\mathbf{r}} \tag{8}$$

of which it is required only that $\hat{\mathbf{n}} \cdot \mathbf{E_o} = 0$. A magnetic companion $\mathbf{H}_i(\mathbf{r})$ has $\mathbf{H}_o = Z_o^{-1}\hat{\mathbf{n}} \times \mathbf{E}_o$. Field (8) is unconstrained as regards its polarization, so that complex values are permitted for both vectors $\mathbf{E}_o$ and $\mathbf{H}_o$.

[4] Later on, in Section 6 on the two-dimensional optical theorem, we shall find it somewhat more convenient to begin with the magnetic field $\mathbf{H}_s(\mathbf{r})$, and this for a reason no more profound than that of mere typographical ease.

In general, the scattered field is fixed against the backdrop of (8) through the imposition of boundary and/or volume-distributed field self-consistency conditions. And, while such determination is ultimately of paramount interest, all of our ensuing considerations are wholly exempt from any need to display such knowledge, save for the implicit recognition that, willy-nilly, it must emerge somewhere in a truly comprehensive analysis. A simple instance therefore is summarized in Eqs. (44), (49)–(50).

3. AN ASYMPTOTIC LEMMA

In both (7) and the quadrature (21) which underlies the excitation power we encounter an integral over solid angle having the generic form

$$Q_{\pm}(\mathbf{r}) = \int_{4\pi} \mathrm{e}^{\pm i\mathbf{K}\cdot\mathbf{r}} f(\mathbf{K}) d^2\Omega_{\hat{\mathbf{K}}} \tag{9}$$

with $f(\mathbf{K})$ being some benign function and radial distance $r \to \infty$. In the interest of logical cohesion we summarize now two distinct methods for estimating $Q_{\pm}(\mathbf{r})$. Each of them begins by rotating coordinates in $\mathbf{K}$ space so as to have the polar axis aligned along $\mathbf{r}$.

In Method I, we first set $\hat{\mathbf{K}} \cdot \hat{\mathbf{r}} = \cos(\vartheta) = \mu$, so that $d^2\Omega_{\hat{\mathbf{K}}} = d\varphi d\mu$, while at the same time

$$\mathrm{e}^{\pm i\mathbf{K}\cdot\mathbf{r}} = \mp \frac{i}{Kr} \frac{d}{d\mu} \mathrm{e}^{\pm iKr\mu}. \tag{10}$$

Repeated integration by parts, with all successive stages discarded, yields then

$$Q_{\pm}(\mathbf{r}) \approx \mp \frac{2\pi i}{Kr} \left[\mathrm{e}^{\pm iKr} f(K\hat{\mathbf{r}}) - \mathrm{e}^{\mp iKr} f(-K\hat{\mathbf{r}})\right] \tag{11}$$

as the dominant asymptotic estimate for $Q_{\pm}(\mathbf{r})$ when $r \to \infty$. We downplay henceforth the asymptotic caveat implicit in the approximate sign $\approx$, simply writing $=$ and permitting the context to arbitrate its shaded meaning wherever necessary.

The simple stratagem of generating an asymptotic estimate through repeated integration by parts comes adorned with a rich pedigree. One need only recall its use in connection with Fresnel's integrals ([3], pp. 430–431, Eqs. (14)–(16)), the exponential integral ([4], pp. 242–243, Eqs. (6-1)–(6-3)), the cosine integral ([4], p. 243, Problem 2), and the complementary error function ([5], p. 230, Problem 17), among others. A similarly overt asymptotic use of integration by parts in the service of acoustic and quantum mechanical scattering can be found respectively in [6], pp. 386–387, Eqs. (III.5)–(III.6), and [7], pp. 191–192 and p. 196. Lemma (11) *per se* is quoted as Equation (107) on p. 658 of [3].

In method II, which is more closely attuned to the classical spirit of stationary phase, an alternative derivation of (11) begins with the premise that, as $r \to \infty$, the dominant contributions to (9) should be sought only from those ϑ neighborhoods wherein variations in the phase of

$$\mathrm{e}^{\pm i\mathbf{K}\cdot\mathbf{r}} = \mathrm{e}^{\pm iKr\cos(\vartheta)} \tag{12}$$

are brought to rest. These are clearly the polar neighborhoods around $\vartheta = 0$ and $\vartheta = \pi$, and so, on setting

$$\cos(\vartheta) \approx 1 - \vartheta^2/2$$

and

$$\sin(\vartheta) \approx \vartheta$$

in the vicinity of $\vartheta = 0$, and similary close to $\vartheta = \pi$, we bring (9) into the form

$$\begin{aligned} Q_\pm(\mathbf{r}) \;=\; 2\pi \Big[&\mathrm{e}^{\pm iKr} f(K\hat{\mathbf{r}}) \int_0^\infty \mathrm{e}^{\mp iKr\vartheta^2/2}\vartheta d\vartheta \\ &+ \mathrm{e}^{\mp iKr} f(-K\hat{\mathbf{r}}) \int_0^\infty \mathrm{e}^{\pm iKr\vartheta^2/2}\vartheta d\vartheta\Big] \end{aligned} \tag{13}$$

whose asymptotic character is placed into evidence on noting the improper integral evaluations

$$\int_0^\infty \mathrm{e}^{\pm iKr\vartheta^2/2}\vartheta d\vartheta = \pm\frac{i}{Kr} \tag{14}$$

which permit (13) to blend into (11). Evaluations (14) are gotten by much the same route of contour swing through $\pm\pi/4$ as is utilized when fixing

$$\int_{-\infty}^\infty \mathrm{e}^{\pm iKr\vartheta^2/2} d\vartheta = \mathrm{e}^{\pm i\pi/4}\sqrt{\frac{2\pi}{Kr}} \tag{15}$$

during a more traditional application of stationary phase. Indeed, on stripping away the multiplier 2π and replacing quadratures (14) with those from (15), we transform (13) into the asymptotic form appropriate to two-dimensional propagation.[5] We make repeated use of structure (13) when thus modified in Section 6 on the two-dimensional optical theorem, signaling its imminent participation with the signpost $\mathcal{ML}$ as a shorthand for *modified lemma.*

[5] Evidently the solid angle measure in (9) gives way then to an azimuthal quadrature over its full, 2π range.

4. FAR FIELDS

When (11) is brought to bear upon (7) we readily find

$$\begin{aligned}\mathbf{E}_s(\mathbf{r}) \;=\; &\pm\frac{\omega\mu_o}{(2\pi)^2k^2r}\int_0^\infty \frac{KdK}{(K^2-k^2)}\left[k^2\mathbf{I}-K^2\hat{\mathbf{r}}\hat{\mathbf{r}}\right]\\ &\cdot\left\{\mathrm{e}^{\mp iKr}\tilde{\mathbf{J}}_\pm(K\hat{\mathbf{r}})-\mathrm{e}^{\pm iKr}\tilde{\mathbf{J}}_\pm(-K\hat{\mathbf{r}})\right\}. \qquad (16)\end{aligned}$$

Contour displacement respectively up or down depending upon the signature $\pm$ of $\exp(\pm iKr)$ sweeps these exponentials into regions of so steep a decay ([8], p. 476, Footnote) that both such integrals can be ignored, save for the residue harvested at the simple pole $K = k - i\varepsilon$, which is encircled clockwise. We thus get

$$\mathbf{E}_s(\mathbf{r}) = -\frac{i\omega\mu_o}{4\pi}\frac{\mathrm{e}^{-ikr}}{r}\left[\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}}\right]\cdot\tilde{\mathbf{J}}_\pm(\pm k\hat{\mathbf{r}}), \qquad (17)$$

or else

$$\mathbf{E}_s(\mathbf{r}) = \frac{\mathrm{e}^{-ikr}}{r}\bar{\mathbf{E}}_s(\hat{\mathbf{r}}), \qquad (18)$$

which latter serves to define a complex, far-field vector amplitude $\bar{\mathbf{E}}_s(\hat{\mathbf{r}})$ (having the dimension of a volt), depending only upon observation direction $\hat{\mathbf{r}}$ and being transverse to it, $\hat{\mathbf{r}}\cdot\bar{\mathbf{E}}_s(\hat{\mathbf{r}}) = 0$. With (17) we are placed on notice that a backward direction will most likely be able to figure within the impending optical theorem, albeit this direction will be retrograde not in ordinary observation space, but rather in its spectral, Fourier conjugate. An equally legitimate contender in spectral space will of course be a forward direction, selected through a simple signature switch.

Similar attributes pertain to magnetic partner

$$\mathbf{H}_s(\mathbf{r}) = \frac{\mathrm{e}^{-ikr}}{r}\bar{\mathbf{H}}_s(\hat{\mathbf{r}}), \qquad (19)$$

whose vector amplitude $\bar{\mathbf{H}}_s(\hat{\mathbf{r}})$ (measured in ampères) follows from Faraday's connection as

$$\bar{\mathbf{H}}_s(\hat{\mathbf{r}}) = Z_o^{-1}\hat{\mathbf{r}}\times\bar{\mathbf{E}}_s(\hat{\mathbf{r}}). \qquad (20)$$

5. A FORWARD/BACKWARD VIEW OF THE OPTICAL THEOREM

With reference to Eq. (I-8) one writes the excitation power[6] as

$$P_{ex} = -\frac{r}{2}\Re \int_{4\pi} \hat{\mathbf{r}} \cdot \left[\mathrm{e}^{ik(r-\hat{\mathbf{n}}\cdot\mathbf{r})} \mathbf{E}_o \times \bar{\mathbf{H}}_s^*(\hat{\mathbf{r}}) - \mathrm{e}^{-ik(r-\hat{\mathbf{n}}\cdot\mathbf{r})} \mathbf{H}_o^* \times \bar{\mathbf{E}}_s(\hat{\mathbf{r}}) \right] d^2\Omega_{\hat{\mathbf{r}}}, \tag{21}$$

which, following a suitable transliteration, is automatically placed beneath the asymptotic auspices of (11), with

$$\begin{aligned} P_{ex} \;=\; & \frac{\pi}{k} \hat{\mathbf{n}} \cdot \Im \left[\mathbf{E}_o \times \left\{ \bar{\mathbf{H}}_s^*(\hat{\mathbf{n}}) + \mathrm{e}^{2ikr} \bar{\mathbf{H}}_s^*(-\hat{\mathbf{n}}) \right\} \right. \\ & \left. + \mathbf{H}_o^* \times \left\{ \bar{\mathbf{E}}_s(\hat{\mathbf{n}}) + \mathrm{e}^{-2ikr} \bar{\mathbf{E}}_s(-\hat{\mathbf{n}}) \right\} \right] \end{aligned} \tag{22}$$

as their outcome,[7] a linear mix of contributions from forward/backward scattering directions $\pm\hat{\mathbf{n}}$. The backward terms, in particular, are burdened with multipliers $\exp(\pm 2ikr)$, whose continued dependence upon radius r militates against our goal of exhibiting P_{ex} as a unique, saturated, global attribute of the scattering process at hand. Their net contribution must clearly vanish.

Such null contribution does indeed obtain, as one sees by noting field orthogonality properties so as to yield

$$\mathbf{E}_o \times \bar{\mathbf{H}}_s^*(-\hat{\mathbf{n}}) = -Z_o^{-1} \hat{\mathbf{n}} \left\{ \mathbf{E}_o \cdot \bar{\mathbf{E}}_s^*(-\hat{\mathbf{n}}) \right\}, \tag{23}$$

whereas

$$\mathbf{H}_o^* \times \bar{\mathbf{E}}_s(-\hat{\mathbf{n}}) = -Z_o^{-1} \hat{\mathbf{n}} \left\{ \mathbf{E}_o^* \cdot \bar{\mathbf{E}}_s(-\hat{\mathbf{n}}) \right\}, \tag{24}$$

so that, taken together, the backward contributions assemble within (22) a real vector which is then simply annihilated by the imaginary filter $\Im$.

Similar steps indicate next that

$$\left. \begin{aligned} \mathbf{E}_o \times \bar{\mathbf{H}}_s^*(\hat{\mathbf{n}}) \;&=\; Z_o^{-1} \hat{\mathbf{n}} \left\{ \mathbf{E}_o \cdot \bar{\mathbf{E}}_s^*(\hat{\mathbf{n}}) \right\} \\ \mathbf{H}_o^* \times \bar{\mathbf{E}}_s(\hat{\mathbf{n}}) \;&=\; -Z_o^{-1} \hat{\mathbf{n}} \left\{ \mathbf{E}_o^* \cdot \bar{\mathbf{E}}_s(\hat{\mathbf{n}}) \right\} \end{aligned} \right\}, \tag{25}$$

[6] Known also as the extinction power.

[7] It is important to keep in mind here that wave number k is now regarded as strictly nondissipative, the real limit attained during the upward approach suggested in connection with (7). While a dissipative k is perhaps tenable in connection with radiated fields (18) and (19), it is certainly not so for the assumed plane wave form (8). This latter caveat, if nothing else, forces us to consider henceforth that k is real. A formal, uniformly applied attempt to entertain a complex, dissipative k, already disallowed for (8) on physical grounds, burdens (21) *et seq.* with perplexing conclusions which we, and, presumably, any other reasonable person, would wish to avoid.

whereupon (22) condenses into a well known structure

$$P_{ex} = \frac{2\pi}{kZ_o} \Im \left[\mathbf{E}_o \cdot \bar{\mathbf{E}}_s^*(\hat{\mathbf{n}}) \right] \tag{26}$$

involving the forward scattering direction $\hat{\mathbf{n}}$ alone.[8] With reference to (17) and (18), and the very definition (1) of a Fourier transform, this further evolves into

$$\begin{aligned} P_{ex} &= \frac{1}{2} \Re \left[\mathbf{E}_o \cdot \tilde{\mathbf{J}}_{\pm}^*(\pm k\hat{\mathbf{n}}) \right] \\ &= \frac{1}{2} \Re \int \mathbf{E}_i(\mathbf{r}) \cdot \mathbf{J}_s^*(\mathbf{r}) d^3\mathbf{r}, \end{aligned} \tag{27}$$

whose second entry gauges the power which incident field (8) expends upon current response $\mathbf{J}_s(\mathbf{r})$. As such it conveys indeed a most gratifying physical assessment, free from any technical ambiguity. Its predecessor, by contrast, exerts but a feeble claim to the honor of being labeled as any sort of backward scattering theorem, simply because it arrives in both forward/backward flavors selected at will on the basis of a superficial technicality. But, if one absolutely insists, a backward direction can be spotted there.

As the scattering obstacle is pushed to the limit of perfect conductivity, its current response $\mathbf{J}_s(\mathbf{r})$ becomes constrained to a thin veneer everywhere tangent to its surface, and then the second line of (27) reverts to (I-14),[9] which is a pleasing result revealed in [1] by a chain of alternate considerations.

[8] Result (26) can already by found, albeit somewhat smothered by intervening algebraic details, as Eq. (26) on p. 569 of [9]. Alternate viewpoints on the electromagnetic optical theorem, of both a physical and mathematical sort, are supplied on pp. 18–20 and 44–47 of [10], and in pp. 419–422 of [11]. On pp. 501–502 of [12] appears a most refreshing derivation based on the vector diffraction integral relating fields to their tangential boundary values. Not to be overlooked finally is the derivation given on pp. 529-530 of [13], wherein the tensor Green's function comes to the fore. A tensor Green's function is of course already resident within our own field expression (7), its latent presence being revealed simply by making explicit the spatial quadrature, akin to that in (1), which underlies the current transform. Close cousins of the electromagnetic optical theorem are also found in quantum mechanics and elastoacoustics, wherein it is normally couched in a normalized form as a statement regarding a scattering cross section. One can begin tracking this consanguinity in [8, 10], and [14–16], among a myriad of sometimes competing, sometimes complementary leads.

[9] Equation (I-14) avails itself of the standard notation $\mathbf{K}_s$ for a surface current, having ampère per meter as its dimension. This usage should of course in no way be confused with that in our (1) *et seq.,* wherein vector $\mathbf{K}$ is simply a spectral coordinate, with inverse meter as its dimension.

6. THE ELECTROMAGNETIC OPTICAL THEOREM IN TWO-DIMENSIONAL SCATTERING

We specialize now to an obstacle which is both geometrically and constitutionally invariant along axis z, and we begin by changing our notation regarding external stimulus (8), demoting unit vector $\hat{\mathbf{n}}$ to the status of a normalized projection, perpendicular to $\hat{\mathbf{e}}_z$, of arrival direction $\hat{\mathbf{p}}$, with $\hat{\mathbf{p}} \cdot \hat{\mathbf{e}}_z = \sin(\phi)$ being its parallel component. Thus

$$\hat{\mathbf{p}} = \cos(\phi)\hat{\mathbf{n}} + \sin(\phi)\hat{\mathbf{e}}_z, \tag{28}$$

and then (8) becomes

$$\mathbf{E}_i(\mathbf{r}) = \mathbf{E}_o \mathrm{e}^{-ik\hat{\mathbf{p}}\cdot\mathbf{r}} \tag{29}$$

with $\hat{\mathbf{p}} \cdot \mathbf{E_o} = 0$, and a magnetic companion having amplitude $\mathbf{H}_o = Z_o^{-1}\hat{\mathbf{p}} \times \mathbf{E}_o$.

On writing $\boldsymbol{\rho}$ for the perpendicular projection of $\mathbf{r}$, so that $\boldsymbol{\rho} \cdot \hat{\mathbf{e}}_z = 0$, and taking note of the fact that the axial wave progression $\exp(-ik\sin(\phi)z)$ is imprinted upon fields and currents across the board, we find the current response decomposed as a product $\mathbf{J}_s(\mathbf{r}) = \mathbf{M}_s(\boldsymbol{\rho})\exp(-ik\sin(\phi)z)$ wherein, by contrast to $\boldsymbol{\rho}$, current density vector $\mathbf{M}_s(\boldsymbol{\rho})$ *per se* clings in general to a nonzero value for each one of its three components. And then, as a direct consequence of all this, current transform $\tilde{\mathbf{J}}_\pm(\mathbf{K})$ itself splits

$$\tilde{\mathbf{J}}_\pm(\mathbf{K}) = 2\pi\delta(K_z \mp k\sin(\phi))\tilde{\mathbf{M}}_\pm(\boldsymbol{\kappa}\ ) \tag{30}$$

into a product of Dirac's delta δ and a two-dimensional Fourier transform[10]

$$\tilde{\mathbf{M}}_\pm(\boldsymbol{\kappa}) = \int \mathrm{e}^{\pm i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\ \mathbf{M}_s(\boldsymbol{\rho}\ )d^2\boldsymbol{\rho} \tag{31}$$

depending upon the perpendicular projection $\boldsymbol{\kappa}$ of $\mathbf{K}$.

On sheer typographical grounds, it is easier to launch the ensuing two-dimensional discussion in the context of the scattered magnetic field[11]

$$\mathbf{H}_s(\mathbf{r}) = \mp i(2\pi)^{-3}\int \mathrm{e}^{\mp i\mathbf{K}\cdot\mathbf{r}}(K^2 - k^2)^{-1}\mathbf{K}\times\tilde{\mathbf{J}}_\pm(\mathbf{K})d^3\mathbf{K} \tag{32}$$

[10] $\tilde{\mathbf{M}}_\pm(\boldsymbol{\kappa})$ carries the dimension of a mere ampère, unlike that of $\tilde{\mathbf{J}}_\pm(\mathbf{K})$, which is ampère meter. It is well to keep these bookkeeping issues in mind when checking the physical consistency of the first member in (27) and its two-dimensional counterpart (43) below.

[11] One could equally well argue that we should perhaps have made the same selection beforehand, replacing $\mathbf{E}_s(\mathbf{r})$ from (7) with $\mathbf{H}_s(\mathbf{r})$ as presently encountered. While a valid point, it would seem to skirt a sort of unspoken, traditional deference to the electric field as being in fact a *primus inter pares,* which should at the very least be mentioned in any electromagnetic discourse.

which follows from an interplay of (7) with Faraday's equation. Acknowledgement of (30) then leads to

$$\begin{aligned}\mathbf{H}_s(\mathbf{r}) \;=\; &\mp i(2\pi)^{-2}\mathrm{e}^{-ik\sin(\phi)z}\int \mathrm{e}^{\mp i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\left(\kappa^2-k^2\cos^2(\phi)\right)^{-1}\\ &\{\boldsymbol{\kappa}\pm k\sin(\phi)\hat{\mathbf{e}}_z\}\times\tilde{\mathbf{M}}_\pm(\boldsymbol{\kappa})d^2\boldsymbol{\kappa}\end{aligned} \tag{33}$$

as a general structure still in need of asymptotic estimates.

Such estimates proceed once again along a two-pronged path which, as regards the angular integral over $\varphi_{\hat{\boldsymbol{\kappa}}}$, draws upon $\mathcal{ML}$, with due regard being paid to any required adaptation of symbols. This gives

$$\begin{aligned}\mathbf{H}_s(\mathbf{r}) \;=\; &\mp i\sqrt{\frac{1}{8\pi^3\rho}}\mathrm{e}^{-ik\sin(\phi)z}\int_0^\infty\sqrt{\kappa}\left(\kappa^2-k^2\cos^2(\phi)\right)^{-1}\\ &\Big[\mathrm{e}^{\mp i(\kappa\rho-\pi/4)}\{\kappa\hat{\boldsymbol{\rho}}\pm k\sin(\phi)\hat{\mathbf{e}}_z\}\times\tilde{\mathbf{M}}_\pm(\kappa\hat{\boldsymbol{\rho}})\\ &-\mathrm{e}^{\pm i(\kappa\rho-\pi/4)}\{\kappa\hat{\boldsymbol{\rho}}\mp k\sin(\phi)\hat{\mathbf{e}}_z\}\times\tilde{\mathbf{M}}_\pm(-\kappa\hat{\boldsymbol{\rho}})\Big]\,d\kappa\end{aligned} \tag{34}$$

and, as was the case beforehand in connection with (16), invites contour deformation up or down depending upon the sign $\pm$ in $\exp(\pm i\kappa\rho)$. Only the downward contour sweep is rewarded with a residue at the pole $\kappa = k\cos(\phi)$, and thus we find

$$\begin{aligned}\mathbf{H}_s(\mathbf{r}) \;=\; &-\sqrt{\frac{ik}{8\pi\rho\cos(\phi)}}\mathrm{e}^{-ik(\cos(\phi)\rho+\sin(\phi)z)}\\ &\{\cos(\phi)\hat{\boldsymbol{\rho}}+\sin(\phi)\hat{\mathbf{e}}_z\}\times\tilde{\mathbf{M}}_\pm(\pm k\cos(\phi)\hat{\boldsymbol{\rho}})\end{aligned} \tag{35}$$

as an analogue to (17). The notation here is usefully compressed on emulating (28) so as to define

$$\hat{\mathbf{g}} = \cos(\phi)\hat{\boldsymbol{\rho}} + \sin(\phi)\hat{\mathbf{e}}_z \tag{36}$$

through a replacement of $\hat{\mathbf{n}}$ by $\hat{\boldsymbol{\rho}}$, whereupon

$$\mathbf{H}_s(\mathbf{r}) = -\sqrt{\frac{ik}{8\pi\rho\cos(\phi)}}\mathrm{e}^{-ik\hat{\mathbf{g}}\cdot\mathbf{r}}\;\hat{\mathbf{g}}\times\tilde{\mathbf{M}}_\pm(\pm k\cos(\phi)\hat{\boldsymbol{\rho}}) \tag{37}$$

as befits a cylindrical, $\sim \rho^{-1/2}$ field spreading outward along a Fock cone with generator $\hat{\mathbf{g}}$. Evidently $\hat{\mathbf{g}}\cdot\mathbf{H}_s(\mathbf{r}) = 0$.

When written next as

$$\mathbf{H}_s(\mathbf{r}) = \sqrt{\frac{1}{\rho}}\mathrm{e}^{-ik\hat{\mathbf{g}}\cdot\mathbf{r}}\bar{\mathbf{H}}_s(\hat{\boldsymbol{\rho}}) \tag{38}$$

Eq. (37) serves to define a magnetic amplitude $\bar{\mathbf{H}}_s(\hat{\boldsymbol{\rho}})$ transverse to $\hat{\mathbf{g}}$. An identical form

$$\mathbf{E}_s(\mathbf{r}) = \sqrt{\frac{1}{\rho}}\mathrm{e}^{-ik\hat{\mathbf{g}}\cdot\mathbf{r}}\bar{\mathbf{E}}_s(\hat{\boldsymbol{\rho}}) \tag{39}$$

governs the asymptotic electric companion, with

$$\bar{\mathbf{E}}_s(\hat{\boldsymbol{\rho}}) = -Z_o\hat{\mathbf{g}} \times \bar{\mathbf{H}}_s(\hat{\boldsymbol{\rho}}). \tag{40}$$

All ingredients are in place to implement a computation of the excitation power, reckoned now as power per unit axial length. Imitation of the pattern found in (21) gives

$$\begin{aligned} P_{ex} \;=\; & -\frac{\sqrt{\rho}}{2}\Re\int_{2\pi}\hat{\boldsymbol{\rho}}\cdot\Big[\mathrm{e}^{ik\cos(\phi)(\rho-\hat{\mathbf{n}}\cdot\boldsymbol{\rho})}\mathbf{E}_o\times\bar{\mathbf{H}}_s^*(\hat{\boldsymbol{\rho}}) \\ & -\mathrm{e}^{-ik\cos(\phi)(\rho-\hat{\mathbf{n}}\cdot\boldsymbol{\rho})}\mathbf{H}_o^*\times\bar{\mathbf{E}}_s(\hat{\boldsymbol{\rho}})\Big]\,d\varphi_{\hat{\boldsymbol{\rho}}} \end{aligned} \tag{41}$$

and then, on appeal once more to $\mathcal{ML}$[12] one duly arrives at the following analogues of (26)

$$P_{ex} = \frac{1}{Z_o}\sqrt{\frac{2\pi\cos(\phi)}{k}}\,\Im\left[\mathrm{e}^{-i\pi/4}\mathbf{E}_o\cdot\bar{\mathbf{E}}_s^*(\hat{\mathbf{n}})\right] \tag{42}$$

and (27)

$$\begin{aligned} P_{ex} &= \frac{1}{2}\Re\left[\mathbf{E}_o\cdot\tilde{\mathbf{M}}_\pm^*(\pm k\cos(\phi)\hat{\mathbf{n}})\right] \\ &= \frac{1}{2}\Re\int\mathbf{E}_i(\mathbf{r})\cdot\mathbf{J}_s^*(\mathbf{r})d^2\boldsymbol{\rho}. \end{aligned} \tag{43}$$

7. A SIMPLE EXAMPLE: PERPENDICULAR INCIDENCE UPON A PERFECTLY CONDUCTING, CIRCULAR CYLINDER

Equation (43) is easily put to the test in the simple context of perpendicular wave impact (*i.e.*, $\phi = 0$) upon a perfectly conducting

[12] Such appeal transforms (41) into a forward/backward structure akin to (22), but with a projection $\Re$ pointing initially toward its real component. Since the forward cone generator $\hat{\mathbf{g}}$ coincides with incidence direction $\hat{\mathbf{p}}$, the forward ingredients are processed much along the lines of (25), but this time with a common sign, and lead ultimately to (42). When tackling the backward ingredients, however, one is obliged to recognize first that the aft generator can be gotten by reversing the sense of $\hat{\mathbf{p}}$, provided only that one compensates with the addition of $2\sin(\phi)\hat{\mathbf{e}}_z$. With this elementary observation, and with the inherent fore/aft field orthogonality properties *vis-à-vis* the respective cone generators kept in mind, all quantities in play are reduced to scalar products with $\hat{\mathbf{e}}_z$ alone. At this point one encounters a purely imaginary backward quantity, and the agency of the real filter is to simply whisk it off the asymptotic stage, just as before in the buildup of (26).

cylinder of radius a, with the electric polarization aligned along the cylinder axis. This relatively trivial scattering problem is resolved in terms of cylindrical modes built up as products of Bessel and Hankel functions $J_m(k\rho)$ and $H_m^{(2)}(k\rho)$, with azimuthal exponentials $\exp(im\varphi_{\hat{\boldsymbol{\rho}}})$. The electric field is here purely axial, while its magnetic partner is purely transverse, whence follows also a strictly axial surface current response. Expansion coefficients for the scattered field are gotten in the usual way by imposing the condition of null total axial electric field along the cylinder surface, it being understood that the incident plane wave has at the same time itself been subjected to mode decomposition. A determination of the total electric field makes definite its magnetic companion and, with it, the surface current response. This problem is briefly summarized, on behalf of another objective, in Eqs. (14)–(16) as listed in [17].

From [17] we thus find

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{e}}_z \frac{2}{\pi a \omega \mu_o} \delta(\rho - a) \sum_{m=-\infty}^{\infty} \left\{ i^m H_m^{(2)}(ka) \right\}^{-1} \exp(im\varphi_{\hat{\boldsymbol{\rho}}}), \tag{44}$$

with azimuth $\varphi_{\hat{\boldsymbol{\rho}}}$ being measured from the direction of incidence and the excitation amplitude having been set at one volt per meter. A standard integral representation for $J_m(\kappa a)$ gives

$$\tilde{\mathbf{M}}_\pm(\boldsymbol{\kappa}) = \hat{\mathbf{e}}_z \frac{4}{\omega \mu_o} \sum_{m=-\infty}^{\infty} (\pm)^m \frac{J_m(\kappa a)}{H_m^{(2)}(ka)} \exp(im\varphi_{\hat{\boldsymbol{\kappa}}}), \tag{45}$$

whence it follows that

$$\tilde{\mathbf{M}}_\pm(\pm k\hat{\mathbf{n}}) = \hat{\mathbf{e}}_z \frac{4}{\omega \mu_o} \sum_{m=-\infty}^{\infty} \frac{J_m(ka)}{H_m^{(2)}(ka)}, \tag{46}$$

which causes the first line of (43) to become

$$P_{ex} = \frac{2}{\omega \mu_o} \sum_{m=-\infty}^{\infty} \left| \frac{J_m(ka)}{H_m^{(2)}(ka)} \right|^2. \tag{47}$$

Since the obstacle here is perfectly conducting, we require that the scattered power

$$P_{sc} = \frac{\rho}{2} \Re \int_{2\pi} \hat{\boldsymbol{\rho}} \cdot [\mathbf{E}_s(\mathbf{r}) \times \mathbf{H}_s^*(\mathbf{r})] \, d\varphi_{\hat{\boldsymbol{\rho}}} \tag{48}$$

exactly match P_{ex} as just now exhibited. This is verified on noting from [17] that

$$\mathbf{E}_s(\mathbf{r}) = -\hat{\mathbf{e}}_z \sum_{m=-\infty}^{\infty} \frac{J_m(ka)}{i^m H_m^{(2)}(ka)} H_m^{(2)}(k\rho) \exp(im\varphi_{\hat{\boldsymbol{\rho}}}) \tag{49}$$

whenever $\rho \geq a$, with

$$\mathbf{E}_s(\mathbf{r}) = -\hat{\mathbf{e}}_z \sqrt{\frac{2i}{\pi k \rho}} \mathrm{e}^{-ik\rho} \sum_{m=-\infty}^{\infty} \frac{J_m(ka)}{H_m^{(2)}(ka)} \exp(im\varphi_{\hat{\boldsymbol{\rho}}}) \tag{50}$$

being its asymptotic form. Quadrature (48) is then immediately transformed into (47) when note is taken of the associated asymptotic form $\mathbf{H}_s(\mathbf{r}) = Z_o^{-1} \hat{\boldsymbol{\rho}} \times \mathbf{E}_s(\mathbf{r})$ for magnetic partner $\mathbf{H}_s(\mathbf{r})$.[13]

The reader may conceivably object that this specialized example, while successful in its limited range, is perhaps too threadbare to serve as a full-throated test of (43). At its maximal scope this test should confront electromagnetically permeable, dissipative, anisotropic and two-dimensionally inhomogeneous obstacles described by a permittivity tensor $\boldsymbol{\epsilon}$ and supporting a current profile $\mathbf{J}_s = i\omega(\boldsymbol{\epsilon} - \epsilon_o \mathbf{I}) \cdot \mathbf{E}$, with $\mathbf{E}$ being the *total* self-consistent field throughout the obstacle interior. Moreover, the angle of incidence ϕ should be unrestricted, save for the natural caveat $\phi \neq \pm\pi/2$ of avoiding propagation along the translational symmetry axis *per se*.[14] But to marshall out any such program, even at some intermediate stage short of its full generality, is clearly to unleash a fount of algebraic minutiae best served by a dedicated essay unto themselves. Some indication of how such problems are tackled in terms of axial Hertz potentials can be found in [18], where additional pointers are listed to a supporting literature.

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[13] This example clearly emulates, in a simpler setting, the treatment which C.-T. Tai provides for a perfectly conducting sphere in his equations (I-16)–(I-43). There, however, the power outflow quadrature is performed upon the obstacle surface as such (*cf.* Equations (I-12), (I-36)), whereas we, in our Eqs. (48)–(50), resort instead to an asymptotic alternative. A surface quadrature option is of course also available here, and, on taking note of a standard Wronskian, it too leads inevitably to the keystone result embodied in Eq. (47).

[14] Clearly, in the degenerate case of axial incidence with $\phi = \pm\pi/2$, the putative stimulus (29) is, so to speak, always "inside" the scattering obstacle, and thus cannot be regarded as a freely mandated exterior datum. With such axial "input" one must perforce shift attention to the concept of some narrowly defined, self-consistent eigenwave. Indeed, in testimony to this exception, much of the preceding analysis fails in the event that $\phi = \pm\pi/2$, simply because wholly unacceptable divisions by zero are then allowed to intrude.

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