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THE VARIATIONAL CLOSED-FORM FORMULAE FOR THE CAPACITANCE OF ONE TYPE OF CONFORMAL COAXIAL LINES

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Abstract—The variational closed-forms of the capacitance formulae of a type of conformal coaxial lines are presented in this paper with the assumption that the equipotential lines are conformal to the contour of the coaxial line. The variational extreme formula of the functional of continuous functions is first obtained. Then the variational stable and analytical expression of the capacitance is deduced. Considering the actual applications, we give the variational stable formulae of the capacitances of the conformal coaxial transmission lines whose contours are homogeneous curves of the 1st, 2nd and n-th order. Examples are given including the conformal regular polygonal, elliptical and high-order elliptical coaxial lines.

1. INTRODUCTION

The fast development of modern microwave theories and technologies urges more scholars to explore coaxial transmission lines with new and special cross-sections. Some papers [1, 2] have reported the progress in this aspect, but little work has been reported to find the capacitance and characteristic impedance of coaxial lines using the variational extreme value theory. A type of conformal coaxial lines made of homogeneous curves is studied in this paper. Assuming the distribution of the equipotential lines is in agreement with that of the conformal curve cluster, a set of analytical formulae are obtained using the variational method. It is well known that the characteristic impedance of the transmission line Z_0 is related to the capacitance per unit length C by

$$Z_0 = \frac{1}{vC} \tag{1}$$

where v is the wave velocity in the transmission line. Once the expression of the capacitance C is obtained, therefore, the characteristic impedance Z_0 is known and then the main characters of this transmission line are known.

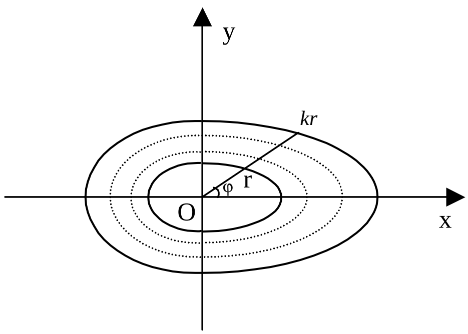

Figure 1. Conformal coaxial line.

2. BASIC FORMULAE

As shown in Fig. 1, the conformal coaxial line studied here has a conformal center and the ratio of the distances from the center to the outer and inner conductor along any radial line starting from the center is a common value k.

The variational functional of the capacitance of a coaxial line expressed in terms of electric potential function Φ is [3]

$$C = \frac{2W}{V^2} = \varepsilon \frac{\iint_s (\nabla\Phi \cdot \hat{r})^2 ds}{\left(\int \nabla\Phi \cdot d\hat{l}\right)^2} \tag{2}$$

where, W is the stored electrostatic field energy and V is the electric potential difference between the inner and outer conductors. According to the principle of minimum energy, the functional expressed by (2) has a minimum value. In other words, (2) gives the variational upper bound of capacitance.

A type of conformal homogeneous curves is expressed in polar coordinates as follows:

$$F(r,\varphi) = r^n s(\varphi) = \text{constant} > 0 \tag{3}$$

We assume that the cluster of equipotential lines u is in agreement with the above conformal lines and is defined as follows:

$$u = \sqrt[n]{F(r,\varphi)} = r\sqrt[n]{s(\varphi)} \tag{4}$$

Now considering the denominator of (2), we have

$$\nabla\Phi \cdot d\vec{l} = \frac{\partial\Phi}{\partial r}dr = \frac{\partial\Phi}{\partial u}du \tag{5}$$

Then the integrand of the numerator in (2) is expressed as

$$(\nabla\Phi)^2 = \left(\frac{\partial\Phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}\right)^2 = \sqrt[n]{s^2(\varphi)}\left(\frac{\partial\Phi}{\partial u}\right)^2\left[1 + \frac{[s'(\varphi)]^2}{n^2 s^2(\varphi)}\right] \tag{6}$$

where $s'(\varphi)$ stands for the derivative of $s(\varphi)$with respect to φ.

From Eq. (4) we get

$$r dr d\varphi = \frac{1}{\sqrt[n]{s^2(\varphi)}} u du d\varphi \tag{7}$$

Let

$$f(u) = \frac{\partial\Phi}{\partial u} \tag{8}$$

and with the substitution of Eqs. (5)–(8) to Eq. (2), a functional of the capacitance C is obtained:

$$C = \varepsilon\int_0^{2\pi}\frac{n^2 s^2(\varphi) + [s'(\varphi)]^2}{n^2 s^2(\varphi)}d\varphi \cdot \frac{\displaystyle\int_{u_1}^{ku_1} f^2(u) u du}{\left[\displaystyle\int_{u_1}^{ku_1} f(u) du\right]^2} \tag{9}$$

where u_1 represents the potential of the inner conductor and ku_1 represents the potential of the outer conductor.

3. VARIATIONAL EXTREME OF THE FUNCTIONAL $\lambda[f(u)]$

Define a variational functional of continuous functions $f(u)$ as follows

$$\lambda[f(u)] = \frac{\left[\int_{u_1}^{ku_1} f(u)du\right]^2}{\int_{u_1}^{ku_1} f^2(u)udu} \tag{10}$$

Discretizing the integrals and assuming h is large enough, the above definition is expressed in terms of Rayleigh quotient:

$$\lambda[f(u)] = \lim_{n\to\infty} \frac{[f(u_1)du\ f(u_2)du\ \cdots\ f(u_n)du]\begin{bmatrix}1\\1\\\cdots\\1\end{bmatrix}[1\ \ 1\ \ \cdots\ \ 1]\begin{bmatrix}f(u_1)du\\f(u_2)du\\\cdots\\f(u_n)du\end{bmatrix}}{[f(u_1)du\ f(u_2)du\ \cdots\ f(u_n)du]\begin{bmatrix}\frac{u_1}{du} & & & \\ & \frac{u_2}{du} & & \\ & & \cdots & \\ & & & \frac{u_n}{du}\end{bmatrix}\begin{bmatrix}f(u_1)du\\f(u_2)du\\\cdots\\f(u_n)du\end{bmatrix}} \tag{11}$$

with $u_n = ku_1$. For simplicity, (11) is rewritten as follows

$$\lambda[f(u)] = \lim_{n\to\infty} \frac{[F]^T[A][F]}{[F]^T[B][F]} \tag{12}$$

with $[F]^T = [f(u_1)du\ f(u_2)du\ \cdots\ f(u_n)du]$, $[A] = a^Ta$, $a = [1\ \ 1\ \ \cdots\ \ 1]$ and $[B]$ is a diagonal matrix. This kind of Rayleigh quotient like Eq. (12) has been proved by D. K. Cheng and F. I. Tseng [4] to have one and only one maximum:

$$\max\lambda[f(u)] = [a][B]^{-1}[a]^T \tag{13}$$

And the corresponding characteristic vector is

$$[F] = [B]^{-1}a^T \tag{14}$$

i.e.,

$$\begin{bmatrix} f(u_1)du \\ f(u_2)du \\ \cdots \\ f(u_n)du \end{bmatrix} = \begin{bmatrix} \dfrac{1}{u_1}du \\ \dfrac{1}{u_2}du \\ \cdots \\ \dfrac{1}{u_n}du \end{bmatrix} \tag{15}$$

It is obvious that we have

$$f(u) = \frac{1}{u} \tag{16}$$

when the functional achieves the variational maximum. And with the substitution of (16) into (10), we have

$$\min \frac{1}{\lambda[f(u)]} = \frac{\displaystyle\int_{u_1}^{ku_1} f^2(u)udu}{\left[\displaystyle\int_{u_1}^{ku_1} f(u)du\right]^2} = \frac{1}{\ln k} \tag{17}$$

We can extend this to the following case:

$$\min \frac{1}{\lambda[f(u)]} = \frac{\displaystyle\int_{u_1}^{ku_1} f^2(u)g(u)du}{\left[\displaystyle\int_{u_1}^{ku_1} f(u)du\right]^2} = \frac{1}{\displaystyle\int_{u_1}^{ku_1} \frac{1}{g(u)}du} \tag{18}$$

And $f(u) = \frac{1}{g(u)}$ when the functional $\lambda[f(u)]$ achieves its variational extreme, where $g(u) > 0$. With the substitution of variational expression (17) into (9), we obtain the capacitance of a type of coaxial lines made of nth-order homogeneous curves (3)

$$C = \frac{\varepsilon}{\ln k} \int_0^{2\pi} \frac{n^2 s^2(\varphi) + [s'(\varphi)]^2}{n^2 s^2(\varphi)} d\varphi \tag{19}$$

4. APPLICATIONS AND EXAMPLES

Now we apply the above common formula to different homogeneous and conformal curves to obtain their corresponding closed-forms of the capacitance formulae.

Case 1: $n = 1$

When $n = 1$, we illustrate this case with the coaxial line of m-side ($m = 3, 4, \cdots$) regular polygon shown in Fig. 2 as an example.

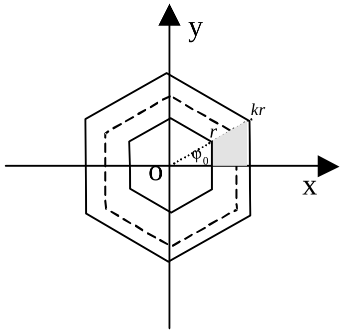

Figure 2. Cross-section of conformal m-side regular polygon coaxial line.

Considering the symmetry of the geometry, we study the part $1/2m$ of the whole shown in Fig. 2. Now,

$$F(r, \varphi) = r \cos \varphi = x \tag{20}$$

and

$$n = 1, \quad s(\varphi) = \cos \varphi, \quad s'(\varphi) = -\sin \varphi \tag{21}$$

With the substitution of (21) into (19), one gets

$$C_m = \frac{2m\varepsilon}{\ln k} \tan\left(\frac{\pi}{m}\right) \tag{22}$$

Specially, when $m = 4$, one gets the variational capacitance formula of the square coaxial line:

$$C_4 = \frac{8\varepsilon}{\ln k} \tag{23}$$

And we have $C_\infty = C_0 = \frac{2\pi\varepsilon}{\ln k}$, that is to say when $m \to \infty$ the capacitance approaches that of the circular coaxial line C_0. Introducing the normalized capacitance

$$\overline{C}_m = \frac{C_m}{C_0} = \frac{\pi}{m} \tan\left(\frac{\pi}{m}\right), \quad m = 3, 4, \cdots \tag{24}$$

the variation of the normalized capacitances with m is shown in Fig. 3.

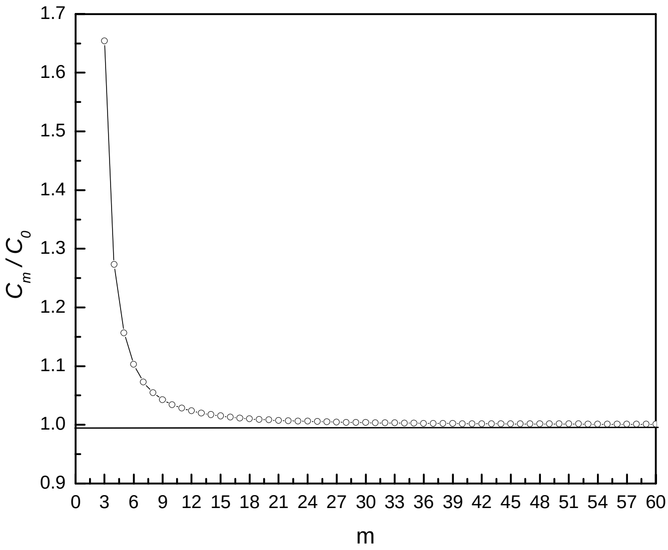

Figure 3. Normalized capacitances of the m-side regular polygon coaxial lines to that of circular coaxial transmission line.

Case two: $n = 2$

As shown in Fig. 4, the elliptical coaxial line is taken as an example to illustrate this case.

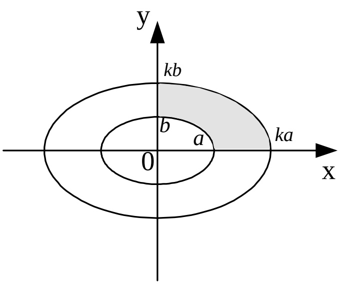

Figure 4. Cross-section of conformal elliptical coaxial line.

Because of the symmetry of the geometric structure, we may consider one-quarter of the structure and have

$$F(r, \varphi) = r^2(b^2 \cos^2 \varphi + a^2 \sin^2 \varphi) \tag{25}$$

then

$$n = 2, \quad s(\varphi) = b^2 \cos^2 \varphi + a^2 \sin^2 \varphi, \quad s'(\varphi) = 2(a^2 - b^2) \sin \varphi \cos \varphi \tag{26}$$

By substituting (26) into (19), one arrives at

$$C_e = \frac{4\varepsilon}{\ln k} \int_0^{\pi/2} \frac{a^4 \sin^2 \varphi + b^4 \cos^2 \varphi}{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} d\varphi = \frac{\pi\varepsilon}{\ln k} \cdot \frac{a^2 + b^2}{ab} \tag{27}$$

The above formula is the variational closed-form of the capacitance of the conformal elliptical coaxial line. Normalizing the capacitance of the elliptical coaxial line to that of the circular coaxial line, one obtains

$$\overline{C}_e = \frac{C_e}{C_0} = \frac{a^2 + b^2}{2ab} \tag{28}$$

The variation of the normalized capacitance with b/a is shown in Fig. 5, where $b/a < 1$ is assumed and will not lose the universality.

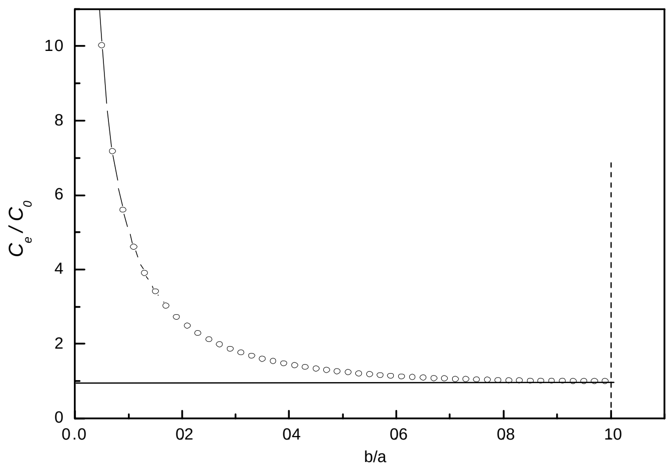

Figure 5. Normalized capacitance of the conformal elliptical coaxial line as function of the axis ratio b/a to that of circular coaxial transmission line.

Case 3: $n = N > 2$

Here high-order "elliptical" coaxial lines of Nth-order homogeneous curves shown in Fig. 6 are studied and the governing equation is:

$$\left(\frac{|x|}{a}\right)^N + \left(\frac{|y|}{b}\right)^N = \text{constant}, \quad N > 2 \tag{29}$$

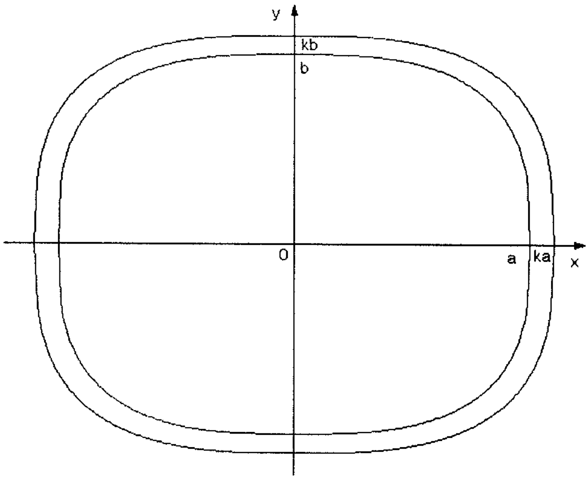

Figure 6. Cross-section of conformal high-order "elliptical" coaxial line of Nth-order homogeneous curve (here, $N = 3$, $b/a = 0.8$, and $k = 1.1$).

For simplicity we consider the part located in the first quadrant and have

$$F(r, \varphi) = r^N (b^N \cos^N \varphi + a^N \sin^N \varphi) \tag{30}$$

Comparing Eq. (30) with Eq. (3), one obtains

$$\begin{aligned} s(\varphi) &= b^N \cos^N \varphi + a^N \sin^N \varphi, \\ s'(\varphi) &= N(-b^N \cos^{N-1} \varphi \sin \varphi + a^N \sin^{N-1} \varphi \cos \varphi) \end{aligned} \tag{31}$$

With the substitution of (31) into (19), we arrive at

$$\begin{aligned} C_h &= \frac{4\varepsilon}{\ln k} \int_0^{\pi/2} \frac{a^{2N} \sin^{2N-2} \varphi + b^{2N} \cos^{2N-2} \varphi}{(a^N \sin^N \varphi + b^N \cos^N \varphi)^2} d\varphi \\ &= \frac{4\varepsilon}{\ln k} \cdot \frac{b}{a} \cdot \int_0^{\infty} \frac{1 + (a/b)^2 z^{2N-2}}{(1 + z^N)^2} dz \end{aligned} \tag{32}$$

where $z = \frac{a}{b} \tan \varphi$.

Let

$$w = \frac{1}{1 + z^N} \tag{33}$$

then the first part of the integral in (32) becomes

$$\int_0^\infty \frac{dz}{(1+z^N)^2} = \frac{1}{N}\int_0^1 w^{1-1/N}(1-w)^{-1+1/N}dw = \frac{1}{N}B\left(2-\frac{1}{N},\frac{1}{N}\right) \tag{34}$$

where the function B is defined by $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$, $(p,q > 0)$ and is related to a Γ function by

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{35}$$

where

$$\Gamma(p) = \int_0^\infty u^{p-1}e^{-u}du \tag{36}$$

Noting that a Γ function has the following properties

$$(1)\Gamma(p+1) = p\Gamma(p);\ (2)\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)},\ 0<p<1;\ (3)\Gamma(2) = 1,$$

Eq. (34) is rewritten as

$$\begin{aligned}\int_0^\infty \frac{dz}{(1+z^N)^2} &= \frac{1}{N}\Gamma\left(2-\frac{1}{N}\right)\Gamma\left(\frac{1}{N}\right)\\ &= \frac{1}{N}\left(1-\frac{1}{N}\right)\Gamma\left(1-\frac{1}{N}\right)\Gamma\left(\frac{1}{N}\right)\\ &= \frac{1}{N}\left(1-\frac{1}{N}\right)\frac{\pi}{\sin(\pi/N)}\end{aligned} \tag{37}$$

As for the second integral in (32), we have

$$\int_0^\infty \frac{z^{2N-2}}{(1+z^N)^2}dz = -\int_0^\infty \frac{1}{(1+z^{-N})^2}d\left(\frac{1}{z}\right) = \int_0^\infty \frac{1}{(1+z^N)^2}dz \tag{38}$$

Then we obtain the variational capacitance formula of the conformal high-order "elliptical" coaxial line of N-th order homogeneous curve

$$C_h = \frac{\pi\varepsilon}{\ln k}\cdot\frac{a^2+b^2}{ab}\cdot\frac{4(N-1)}{N^2\sin(\pi/N)} \tag{39}$$

The above expression is rewritten as

$$C_h = C_e\cdot G(N) \tag{40}$$

where C_e is the capacitance of the conformal elliptical coaxial line and

$$G(N) = \frac{4(N-1)}{N^2 \sin(\pi/N)} \tag{41}$$

If one normalizes the capacitance of the conformal high-order "elliptical" coaxial line C_h to that of the conformal elliptical coaxial line C_e, he gets

$$\overline{C}_h = C_h/C_e = G(N) \tag{42}$$

It is obvious that the normalized capacitance $\overline{C}_h$ depends only on N. The variation of $\overline{C}_h$ with the order N is shown in Fig. 7.

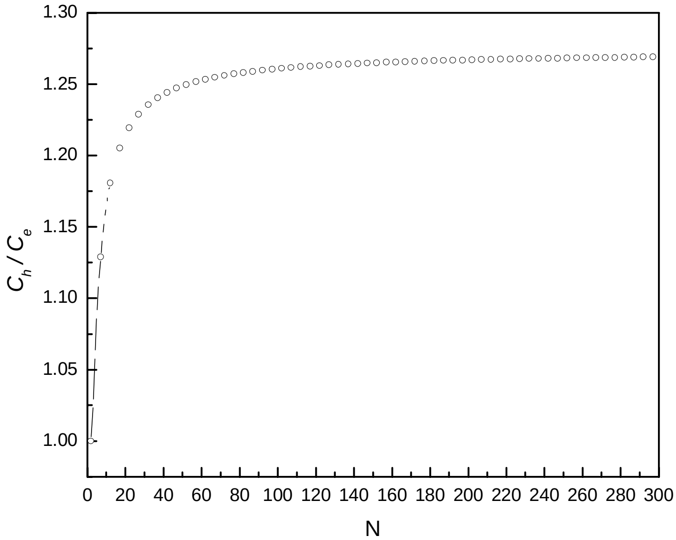

Figure 7. Normalized capacitance of conformal high-order homogeneous "elliptical" coaxial line to that of conformal elliptical coaxial line as the function of the order N.

Table 1 gives some capacitance values when $b = a$ and $n \geq 2$.

Table 1. Capacitance values when $b = a$ and $n \geq 2$.

n	2	3	4	8	12	∞
$C \cdot \ln k/\varepsilon$	2π	6.449064	6.664324	7.183205	7.417778	8.00000

It is obviously seen from table 1 that the capacitance when $n = 2$ is that of the circular coaxial line and when $n = \infty$ that of the conformal square coaxial line. Moreover, expression (39) becomes the capacitance formula of the conformal rectangular coaxial line when $N = \infty$. Therefore, the introduction of high-order homogeneous "elliptical" coaxial lines unifies the capacitance formulae of the conformal circular, square, rectangular and elliptical coaxial lines.

To test the correctness of our results, comparison is made between our variational closed-form results and numerical results of Boundary Element Method (BEM) with the elliptical coaxial line as an example. As shown in Table 2, good agreement is achieved.

Table 2. Comparison between variational results and numerical results of BEM.

Dimensions of inner ellipse (meter)	$a = 1.25$	$a = 2.5$	$a = 2.0$
	$b = 0.75$	$b = 1.5$	$b = 1.6$
K	2.07	2.0	2.5
C/ε (Variational)	10.273361	10.273361	7.066682
C/ε (BEM)	10.106847	10.107461	6.997397

5. CONCLUSION

The variational closed-forms of the capacitance formulae of a type of conformal coaxial lines made of homogeneous curves are deduced. Because of the proper assumption of the distribution of equipotential lines and the application of variational extreme formula of the functional of continuous functions, the resultant expressions are not only concise but also of high accurateness. The presented method may be extended to the case of conformal coaxial lines made of inhomogeneous curves.

REFERENCES

1. Bancroft, R., "A note on the moment method solution for the capacitance of a conducting flat plate," *IEEE Trans. on Antennas Propagat.*, Vol. AP-45, 1704, 1997.
2. Kuo, J. T. and K.-Y. Su, "Analytical evaluation of the MoM matrix elements for the capacitance of a charged plate," *IEEE Trans. Microwave Theory Tech.*, Vol. 50, 1435–1436, 2002.

3. Collin, R. E., *Field Theory of Guided Waves*, McGraw-Hill, New York, 1966.
4. Cheng, D. K. and F. I. Tseng, "Gain optimization for arbitrary antenna arrays," *IEEE Trans. on AP*, Vol. AP-13, 973–974, 1965.

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