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THEORY OF DISPERSION-MANAGED OPTICAL SOLITONS

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Abstract—The variational principle is employed to study chirped solitons that propagate through optical fibers and is governed by the dispersion-managed nonlinear Schrödinger's equation. Here, in this paper, the polarization-preserving fibers, birefringent fibers as well as multiple channels have been considered. The study is extended to obtain the adiabatic evolution of soliton parameters in presence of perturbation terms for such fibers. Both Gaussian and super-Gaussian solitons have been considered.

1. INTRODUCTION

The propagation of solitons through optical fibers has been a major area of research given its potential applicability in all optical communication systems. The field of telecommunications has undergone a substantial evolution in the last couple of decades due to the impressive progress in the development of optical fibers, optical amplifiers as well as transmitters and receivers. In a modern optical communication system, the transmission link is composed of optical fibers and amplifiers that replace the electrical regenerators. But the amplifiers introduce some noise and signal distortion that limit the system capacity. Presently the optical systems that show the best characteristics in terms of simplicity, cost and robustness against the degrading effects of a link are those based on intensity modulation with direct detection (IM-DD). Conventional IM-DD systems are based on non-return-to-zero (NRZ) format, but for transmission at higher data rate the return-to-zero (RZ) format is preferred. When the data rate is quite high, soliton transmission can be used. It allows the exploitation of the fiber capacity much more, but the NRZ signals offer very high potential especially in terms of simplicity.

There are limitations, however, on the performance of optical system due to several effects that are present in optical fibers and amplifiers. Signal propagation through optical fibers can be affected by group velocity dispersion (GVD), polarization mode dispersion (PMD) and the nonlinear effects. The chromatic dispersion that is essentially the GVD when waveguide dispersion is negligible, is a linear effect that introduces pulse broadening generates intersymbol interference. The PMD arises due the fact that optical fibers for telecommunications have have two polarization modes, in spite of the fact that they are called monomode fibers. These modes have two different group velocities that induce pulse broadening depending on the input signal state of polarization. The transmission impairment due to PMD looks similar to that of the GVD. However, PMD is a random process as compared to the GVD that is a deterministic process. So PMD cannot be controlled at the receiver. Newly installed optical fibers have quite low values of PMD that is about 0.1 $ps/\sqrt{km}$.

The main nonlinear effects that arises in monomode fibers are the Brillouin scattering, Raman scattering and the Kerr effect. Brillouin is a backward scattering that arises from acoustic waves and can generate forward noise at the receiver. Raman scattering is a forward scattering from silica molecules. The Raman gain response is characterized by low gain and wide bandwidth namely about 5 THz. The Raman threshold in conventional fibers is of the order of 500 mW for copolarized pump and Stokes' wave (that is about 1 W for random polarization), thus making Raman effect negligible for a single channel signal. However, it becomes important for multichannel wavelength-division-multiplexed (WDM) signal due to an extremely wide band of wide gain curve.

The Kerr effect of nonlinearity is due to the dependence of the fiber refractive index on the field intensity. This effect mainly manifests as a new frequency when an optical signal propagates through a fiber. In a single channel the Kerr effect induces a spectral broadening and the phase of the signal is modulated according to its power profile. This effect is called self-phase modulation (SPM). The SPM-induced chirp combines with the linear chirp generated by the chromatic dispersion. If the fiber dispersion coefficient is positive namely in the normal dispersion regime, linear and nonlinear chirps have the same sign while in the anomalous dispersion regime they are of opposite signs. In the former case, pulse broadening is enhanced by SPM while in the later case it is reduced. In the anomalous dispersion case the Kerr nonlinearity induces a chirp that can compensate the degradation induced by GVD. Such a compensation is total if soliton signals are used.

If multichannel WDM signals are considered, the Kerr effect can be more degrading since it induces nonlinear cross-talk among the channels that is known as the cross-phase modulation (XPM). In addition WDM generates new frequencies called the Four-Wave mixing (FWM). The other issue in the WDM system is the collision-induced timing jitter that is introduced due to the collision of solitons in different channels. The XPM causes further nonlinear chirp that interacts with the fiber GVD as in the case of SPM. The FWM is a parametric interaction among waves satisfying a particular relationship called phase-matching that lead to power transfer among different channels.

To limit the FWM effect in a WDM it is preferable to operate with a local high GVD that is periodically compensated by devices having an opposite sign of GVD. One such device is a simple optical fiber with opportune GVD and the method is commonly known as the dispersion-management. With this approach the accumulated GVD can be very low and at the same time FWM effect is strongly limited. Through dispersion-management it is possible to achieve highest capacity for both RZ as well as NRZ signals. In that case the overall link dispersion has to be kept very close to zero, while a small amount of chromatic anomalous dispersion is useful for the efficient propagation of a soliton signal. It has been demonstrated that with soliton signals, the dispersion-management is very useful since it reduces collision induced timing jitter [3] and also the pulse interactions. It thus permits the achievement of higher capacities as compared to the link having constant chromatic dispersion.

In this paper, the dynamics of dispersion-managed (DM) solitons propagating through an optical fiber will be studied in presence of perturbation terms. Both Gaussian and super-Gaussian type solitons will be considered for completeness.

2. GOVERNING EQUATIONS

The relevant equation is the nonlinear Schrödinger's equation (NLSE) with damping and periodic amplification [2, 3, 8, 9]

$$iu_z + \frac{D(z)}{2} u_{tt} + |u|^2 u = -i\Gamma u + i\left[e^{\Gamma z_a} - 1\right] \sum_{n=1}^{N} \delta(z - nz_a) u \tag{1}$$

Here, Γ is the normalized loss coefficient, z_a is the normalized characteristic amplifier spacing and z and t represent the normalized propagation distance and the normalized time, respectively, while u represents the wave profile expressed in the nondimensional units.

Also, $D(z)$ is used to model strong dispersion management. The fiber dispersion $D(z)$ is decomposed into two components namely a path-averaged constant value δ_a and a term representing the large rapid variation due to large local values of the dispersion [1–3]. Thus,

$$D(z) = \delta_a + \frac{1}{z_a} \Delta(\zeta) \tag{2}$$

where $\zeta = z/z_a$. The function $\Delta(\zeta)$ is taken to have average zero over an amplification period namely

$$\langle \Delta \rangle = \frac{1}{z_a} \int_0^{z_a} \Delta \left(\frac{z}{z_a} \right) dz = 0 \tag{3}$$

so that the path-averaged dispersion D will have an average δ_a namely

$$\langle D \rangle = \frac{1}{z_a} \int_0^{z_a} D(z) dz = \delta_a \tag{4}$$

The proportionality factor in front of $\Delta(\zeta)$, in (2), is chosen so that both δ_a and $\Delta(\zeta)$ are quantities of order one. In practical situations, dispersion management is often performed by concatenating together two or more sections of given length with different values of fiber dispersion. In the special case of a two-step map it is convenient to write the dispersion map as a periodic extension of [2]

$$\Delta(\zeta) = \begin{cases} \Delta_1 & : \quad 0 \le |\zeta| < \frac{\theta}{2} \\ \Delta_2 & : \quad \frac{\theta}{2} \le |\zeta| < \frac{1}{2} \end{cases} \tag{5}$$

where Δ_1 and Δ_2 are given by

$$\Delta_1 = \frac{2s}{\theta} \tag{6}$$

and

$$\Delta_2 = -\frac{2s}{1-\theta} \tag{7}$$

with the map strength, s, being defined as

$$s = \frac{\theta\Delta_1 - (1-\theta)\Delta_2}{4} \tag{8}$$

Conversely,

$$s = \frac{\Delta_1\Delta_2}{4(\Delta_2 - \Delta_1)} \tag{9}$$

and

$$\theta = \frac{\Delta_2}{\Delta_2 - \Delta_1} \tag{10}$$

A typical two-step dispersion map is shown in the following figure

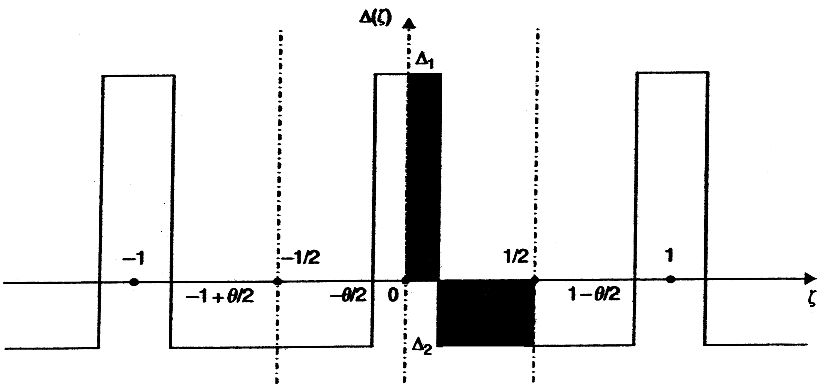

Figure 1. Schematic diagram of a two-step map.

Taking into account the loss and amplification cycles by looking for a solution of (1) of the form $u(z,t) = P(z)q(z,t)$ for real P and letting P satisfy

$$P_z + \Gamma P - \left[e^{\Gamma z_a} - 1\right]\sum_{n=1}^{N}\delta(z - nz_a)P = 0 \tag{11}$$

equation (1) transforms to

$$iq_z + \frac{D(z)}{2}q_{tt} + g(z)|q|^2 q = 0 \tag{12}$$

where

$$g(z) = P^2(z) = a_0^2 e^{-2\Gamma(z - nz_a)} \tag{13}$$

for $z \in [nz_a, (n+1)z_a)$ and $n > 0$ and also

$$a_0 = \left[\frac{2\Gamma z_a}{1 - e^{-2\Gamma z_a}}\right]^{\frac{1}{2}} \tag{14}$$

so that over each amplification period

$$\langle g(z) \rangle = \frac{1}{z_a} \int_0^{z_a} g(z)dz = 1 \tag{15}$$

Equation (12) is commonly known as the Dispersion-Managed Nonlinear Schrödinger's equation (DMNLSE) and it governs the propagation of a dispersion-managed soliton through a polarization preserved optical fiber with damping and periodic amplification [1, 13].

The following figures show a direct numerical simulations of (12). Figure 2 illustrates the profile of the pulse as the map strength, s, varies from 0 to 16. However, Figures 3(a) and (b) are profiles of DM solitons in the linear and logarithmic scales respectively.

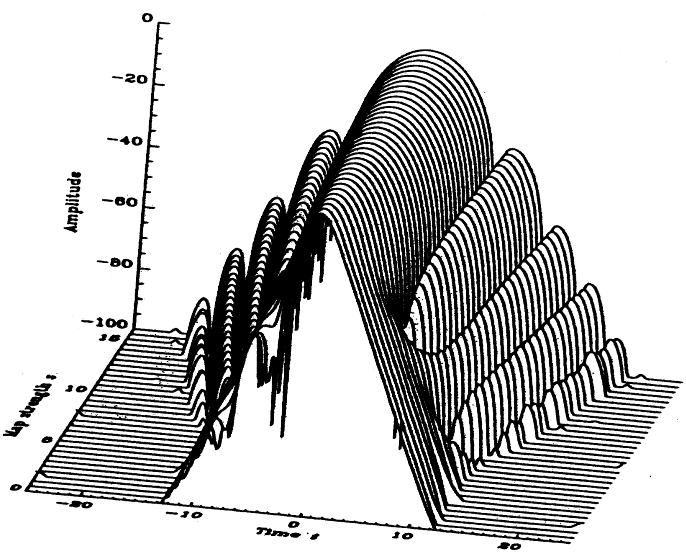

Figure 2. Pulse profile.

Equation (12) will now be studied approximately by variational method based on the observation that it supports well-defined chirped

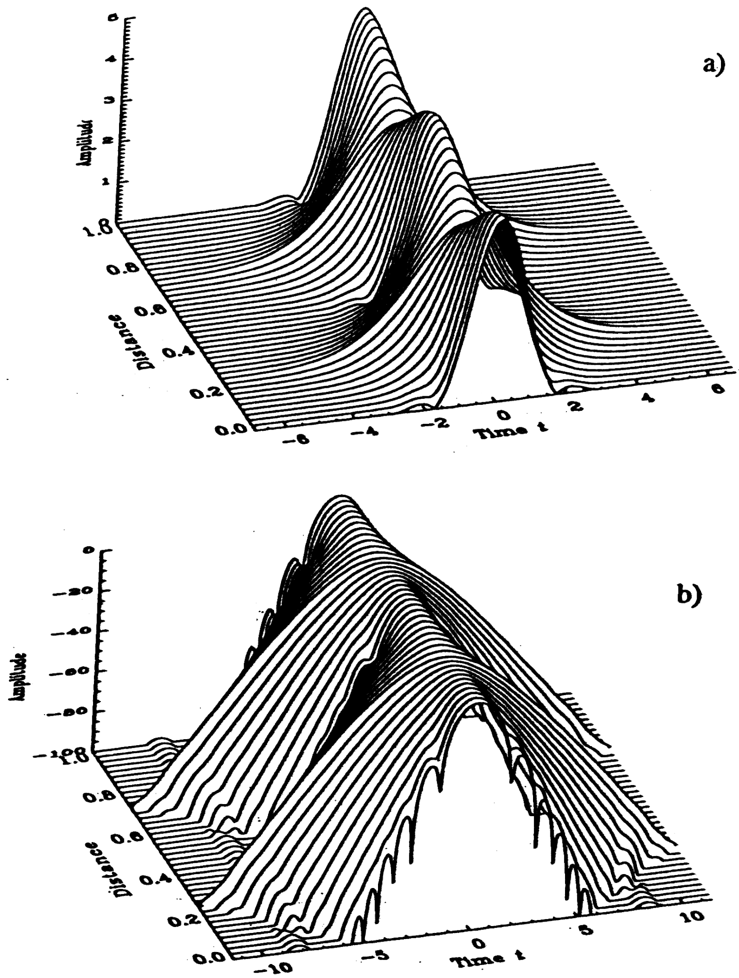

Figure 3. DM soliton profile (a) linear scale (b) logarithmic scale.

soliton solution whose shape is close to that of a Gaussian [3, 5, 10, 11]. These pulses deviate from a conventional soliton. However, Gaussian pulses have relatively broad leading and trailing edges. In general, a soliton with leading and trailing edges broadens more rapidly as it propagates since such a pulse has a wider spectrum to start with. Pulses emitted by directly modulated semiconductor lasers fall in this category and cannot generally be approximated by a Gaussian soliton.

A hyper-Gaussian, also known as a super-Gaussian (SG) soliton can be used to model the effects of steep leading and trailing edges on dispersion-induced pulse broadening [3]. It is to be noted here that these pulses are solitary waves and are not strictly solitons as it is not yet established whether they regain their form after interaction. Henceforth, these solitary waves will be simply referred to as pulses.

3. VARIATIONAL PRINCIPLE

For a finite dimensional problem of a single particle, the temporal development of its position is given by the Hamilton's principle of least action [8]. It states that the action given by the time integral of the Lagrangian is an extremum, namely

$$\delta \int_{t_1}^{t_2} L(x, \dot{x}) dt = 0 \tag{16}$$

where x is the position of the particle and $\dot{x} = dx/dt$. The variational problem (12) then leads to the familiar Euler-Lagrange's (EL) equation [5, 6, 8–10]

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{17}$$

In this procedure, an intelligent guess is made for the evolution of $q(z, t)$ in the sense that the form of q is modeled in terms of certain parameter functions, r that characterize the crucial features of the solutions namely the amplitude, spatial width, phase variations and others. The parameters of this trial function are allowed to be functions of z namely $r = r(z)$. Inserting the trial function into the variational integral, the spatial integration can be performed and a reduced variational problem for the parameter functions $r(z)$ is obtained.

An advantage of this method is that it is applicable to a perturbation problem where the unperturbed system may not be integrable. This method only requires that the unperturbed system admits a well defined solution such as a soliton or a solitary wave. The nontrivial applications of this method may be found elsewhere [20].

By the principle of least action, namely (16), the EL equation is [8]

$$\frac{\partial L}{\partial r} - \frac{d}{dz}\left(\frac{\partial L}{\partial r_z}\right) = 0 \tag{18}$$

where r is one of the six soliton parameters. Using the variational principle, a set of evolution equations for the pulse parameters is derived. This approach is only approximate and does not account for characteristics such as energy loss due to continuum radiation, damping of the amplitude oscillations and changing of the pulse shape.

4. POLARIZATION PRESERVING FIBERS

In a polarization preserved optical fiber, it is shown in the last section that the propagation of solitons is governed by the scalar DMNLSE given by (12). In (12), if $D(z) = g(z) = 1$, the NLSE is recovered. It is possible to integrate NLSE by the method of Inverse Scattering Transform (IST). Thus, this case falls into the category of S-integrable partial differential equations [8, 9, 19]. The IST is the nonlinear analogue of the Fourier Transform that is used to solve the linear partial differential equations and thus the two methods are schematically similar [8, 19].

4.1. Integrals of Motion

Equation, (12) does not contain an infinite number of integrals of motion. In fact, there are as few as two of them. They are energy (E), also known as the L_2 norm, and linear momentum (M) that are respectively given by

$$E = \int_{-\infty}^{\infty} |q|^2 dt \tag{19}$$

$$M = \frac{i}{2} D(z) \int_{-\infty}^{\infty} (q^* q_t - q q_t^*) dt \tag{20}$$

The Hamiltonian (H) which is given by

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \left(D(z)|q_t|^2 - g(z)|q|^4 \right) dt \tag{21}$$

is however not a constant of motion, in general. The case $D(z)$ and $g(z)$ a constant makes the Hamiltonian a conserved quantity.

Now, it is assumed that the solution of (12) is given by a chirped pulse of the form [6, 10]

$$\begin{aligned} q(z,t) = {} & A(z) f\left[B(z)\left\{t - \bar{t}(z)\right\}\right] \\ & \exp\left[iC(z)\left\{t - \bar{t}(z)\right\}^2 - i\kappa(z)\left\{t - \bar{t}(z)\right\} + i\theta(z)\right] \end{aligned} \tag{22}$$

where f represents the shape of the pulse. It could be a Gaussian type or a SG type pulse. Also, here the parameters $A(z)$, $B(z)$, $C(z)$, $\kappa(z)$, $\bar{t}(z)$ and $\theta(z)$ respectively represent the soliton amplitude, the inverse width of the pulse, chirp, frequency, the center of the pulse and the phase of the pulse. For convenience, the following integral is defined

$$I_{a,b,c} = \int_{-\infty}^{\infty} \tau^a f^b(\tau) \left(\frac{df}{d\tau}\right)^c d\tau \tag{23}$$

where a, b and c are positive integers and

$$\tau = B(z)\left\{t - \bar{t}(z)\right\} \tag{24}$$

For such a pulse form given by (22), the integrals of motion are

$$E = \int_{-\infty}^{\infty} |q|^2 dt = \frac{A^2}{B} I_{0,2,0} \tag{25}$$

$$M = \frac{i}{2} D(z) \int_{-\infty}^{\infty} (q^* q_t - q q_t^*) dt = -\kappa D(z) \frac{A^2}{B} I_{0,2,0} \tag{26}$$

while the Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2} \int_{-\infty}^{\infty} \left(D(z)|q_t|^2 - g(z)|q|^4\right) dt \\ &= \frac{D(z)}{2B^3}\left(A^2 B^3 I_{0,0,2} + 4A^2C^2 I_{2,2,0} + \kappa^2 A^2 B^2 I_{0,2,0}\right) - \frac{g(z)}{2}\frac{A^4}{B} I_{0,4,0} \end{aligned} \tag{27}$$

4.2. Variational Formulation

Equation (12) will now be studied by means of variational method based on the observation that it cannot be integrated by IST. This equation supports well-defined chirped soliton solution whose shape is close to that of a Gaussian or a SG. [4]. The Lagrangian, for (12), is

$$L = \frac{1}{2} \int_{-\infty}^{\infty} \left[i\left(q^* q_z - q q_z^*\right) - D(z)|q_t|^2 + g(z)|q|^4\right] dt \tag{28}$$

Now, using (23), the Lagrangian given by (28), reduces to

$$\begin{aligned} L = &-D(z)\frac{A^2}{2B^3}\left(B^4 I_{0,0,2} + 4C^2 I_{2,2,0} + \kappa^2 B^2 I_{0,2,0}\right) \\ &+ g(z)\frac{A^4}{2B} I_{0,4,0} - \frac{A^2}{B^3} I_{2,2,0}\frac{dC}{dz} + \frac{A^2}{B} I_{0,2,0}\left(\bar{t}\frac{d\kappa}{dz} - \frac{d\theta}{dz}\right) \end{aligned} \tag{29}$$

In the EL equation, namely (18), substituting $A, B, C, \kappa, \bar{t}$ and θ for r yields the following parameter dynamics for the solitons.

$$\frac{dA}{dz} = -ACD(z) \tag{30}$$

$$\frac{dB}{dz} = -2BCD(z) \tag{31}$$

$$\frac{dC}{dz} = \left(\frac{B^4}{2}\frac{I_{0,0,2}}{I_{2,2,0}} - 2C^2\right)D(z) - \frac{g(z)A^2B^2}{4}\frac{I_{0,4,0}}{I_{2,2,0}} \tag{32}$$

$$\frac{d\kappa}{dz} = 0 \tag{33}$$

$$\frac{d\bar{t}}{dz} = -\kappa D(z) \tag{34}$$

$$\frac{d\theta}{dz} = \left(\frac{\kappa^2}{2} - \frac{I_{0,0,2}}{I_{0,2,0}}B^2\right)D(z) + \frac{5g(z)A^2}{4}\frac{I_{0,4,0}}{I_{0,2,0}} \tag{35}$$

Now, from (30) and (31), $A = K\sqrt{B}$ where the constant K is proportional to the square root of the energy as seen from (25). So, the number of parameters reduces by one. Thus, (30)-(35), respectively, modify to

$$\frac{dB}{dz} = -2BCD(z) \tag{36}$$

$$\frac{dC}{dz} = \left(\frac{B^4}{2}\frac{I_{0,0,2}}{I_{2,2,0}} - 2C^2\right)D(z) - \frac{K^2g(z)B^3}{4}\frac{I_{0,4,0}}{I_{2,2,0}} \tag{37}$$

$$\frac{d\kappa}{dz} = 0 \tag{38}$$

$$\frac{d\bar{t}}{dz} = -\kappa D(z) \tag{39}$$

$$\frac{d\theta}{dz} = \left(\frac{\kappa^2}{2} - \frac{I_{0,0,2}}{I_{0,2,0}}B^2\right)D(z) + \frac{5Kg(z)B}{4}\frac{I_{0,4,0}}{I_{0,2,0}} \tag{40}$$

4.2.1. Gaussian Pulses

For a pulse of Gaussian type, $f(\tau) = e^{-\frac{1}{2}\tau^2}$. In this case, the conserved quantities respectively reduce to

$$E = \int_{-\infty}^{\infty} |q|^2 dt = \frac{A^2}{B}\sqrt{\frac{\pi}{2}} = K^2\sqrt{\frac{\pi}{2}} \tag{41}$$

$$\begin{aligned} M &= \frac{i}{2}D(z)\int_{-\infty}^{\infty}(q^* q_t - q q_t^*)dt \\ &= -\kappa D(z)\frac{A^2}{B}\sqrt{\frac{\pi}{2}} = -\kappa D(z)K^2\sqrt{\frac{\pi}{2}} \end{aligned} \tag{42}$$

while the Hamiltonian is

$$
\begin{aligned}
H &= \frac{1}{2}\int_{-\infty}^{\infty}\left(D(z)|q_t|^2 - g(z)|q|^4\right)dt \\
&= \frac{D(z)K^2}{2}\left(B^2+\frac{C^2}{B^2}+\kappa^2\right)\sqrt{\frac{\pi}{2}} - \frac{\sqrt{\pi}}{4}g(z)BK^4 \qquad (43)
\end{aligned}
$$

Also, the parameter dynamics given by (36)–(40) respectively are

$$\frac{dB}{dz} = -2BCD(z) \qquad (44)$$

$$\frac{dC}{dz} = 2D(z)\left(B^4 - C^2\right) - \frac{\sqrt{2}}{2}g(z)B^3K^2 \qquad (45)$$

$$\frac{d\kappa}{dz} = 0 \qquad (46)$$

$$\frac{d\bar{t}}{dz} = -\kappa D(z) \qquad (47)$$

$$\frac{d\theta}{dz} = \frac{D(z)}{2}\left(\kappa^2 - B^2\right) + \frac{5\sqrt{2}}{8}g(z)BK \qquad (48)$$

4.2.2. Super-Gaussian Pulses

For SG pulse, $f(\tau) = e^{-\frac{1}{2}\tau^{2p}}$ with $p \geq 1$ where the parameter p controls the degree of edge sharpness. With $p = 1$, the case of chirped Gaussian pulse is recovered, while for larger values of p the pulse gradually becomes square shaped with sharper leading and trailing edges [4]. Figure 4 below, represent the shape of the pulses as the parameter p varies.

For a SG pulse, the integrals of motion respectively are

$$E = \int_{-\infty}^{\infty}|q|^2dt = \frac{A^2}{B}\frac{1}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right) = \frac{K^2}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right) \qquad (49)$$

$$
\begin{aligned}
M &= \frac{i}{2}D(z)\int_{-\infty}^{\infty}(q^*q_t - qq_t^*)dt \\
&= -\kappa D(z)\frac{A^2}{B}\frac{1}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right) = -\frac{\kappa D(z)K^2}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right) \qquad (50)
\end{aligned}
$$

while the Hamiltonian is

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left(D(z)|q_t|^2 - g(z)|q|^4\right)dt$$

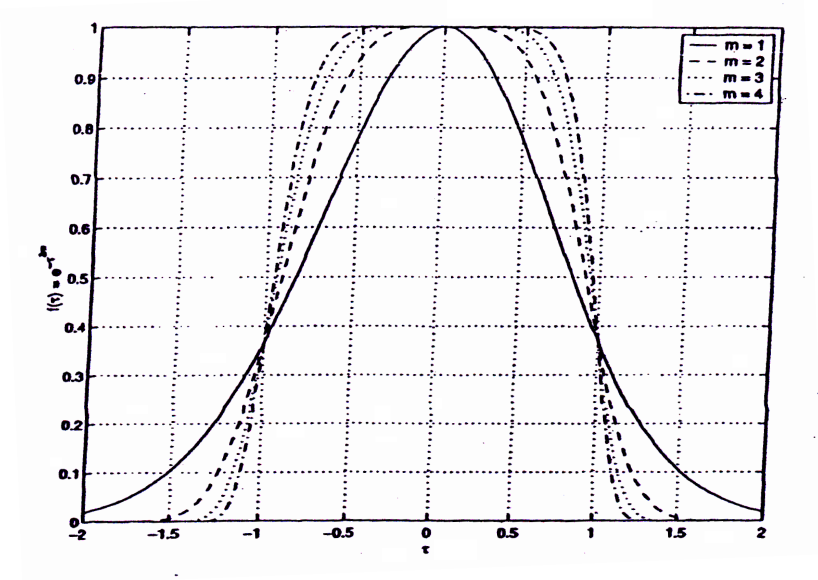

Figure 4. SG pulse with the variation of the parameter m.

$$= D(z)K^2\left\{B^2\frac{p}{2^{\frac{2p-1}{2p}}}\Gamma\left(\frac{4p-1}{2p}\right)+\frac{C^2}{B^2}\frac{2^{\frac{2m-3}{2m}}}{m}\Gamma\left(\frac{3}{2p}\right)\right.$$
$$\left.+\kappa^2\frac{1}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{2p}\right)\right\}-g(z)\frac{BK^4}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{p}\right) \tag{51}$$

Also, the evolution equations for the pulse parameters (36)-(40) respectively reduce to

$$\frac{dB}{dz} = -2BCD(z) \tag{52}$$

$$\frac{dC}{dz} = \left\{B^4\frac{p^2}{2^{\frac{p-2}{p}}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}-2C^2\right\}D(z)-\frac{1}{2^{\frac{4p+1}{2p}}}\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}g(z)B^3K^2 \tag{53}$$

$$\frac{d\kappa}{dz} = 0 \tag{54}$$

$$\frac{d\bar{t}}{dz} = -\kappa D(z) \tag{55}$$

$$\frac{d\theta}{dz} = \left\{\frac{\kappa^2}{2}-p^2 2^{\frac{1}{p}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)}B^2\right\}D(z)+\frac{5}{2^{\frac{4p+1}{2p}}}g(z)BK \tag{56}$$

Here, for $p = 1$, (49)–(56) reduce to (41)–(48) respectively for Gaussian pulses.

5. BIREFRINGENT FIBERS

A single mode fiber supports two degenerate modes that are polarized in two orthogonal directions. Under ideal conditions of perfect cylindrical geometry and isotropic material, a mode excited with its polarization in one direction would not couple with the mode in the orthogonal direction. However, small deviations from the cylindrical geometry or small fluctuations in material anisotropy result in a mixing of the two polarization states and the mode degeneracy is broken. Thus the mode propagation constant becomes slightly different for the modes polarized in orthogonal directions. This property is referred to as modal birefringence [8, 15, 16]. Birefringence can also be introduced artificially in optical fibers.

The propagation of solitons in birefringent nonlinear fibers has attracted much attention in recent years. It has potential applications in optical communications and optical logic devices. The equations that describe the pulse propagation through these fibers was originally derived by Menyuk [16]. They can be solved approximately in certain special cases only. The localized pulse evolution in a birefringent fiber has been studied analytically, numerically and experimentally [16] on the basis of a simplified chirp-free model without oscillating terms under the assumptions that the two polarizations exhibit different group velocities. The dimensionless equations that describe the pulse propagation in birefringent fibers are

$$i(u_z + \delta u_t) + \beta u + \frac{D(z)}{2} u_{tt} + g(z)\left(|u|^2 + \alpha|v|^2\right) u + \gamma v^2 u^* = 0 \quad (57)$$

$$i(v_z - \delta v_t) + \beta v + \frac{D(z)}{2} v_{tt} + g(z)\left(|v|^2 + \alpha|u|^2\right) v + \gamma u^2 v^* = 0 \quad (58)$$

Equations (57) and (58) are known as the Dispersion Managed Vector Nonlinear Schrodinger's Equation (DM-VNLSE). Here, u and v are slowly varying envelopes of the two linearly polarized components of the field along the x and y axis. Also, δ is the group velocity mismatch between the two polarization components and is called the birefringence parameter, β corresponds to the difference between the propagation constants, α is the cross-phase modulation coefficient and γ is the coefficient of the coherent energy coupling term. These equations are, in general, not integrable. However, they can be solved analytically only for certain specific cases [31].

In this paper, the terms with δ will be neglected as $\delta \leq 10^{-3}$ [6]. Also, neglecting β and the coherent energy coupling given by the coefficient of the γ terms, the reduced DM-VNLSE is

$$iu_z + \frac{D(z)}{2}u_{tt} + g(z)\left(|u|^2 + \alpha|v|^2\right)u = 0 \tag{59}$$

$$iv_z + \frac{D(z)}{2}v_{tt} + g(z)\left(|v|^2 + \alpha|u|^2\right)v = 0 \tag{60}$$

Equations (59) and (60) will now be studied in this paper by means of the variational principle as this is not integrable by the IST.

5.1. Integrals of Motion

The two integrals of motion of (57) and (58) are the energy (E) and the linear momentum (M) of the pulse that are respectively given by

$$E = \int_{-\infty}^{\infty}\left(|u|^2 + |v|^2\right)dt \tag{61}$$

$$M = \frac{i}{2}D(z)\int_{-\infty}^{\infty}\left(u^* u_t - uu_t^* + v^* v_t - vv_t^*\right)dt \tag{62}$$

By Noether's theorem, each of these two conserved quantities corresponds to a symmetry of the system. The conservation of energy is a result of the translational invariance of (57) and (58) relative to phase shifts, while the conservation of the momentum is a consequence of the translational invariance in t [8]. The Hamiltonian (H) of (57) and (58) is given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\Big[D(z)\left(|u_t|^2 + |v_t|^2\right) - 2\beta\left(|u|^2 - |v|^2\right) - g(z)\left(|u|^4 + |v|^4\right) \\ -i\delta\left(u^* u_t - uu_t^* + v^* v_t - vv_t^*\right) - 2\alpha|u|^2|v|^2 - (1-\alpha)\left(u^2 v^{*2} + v^2 u^{*2}\right)\Big]dt \tag{63}$$

while that of the reduced DM-VNLSE, given by (59) and (60), the Hamiltonian is

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\Big[D(z)\left(|u_t|^2 + |v_t|^2\right) \\ -g(z)\left(|u|^4 + |v|^4\right) - 2\alpha|u|^2|v|^2 - (1-\alpha)\left(u^2 v^{*2} + v^2 u^{*2}\right)\Big]dt \tag{64}$$

which is, however, not a constant of motion, unless $D(z)$ and $g(z)$ are constants in which case it is a consequence of the translational invariance in z.

Now, the solutions of (59) and (60) are assumed to be given by chirped pulses of the form

$$
\begin{aligned}
u(z,t) = {} & A_1(z) f\left[B_1(z)\left\{t - t_1(z)\right\}\right] \\
& \exp\left[iC_1(z)\left\{t - t_1(z)\right\}^2 - i\kappa_1(z)\left\{t - t_1(z)\right\} + i\theta_1(z)\right] \qquad (65)
\end{aligned}
$$

and

$$
\begin{aligned}
v(z,t) = {} & A_2(z) f\left[B_2(z)\left\{t - t_2(z)\right\}\right] \\
& \exp\left[iC_2(z)\left\{t - t_2(z)\right\}^2 - i\kappa_2(z)\left\{t - t_2(z)\right\} + i\theta_2(z)\right] \qquad (66)
\end{aligned}
$$

where f represents the shape of the pulse. Also, here the parameters $A_j(z)$, $B_j(z)$, $C_j(z)$, $\kappa_j(z)$, $t_j(z)$ and $\theta_j(z)$ (for $j = 1, 2$) respectively represent the pulse amplitudes, the inverse width of the pulses, chirps, frequencies, the centers of the pulses and the phases of the pulses respectively. Using the variational principle, a set of evolution equations for these pulse parameters will be derived. Also, for such pulse forms, the integrals of motion are

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty}\left(|u|^2 + |v|^2\right) dt = \frac{A_1^2}{B_1} I_{0,2,0}^{(1)} + \frac{A_2^2}{B_2} I_{0,2,0}^{(2)} \qquad (67) \\
M &= \frac{i}{2} D(z) \int_{-\infty}^{\infty} \left(u^* u_t - u u_t^* + v^* v_t - v v_t^*\right) dt \\
&= -D(z)\left\{\kappa_1 \frac{A_1^2}{B_1} I_{0,2,0}^{(1)} + \kappa_2 \frac{A_2^2}{B_2} I_{0,2,0}^{(2)}\right\} \qquad (68)
\end{aligned}
$$

The Hamiltonian is now given by

$$
\begin{aligned}
H = {} & \frac{1}{2}\int_{-\infty}^{\infty}\Big[D(z)\left(|u_t|^2 + |v_t|^2\right) - 2\beta\left(|u|^2 - |v|^2\right) - g(z)\left(|u|^4 + |v|^4\right) \\
& -i\delta\left(u^* u_t - u u_t^* + v^* v_t - v v_t^*\right) - 2\alpha |u|^2 |v|^2 - (1-\alpha)\left(u^2 v^{*2} + v^2 u^{*2}\right)\Big] dt \\
= {} & \frac{D(z)}{2}\left\{A_1^2 B_1 I_{0,0,2}^{(1)} + 4\frac{A_1^2 C_1^2}{B_1^3} I_{2,2,0}^{(1)} + \frac{\kappa_1^2 A_1^2}{B_1} I_{0,2,0}^{(1)}\right\} - \frac{g(z)}{2}\frac{A_1^4}{B_1} I_{0,4,0}^{(1)} \\
& + \frac{D(z)}{2}\left\{A_2^2 B_2 I_{0,0,2}^{(2)} + 4\frac{A_2^2 C_2^2}{B_2^3} I_{2,2,0}^{(2)} + \frac{\kappa_2^2 A_2^2}{B_2} I_{0,2,0}^{(2)}\right\} - \frac{g(z)}{2}\frac{A_2^4}{B_2} I_{0,4,0}^{(2)} \\
& -\beta\left\{\frac{A_1^2}{B_1} I_{0,2,0}^{(1)} - \frac{A_2^2}{B_2} I_{0,2,0}^{(2)}\right\} - \delta\left\{\kappa_1 \frac{A_1^2}{B_1} I_{0,2,0}^{(1)} + \kappa_2 \frac{A_2^2}{B_2} I_{0,2,0}^{(2)}\right\} \\
& -\alpha A_1^2 A_2^2 J_{2,2,0,0}^{0,0} + (1-\alpha) A_1 A_2 J_{1,1,0,0}^{1,0} \qquad (69)
\end{aligned}
$$

while that of the reduced DM-VNLSE the Hamiltonian simplifies to

$$
\begin{aligned}
H = &\frac{1}{2}\int_{-\infty}^{\infty}\Big[D(z)\left(|u_t|^2+|v_t|^2\right)-g(z)\left(|u|^4+|v|^4\right)\\
&-2\alpha|u|^2|v|^2-(1-\alpha)\left(u^2v^{*2}+v^2u^{*2}\right)\Big]\,dt\\
=&\frac{D(z)}{2}\left\{A_1^2B_1I^{(1)}_{0,0,2}+4\frac{A_1^2C_1^2}{B_1^3}I^{(1)}_{2,2,0}+\frac{\kappa_1^2A_1^2}{B_1}I^{(1)}_{0,2,0}\right\}\\
&+\frac{D(z)}{2}\left\{A_2^2B_2I^{(2)}_{0,0,2}+4\frac{A_2^2C_2^2}{B_2^3}I^{(2)}_{2,2,0}+\frac{\kappa_2^2A_2^2}{B_2}I^{(2)}_{0,2,0}\right\}\\
&-\frac{g(z)}{2}\left\{\frac{A_1^4}{B_1}I^{(1)}_{0,4,0}+\frac{A_2^4}{B_2}I^{(2)}_{0,4,0}\right\}-\alpha A_1^2A_2^2J^{0,0}_{2,2,0,0}+(1-\alpha)A_1A_2J^{1,0}_{1,1,0,0}
\end{aligned}
\tag{70}
$$

where

$$
I^{(l)}_{a,b,c}=\int_{-\infty}^{\infty}\tau_l^a f^b(\tau_l)\left(\frac{df}{d\tau_l}\right)^c d\tau_l \tag{71}
$$

and

$$
\begin{aligned}
J^{\alpha,\beta}_{a,b,m,n} = &\int_{-\infty}^{\infty} f^a(\tau_1)\,f^b(\tau_2)\left(\frac{df}{d\tau_1}\right)^m\left(\frac{df}{d\tau_2}\right)^n\\
&\cdot\cos\left[\alpha\left(\phi_1-\phi_2\right)\right]\sin^{\beta}\left(\phi_1-\phi_2\right)dt
\end{aligned}
\tag{72}
$$

along with

$$
\tau_l = B_l(z)\left\{t-t_l(z)\right\} \tag{73}
$$

and

$$
\phi_l = C_l\left(t-t_l\right)^2-\kappa_l\left(t-t_l\right)+\theta_l \tag{74}
$$

for positive integers a, b, c, m, n, α and β while $l=1,2$.

5.2. Variational Formulation

Since, there is no inverse scattering solution to (59) and (60) these equations will be studied approximately by means of variational method based on the observation that it supports well-defined chirped soliton solution whose shape is that of a Gaussian or a SG. For the EL equation that is given by (18), the Lagrangian is given by

$$\begin{aligned}L = \frac{1}{2}\int_{-\infty}^{\infty} &\left[i\left(u^* u_z - u u_z^*\right) + i\left(v^* v_z - v v_z^*\right)\right.\\
&+i\delta\left(v^* u_t - u v_t^*\right) + i\delta\left(u^* v_t - v u_t^*\right)\\
&-D(z)\left(|u_t|^2 + |v_t|^2\right) + g(z)\left(|u|^4 + |v|^4\right)\\
&\left.+2\alpha g(z)|u|^2|v|^2 + 2\beta\left(u^* v + u v^*\right) + \gamma\left(u^2 v^{*2} + v^2 u^{*2}\right)\right] dt \end{aligned} \quad (75)$$

and this simplifies to

$$\begin{aligned}L = &-D(z)\frac{A_1^2}{2B_1^3}\left\{B_1^4 I^{(1)}_{0,0,2} + 4C_1^2 I^{(1)}_{2,2,0} + \kappa_1^2 B_1^2 I^{(1)}_{0,2,0}\right\}\\
&+\frac{g(z)A_1^4}{2B_1} I^{(1)}_{0,4,0} - \frac{A_1^2}{B_1^3} I^{(1)}_{2,2,0}\frac{dC_1}{dz} + \frac{A_1^2}{B_1} I^{(1)}_{0,2,0}\left(t_1\frac{d\kappa_1}{dz} - \frac{d\theta_1}{dz}\right)\\
&-D(z)\frac{A_2^2}{2B_2^3}\left\{B_2^4 I^{(2)}_{0,0,2} + 4C_2^2 I^{(2)}_{2,2,0} + \kappa_2^2 B_2^2 I^{(2)}_{0,2,0}\right\}\\
&+\frac{g(z)A_2^4}{2B_2} I^{(2)}_{0,4,0} - \frac{A_2^2}{B_2^3} I^{(2)}_{2,2,0}\frac{dC_2}{dz} + \frac{A_2^2}{B_2} I^{(2)}_{0,2,0}\left(t_2\frac{d\kappa_2}{dz} - \frac{d\theta_2}{dz}\right)\\
&+2\delta A_1 A_2\left\{\left(\kappa_1 + \kappa_2\right) J^{1,0}_{1,1,0,0} + B_2 J^{0,1}_{1,0,0,1} + B_1 J^{0,1}_{0,1,1,0}\right\}\\
&+4\beta A_1 A_2 J^{1,0}_{1,1,0,0} + 2\gamma A_1^2 A_2^2 J^{1,0}_{2,2,0,0} + \alpha g(z) A_1^2 A_2^2 J^{0,0}_{2,2,0,0}\end{aligned} \quad (76)$$

For the reduced DM-VNLSE, the Lagrangian is given by

$$\begin{aligned}L = \frac{1}{2}\int_{-\infty}^{\infty} &\left[i\left(u^* u_z - u u_z^*\right) + i\left(v^* v_z - v v_z^*\right)\right.\\
&\left.-D(z)\left(|u_t|^2 + |v_t|^2\right) + g(z)\left(|u|^4 + |v|^4\right) + 2\alpha g(z)|u|^2|v|^2\right] dt\end{aligned} \quad (77)$$

which simplifies to

$$\begin{aligned}L = &-D(z)\frac{A_1^2}{2B_1^3}\left\{B_1^4 I^{(1)}_{0,0,2} + 4C_1^2 I^{(1)}_{2,2,0} + \kappa_1^2 B_1^2 I^{(1)}_{0,2,0}\right\}\\
&+\frac{g(z)A_1^4}{2B_1} I^{(1)}_{0,4,0} - \frac{A_1^2}{B_1^3} I^{(1)}_{2,2,0}\frac{dC_1}{dz} + \frac{A_1^2}{B_1} I^{(1)}_{0,2,0}\left(t_1\frac{d\kappa_1}{dz} - \frac{d\theta_1}{dz}\right)\\
&-D(z)\frac{A_2^2}{2B_2^3}\left\{B_2^4 I^{(2)}_{0,0,2} + 4C_2^2 I^{(2)}_{2,2,0} + \kappa_2^2 B_2^2 I^{(2)}_{0,2,0}\right\}\\
&+\frac{g(z)A_2^4}{2B_2} I^{(2)}_{0,4,0} - \frac{A_2^2}{B_2^3} I^{(2)}_{2,2,0}\frac{dC_2}{dz} + \frac{A_2^2}{B_2} I^{(2)}_{0,2,0}\left(t_2\frac{d\kappa_2}{dz} - \frac{d\theta_2}{dz}\right)\\
&+\alpha g(z) A_1^2 A_2^2 J^{0,0}_{2,2,0,0}\end{aligned} \quad (78)$$

The EL equation given by (18) shall now be utilized to obtain the dynamics of pulse parameters for birefringent fibers. In the EL equation, p represents one of the twelve soliton parameters. Substituting A_j, B_j, C_j, κ_j, t_j and θ_j $(j = 1, 2)$ for r in (18) the following set of equations are obtained

$$\frac{dA_1}{dz} = -D(z)A_1C_1 \tag{79}$$

$$\frac{dB_1}{dz} = -2D(z)B_1C_1 \tag{80}$$

$$\frac{dC_1}{dz} = D(z)\left\{\frac{B_1^4}{2}\frac{I_{0,0,2}^{(1)}}{I_{2,2,0}^{(1)}} - 2C_1^2\right\} - \frac{g(z)A_1^2B_1^2}{4}\frac{I_{0,4,0}^{(1)}}{I_{2,2,0}^{(1)}} - \frac{\alpha g(z)}{2}A_2^2B_1^3\frac{J_{2,2,0,0}^{0,0}}{I_{2,2,0}^{(1)}} \tag{81}$$

$$\frac{d\kappa_1}{dz} = 0 \tag{82}$$

$$\frac{dt_1}{dz} = -D(z)\kappa_1 \tag{83}$$

$$\frac{d\theta_1}{dz} = D(z)\left\{\frac{\kappa_1^2}{2} - B_1^2\frac{I_{0,0,2}^{(1)}}{I_{0,2,0}^{(1)}}\right\} + \frac{5g(z)A_1^2}{4}\frac{I_{0,4,0}^{(1)}}{I_{0,2,0}^{(1)}} + \frac{3}{2}\alpha g(z)A_2^2B_1\frac{J_{2,2,0,0}^{0,0}}{I_{0,2,0}^{(1)}} \tag{84}$$

$$\frac{dA_2}{dz} = -D(z)A_2C_2 \tag{85}$$

$$\frac{dB_2}{dz} = -2D(z)B_2C_2 \tag{86}$$

$$\frac{dC_2}{dz} = D(z)\left\{\frac{B_2^4}{2}\frac{I_{0,0,2}^{(2)}}{I_{2,2,0}^{(2)}} - 2C_2^2\right\} - \frac{g(z)A_2^2B_2^2}{4}\frac{I_{0,4,0}^{(2)}}{I_{2,2,0}^{(2)}} - \frac{\alpha g(z)}{2}A_1^2B_2^3\frac{J_{2,2,0,0}^{0,0}}{I_{2,2,0}^{(2)}} \tag{87}$$

$$\frac{d\kappa_2}{dz} = 0 \tag{88}$$

$$\frac{dt_2}{dz} = -D(z)\kappa_2 \tag{89}$$

$$\frac{d\theta_2}{dz} = D(z)\left\{\frac{\kappa_2^2}{2} - B_2^2\frac{I_{0,0,2}^{(2)}}{I_{0,2,0}^{(2)}}\right\} + \frac{5g(z)A_2^2}{4}\frac{I_{0,4,0}^{(2)}}{I_{0,2,0}^{(2)}} + \frac{3}{2}\alpha g(z)A_1^2 B_2\frac{J_{2,2,0,0}^{0,0}}{I_{0,2,0}^{(2)}} \tag{90}$$

Now, from (79) and (80) it is easy to see that $A_1 = K_1\sqrt{B_1}$ for some constant K_1 and again from (85) and (86), $A_2 = K_2\sqrt{B_2}$ for some constant K_2. So, the number of parameters reduces by two. Therefore, (79)–(90) respectively modify to

$$\frac{dB_1}{dz} = -2D(z)B_1C_1 \tag{91}$$

$$\frac{dC_1}{dz} = D(z)\left\{\frac{B_1^4}{2}\frac{I_{0,0,2}^{(1)}}{I_{2,2,0}^{(1)}} - 2C_1^2\right\} - \frac{K_1^2 g(z)B_1^3}{4}\frac{I_{0,4,0}^{(1)}}{I_{2,2,0}^{(1)}} - \frac{\alpha g(z)}{2}K_2^2 B_1^3 B_2\frac{J_{2,2,0,0}^{0,0}}{I_{2,2,0}^{(1)}} \tag{92}$$

$$\frac{d\kappa_1}{dz} = 0 \tag{93}$$

$$\frac{dt_1}{dz} = -D(z)\kappa_1 \tag{94}$$

$$\frac{d\theta_1}{dz} = D(z)\left\{\frac{\kappa_1^2}{2} - \frac{I_{0,0,2}^{(1)}}{I_{0,2,0}^{(1)}}B_1^2\right\} + \frac{5g(z)K_1^2 B_1}{4}\frac{I_{0,4,0}^{(1)}}{I_{0,2,0}^{(1)}} + \frac{3}{2}\alpha g(z)K_2^2 B_1 B_2\frac{J_{2,2,0,0}^{0,0}}{I_{0,2,0}^{(1)}} \tag{95}$$

$$\frac{dB_2}{dz} = -2D(z)B_2C_2 \tag{96}$$

$$\frac{dC_2}{dz} = D(z)\left\{\frac{B_2^4}{2}\frac{I_{0,0,2}^{(2)}}{I_{2,2,0}^{(2)}} - 2C_2^2\right\} - \frac{K_2^2 g(z)B_2^3}{4}\frac{I_{0,4,0}^{(2)}}{I_{2,2,0}^{(2)}} - \frac{\alpha g(z)}{2}K_1^2 B_2^3 B_1\frac{J_{2,2,0,0}^{0,0}}{I_{2,2,0}^{(2)}} \tag{97}$$

$$\frac{d\kappa_2}{dz} = 0 \tag{98}$$

$$\frac{dt_2}{dz} = -D(z)\kappa_2 \tag{99}$$

$$\begin{aligned}\frac{d\theta_2}{dz} &= D(z)\left\{\frac{\kappa_2^2}{2}-\frac{I_{0,0,2}^{(2)}}{I_{0,2,0}^{(2)}}B_2^2\right\}+\frac{5g(z)K_2^2B_2}{4}\frac{I_{0,4,0}^{(2)}}{I_{0,2,0}^{(2)}}\\&\quad+\frac{3}{2}\alpha g(z)K_1^2B_1B_2\frac{J_{2,2,0,0}^{0,0}}{I_{0,2,0}^{(2)}}\end{aligned} \tag{100}$$

In the following two subsections, the explicit parameter dynamics of Gaussian and SG solitons will be obtained.

5.2.1. Gaussian Pulses

For a Gaussian pulse, $f(\tau)=e^{-\tau^2/2}$ so that the conserved quantities are

$$\begin{aligned}E &= \int_{-\infty}^{\infty}\left(|u|^2+|v|^2\right)dt=\left(\frac{A_1^2}{B_1}+\frac{A_2^2}{B_2}\right)\sqrt{\frac{\pi}{2}}\\&=\left(K_1^2+K_2^2\right)\sqrt{\frac{\pi}{2}}\end{aligned} \tag{101}$$

$$\begin{aligned}M &= \frac{i}{2}D(z)\int_{-\infty}^{\infty}(u^*u_t-uu_t^*+v^*v_t-vv_t^*)dt\\&=-D(z)\left(\kappa_1\frac{A_1^2}{B_1}+\kappa_2\frac{A_2^2}{B_2}\right)\sqrt{\frac{\pi}{2}}\\&=-D(z)\left(\kappa_1K_1^2+\kappa_2K_2^2\right)\sqrt{\frac{\pi}{2}}\end{aligned} \tag{102}$$

while the Hamiltonian is given by

$$\begin{aligned}H&=\frac{1}{2}\int_{-\infty}^{\infty}\Big[D(z)\left(|u_t|^2+|v_t|^2\right)-2\beta\left(|u|^2-|v|^2\right)-g(z)\left(|u|^4+|v|^4\right)\\&\quad-i\delta\left(u^*u_t-uu_t^*+v^*v_t-vv_t^*\right)-2\alpha|u|^2|v|^2-(1-\alpha)\left(u^2v^{*2}+v^2u^{*2}\right)\Big]dt\\&=\frac{D(z)K_1^2}{2}\left(B_1^2+\frac{C_1^2}{B_1^2}+\kappa_1^2\right)\sqrt{\frac{\pi}{2}}-g(z)B_1K_1^4\frac{\sqrt{\pi}}{4}\\&\quad+\frac{D(z)K_2^2}{2}\left(B_2^2+\frac{C_2^2}{B_2^2}+\kappa_2^2\right)\sqrt{\frac{\pi}{2}}-g(z)B_2K_2^4\frac{\sqrt{\pi}}{4}\\&\quad-\beta\sqrt{\pi}\left\{\frac{A_1^2}{B_1}-\frac{A_2^2}{B_2}\right\}-\delta\sqrt{\pi}\left\{\kappa_1\frac{A_1^2}{B_1}+\kappa_2\frac{A_2^2}{B_2}\right\}\\&\quad-\alpha A_1^2A_2^2\sqrt{\frac{\pi}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2}\end{aligned}$$

$$+(1-\alpha)A_1A_2\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^2\left(t-t_1\right)^2+B_2^2\left(t-t_2\right)^2\right\}\right]$$
$$\cos\left[C_1\left(t-t_1\right)^2-C_2\left(t-t_2\right)^2+\kappa_2(t-t_2)-\kappa_1(t-t_1)+(\theta_1-\theta_2)\right]dt \tag{103}$$

while for the reduced DM-VNLSE the Hamiltonian is

$$\begin{aligned}
H &= \frac{1}{2}\int_{-\infty}^{\infty}\left[D(z)\left(|u_t|^2+|v_t|^2\right)-g(z)\left(|u|^4+|v|^4\right)\right.\\
&\quad\left.-2\alpha|u|^2|v|^2-(1-\alpha)\left(u^2v^{*2}+v^2u^{*2}\right)\right]dt\\
&= \frac{D(z)}{2}\sqrt{\frac{\pi}{2}}\left\{K_1^2\left(B_1^2+\frac{C_1^2}{B_1^2}+\kappa_1^2\right)+K_2^2\left(B_2^2+\frac{C_2^2}{B_2^2}+\kappa_2^2\right)\right\}\\
&\quad-g(z)\frac{\sqrt{\pi}}{4}\left(B_1K_1^4+B_2K_2^4\right)-\alpha A_1^2A_2^2\sqrt{\frac{\pi}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2}\\
&\quad+(1-\alpha)A_1A_2\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^2\left(t-t_1\right)^2+B_2^2\left(t-t_2\right)^2\right\}\right]\\
&\quad\cos\left[C_1\left(t-t_1\right)^2-C_2\left(t-t_2\right)^2+\kappa_2(t-t_2)-\kappa_1(t-t_1)+(\theta_1-\theta_2)\right]dt
\end{aligned} \tag{104}$$

The parameter evolution equations (91)-(100) respectively reduce to

$$\frac{dB_1}{dz} = -2D(z)B_1C_1 \tag{105}$$

$$\begin{aligned}\frac{dC_1}{dz} &= 2D(z)\left(B_1^4-C_1^2\right)-\frac{\sqrt{2}}{2}g(z)K_1^2B_1^3\\
&\quad -2\alpha g(z)K_2^2B_1^3B_2\sqrt{\frac{2}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2}\end{aligned} \tag{106}$$

$$\frac{d\kappa_1}{dz} = 0 \tag{107}$$

$$\frac{dt_1}{dz} = -D(z)\kappa_1 \tag{108}$$

$$\begin{aligned}\frac{d\theta_1}{dz} &= \frac{D(z)}{2}\left(\kappa_1^2-B_1^2\right)+\frac{5}{4\sqrt{2}}g(z)K_1^2B_1\\
&\quad +\frac{3}{2}\alpha g(z)K_2^2B_1B_2\sqrt{\frac{2}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2}\end{aligned} \tag{109}$$

$$\frac{dB_2}{dz} = -2D(z)B_2C_2 \tag{110}$$

$$\begin{aligned}\frac{dC_2}{dz} &= 2D(z)\left(B_2^4 - C_2^2\right) - \frac{\sqrt{2}}{2}g(z)K_2^2B_2^3 \\ &\quad -2\alpha g(z)K_1^2B_2^3B_1\sqrt{\frac{2}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2} \qquad (111)\\ \frac{d\kappa_2}{dz} &= 0 \qquad (112)\\ \frac{dt_2}{dz} &= -D(z)\kappa_2 \qquad (113)\\ \frac{d\theta_2}{dz} &= \frac{D(z)}{2}\left(\kappa_2^2 - B_2^2\right) + \frac{5}{4\sqrt{2}}g(z)K_2^2B_2 \\ &\quad +\frac{3}{2}\alpha g(z)K_1^2B_1B_2\sqrt{\frac{2}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2} \qquad (114)\end{aligned}$$

These equations are useful in studying the various physical aspects of the solitons in birefringent optical fibers namely the timing, amplitude or the frequency jitter, the evolution of the coherent energy [14] and much more.

5.2.2. Super-Gaussian Pulses

For SG pulse, $f(\tau) = e^{-\frac{1}{2}\tau^{2p}}$ as in the case of a polarization preserving fiber. The conserved quantities for SG pulses are

$$\begin{aligned} E &= \int_{-\infty}^{\infty}\left(|u|^2+|v|^2\right)dt \\ &= D(z)\left(\frac{A_1^2}{B_1}+\frac{A_2^2}{B_2}\right)\sqrt{\frac{\pi}{2}} = \frac{D(z)}{p2^{\frac{1}{2p}}}\left(K_1^2+K_2^2\right)\Gamma\left(\frac{1}{2p}\right) \qquad (115)\\ M &= \frac{i}{2}D(z)\int_{-\infty}^{\infty}(u^*u_t - uu_t^* + v^*v_t - vv_t^*)dt \\ &= -D(z)\left(\kappa_1\frac{A_1^2}{B_1}+\kappa_2\frac{A_2^2}{B_2}\right)\sqrt{\frac{\pi}{2}} = -\frac{D(z)}{p2^{\frac{1}{2p}}}\left(\kappa_1K_1^2+\kappa_2K_2^2\right)\Gamma\left(\frac{1}{2p}\right) \\ & \qquad (116)\end{aligned}$$

while the Hamiltonian here is

$$\begin{aligned} H = \frac{1}{2}\int_{-\infty}^{\infty}\Big[&D(z)\left(|u_t|^2+|v_t|^2\right) - 2\beta\left(|u|^2-|v|^2\right) - g(z)\left(|u|^4+|v|^4\right) \\ &-i\delta\left(u^*u_t - uu_t^* + v^*v_t - vv_t^*\right) - 2\alpha|u|^2|v|^2 - (1-\alpha)\left(u^2v^{*2}+v^2u^{*2}\right)\Big]dt\end{aligned}$$

$$
\begin{aligned}
&= D(z)K_1^2\left[B_1^2\frac{p}{2^{\frac{2p-1}{2p}}}\Gamma\left(\frac{4p-1}{2p}\right)+\frac{C_1^2}{B_1^2}\frac{2^{\frac{2p-3}{2p}}}{p}\Gamma\left(\frac{3}{2p}\right)+\kappa_1^2\frac{1}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{2p}\right)\right]\\
&+D(z)K_2^2\left[B_2^2\frac{p}{2^{\frac{2p-1}{2p}}}\Gamma\left(\frac{4p-1}{2p}\right)+\frac{C_2^2}{B_2^2}\frac{2^{\frac{2p-3}{2p}}}{p}\Gamma\left(\frac{3}{2p}\right)+\kappa_2^2\frac{1}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{2p}\right)\right]\\
&-g(z)\left(B_1K_1^4+B_2K_2^4\right)\frac{1}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{p}\right)\\
&-\frac{\beta}{p}\left\{\frac{A_1^2}{B_1}-\frac{A_2^2}{B_2}\right\}\Gamma\left(\frac{1}{p}\right)-\frac{\delta}{p}\left\{\kappa_1\frac{A_1^2}{B_1}-\kappa_2\frac{A_2^2}{B_2}\right\}\Gamma\left(\frac{1}{p}\right)\\
&-\alpha A_1^2A_2^2\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p}+B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]dt\\
&-(1-\alpha)A_1A_2\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p}+B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]\\
&\cos\left[C_1\left(t-t_1\right)^2-C_2\left(t-t_2\right)^2+\kappa_2(t-t_2)-\kappa_1(t-t_1)+(\theta_1-\theta_2)\right]dt
\end{aligned}
\tag{117}
$$

and for the reduced DM-VNLSE the Hamiltonian simplifies to

$$
\begin{aligned}
H=&\frac{1}{2}\int_{-\infty}^{\infty}\left[D(z)\left(|u_t|^2+|v_t|^2\right)-g(z)\left(|u|^4+|v|^4\right)\right.\\
&\left.-2\alpha|u|^2|v|^2-(1-\alpha)\left(u^2v^{*2}+v^2u^{*2}\right)\right]dt\\
=&D(z)K_1^2\left[B_1^2\frac{p}{2^{\frac{2p-1}{2p}}}\Gamma\left(\frac{4p-1}{2p}\right)+\frac{C_1^2}{B_1^2}\frac{2^{\frac{2p-3}{2p}}}{p}\Gamma\left(\frac{3}{2p}\right)+\kappa_1^2\frac{1}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{2p}\right)\right]\\
&+D(z)K_2^2\left[B_2^2\frac{p}{2^{\frac{2p-1}{2p}}}\Gamma\left(\frac{4p-1}{2p}\right)+\frac{C_2^2}{B_2^2}\frac{2^{\frac{2p-3}{2p}}}{p}\Gamma\left(\frac{3}{2p}\right)+\kappa_2^2\frac{1}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{2p}\right)\right]\\
&-g(z)\left(B_1K_1^4+B_2K_2^4\right)\frac{1}{p2^{\frac{2p+1}{p}}}\Gamma\left(\frac{1}{p}\right)\\
&-\alpha A_1^2A_2^2\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p}+B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]dt\\
&-(1-\alpha)A_1A_2\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p}+B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]\\
&\cos\left[C_1\left(t-t_1\right)^2-C_2\left(t-t_2\right)^2+\kappa_2(t-t_2)-\kappa_1(t-t_1)+(\theta_1-\theta_2)\right]dt
\end{aligned}
\tag{118}
$$

Also, the parameter dynamics given by (91)–(100) respectively reduce to

$$\frac{dB_1}{dz} = -2D(z)B_1C_1 \tag{119}$$

$$\begin{aligned}\frac{dC_1}{dz} = {} & D(z)\left\{\frac{p^2}{2^{\frac{p-2}{p}}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}B_1^4 - 2C_1^2\right\} \\ & -g(z)K_1^2B_1^3\frac{1}{2^{\frac{4p+1}{2p}}}\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)} - \frac{p}{2^{\frac{2p-3}{p}}}\alpha g(z)\frac{K_2^2B_1^3B_2}{\Gamma\left(\frac{3}{2p}\right)} \\ & \int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p} + B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]dt \end{aligned} \tag{120}$$

$$\frac{d\kappa_1}{dz} = 0 \tag{121}$$

$$\frac{dt_1}{dz} = -D(z)\kappa_1 \tag{122}$$

$$\begin{aligned}\frac{d\theta_1}{dz} = {} & D(z)\left\{\frac{\kappa_1^2}{2} - p^2 2^{\frac{1}{p}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)}B_1^2\right\} \\ & +\frac{5}{2^{\frac{4p+1}{2p}}}g(z)K_1B_1 + \frac{3p}{2^{\frac{2p-1}{2p}}}\alpha g(z)\frac{K_2^2B_1B_2}{\Gamma\left(\frac{1}{2p}\right)} \\ & \int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p} + B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]dt \end{aligned} \tag{123}$$

$$\frac{dB_2}{dz} = -2D(z)B_2C_2 \tag{124}$$

$$\begin{aligned}\frac{dC_2}{dz} = {} & D(z)\left\{\frac{p^2}{2^{\frac{p-2}{p}}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}B_2^4 - 2C_2^2\right\} \\ & -g(z)K_2^2B_2^3\frac{1}{2^{\frac{4p+1}{2p}}}\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)} - \frac{p}{2^{\frac{2p-3}{p}}}\alpha g(z)\frac{K_1^2B_2^3B_1}{\Gamma\left(\frac{3}{2p}\right)} \\ & \int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p} + B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]dt \end{aligned} \tag{125}$$

$$\frac{d\kappa_2}{dz} = 0 \tag{126}$$

$$\frac{dt_2}{dz} = -D(z)\kappa_2 \tag{127}$$

$$\begin{aligned}\frac{d\theta_2}{dz} &= D(z)\left\{\frac{\kappa_2^2}{2} - p^2 2^{\frac{1}{p}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)}B_2^2\right\} \\ &\quad +\frac{5}{2^{\frac{4p+1}{2p}}}g(z)K_2B_2 + \frac{3p}{2^{\frac{2p-1}{2p}}}\alpha g(z)\frac{K_1^2B_1B_2}{\Gamma\left(\frac{1}{2p}\right)} \\ &\quad \int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}\left(t-t_1\right)^{2p} + B_2^{2p}\left(t-t_2\right)^{2p}\right\}\right]dt \quad (128)\end{aligned}$$

6. MULTIPLE CHANNELS

The successful design of low-loss dispersion-shifted and dispersion-flattened optical fibers with low dispersion over relatively large wavelength range can be used to reduce or completely eliminate the group velocity mismatch for the multi-channel WDM systems resulting in the desirable simultaneous arrival of time aligned bit pulses, thus creating a new class of bit-parallel wavelength links that is used in high speed single fiber computer buses. In spite of the intrinsically small value of the nonlinearity-induced change in the refractive index of fused silica, nonlinear effects in optical fibers cannot be ignored even at relatively low powers. In particular, in WDM systems with simultaneous transmission of pulses of different wavelengths, the cross-phase modulation (XPM) effects needs to be taken into account. Although the XPM will not cause the energy to be exchanged among the different wavelengths, it will lead to the interaction of pulses and thus the pulse positions and shapes gets altered significantly.

The multi-channel WDM transmission of co-propagating wave envelopes in a nonlinear optical fiber, including the XPM effect, can be modeled [7, 18] by the following N-coupled NLSE in the dimensionless form

$$iq_z^{(l)} + \frac{D(z)}{2}q_{tt}^{(l)} + g(z)\left\{\left|q^{(l)}\right|^2 + \sum_{m\neq l}^{N}\alpha_{lm}\left|q^{(m)}\right|^2\right\}q^{(l)} = 0 \qquad (129)$$

where $1 \leq l \leq N$. Equation (129) is the model for bit-parallel WDM soliton transmission. Here α_{lm} are known as the XPM coefficients. It is well known [7, 17] that the straightforward use of this system for description of WDM transmission could potentially give incorrect results. However, this model can be applied to describe WDM transmission for dispersion flattened fibers, the dispersion of which weakly depends on the operating wavelength.

Another important medium in which the model given by (129) arises is the photorefractive medium [12]. In the case of incoherent beam propagation in a biased photorefractive crystal, which is a noninstaneous nonlinear media, the diffraction behaviour of that incoherent beam is to be treated somewhat differently. The diffraction behaviour of an incoherent beam can be effectively described by the sum of the intensity contributions from all its coherent components. Then the governing equation of N self-trapped mutually incoherent wave packets in such a media is given by (129).

Equation (129) is, in general, not integrable. However, it can be solved analytically for certain very specific cases, namely when $D(z) = g(z) = 1$ alongwith $\alpha_{lm} = 1$, $\forall m, n$ with $N = 2$. [15, 16].

6.1. Integrals of Motion

Equation (129) does not have infinitely many conservation laws either. In fact, it has at least two integrals of motion and they are the energy (E) and the linear momentum (M) that are respectively given by

$$E = \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left| q^{(l)} \right|^2 dt \tag{130}$$

and

$$M = \frac{i}{2} D(z) \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left(q^{(l)*} q_t^{(l)} - q^{(l)} q_t^{(l)*} \right) dt \tag{131}$$

The Hamiltonian (H) given by

$$H = \frac{1}{2} \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left\{ D(z) \left| q_t^{(l)} \right|^2 - g(z) \sum_{m \neq l}^{N} \alpha_{lm} \left| q^{(l)} \right|^2 \left| q^{(m)} \right|^2 \right\} dt \tag{132}$$

is, however, not a conserved quantity unless, in addition to $D(z)$ and $g(z)$ being constants, the matrix of XPM coefficients $\Lambda = (\alpha_{ij})_{N \times N}$ is a symmetric matrix namely $\alpha_{ij} = \alpha_{ji}$ for $1 \leq i, j \leq N$. Thus, for a birefringent fiber, the matrix should be of the form

$$\Lambda = \begin{bmatrix} 0 & \alpha_{12} \\ \alpha_{12} & 0 \end{bmatrix} \tag{133}$$

while for a triple channeled fiber,

$$\Lambda = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & 0 & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & 0 \end{bmatrix} \tag{134}$$

and so on. Now, the solution of (129) is assumed to be given by a chirped pulse, in the lth core, of the form [6, 10, 11]

$$q^{(l)}(z,t) = A_l(z) f\left[B_l(z)\left\{t - t_l(z)\right\}\right] \exp\left[iC_l(z)\left\{t - t_l(z)\right\}^2 - i\kappa_l(z)\left\{t - t_l(z)\right\} + i\theta_l(z)\right] \quad (135)$$

where f represents the shape of the pulse. It could be a Gaussian type or a SG type pulse. Also, here the parameters $A_l(z)$, $B_l(z)$, $C_l(z)$, $\kappa_l(z)$, $t_l(z)$ and $\theta_l(z)$ respectively represent the soliton amplitude, the inverse width of the pulse, chirp, frequency, the center of the pulse and the phase of the pulse in the lth channel. Using the variational principle, a set of evolution equations for the pulse parameters will be derived. For convenience, the following integral is defined

$$J_{l,m} = \prod_{j=l,m} \int_{-\infty}^{\infty} f^2\left[B_j(z)\left(t - t_j(z)\right)\right] dt \quad (136)$$

where $1 \leq l, m \leq N$ while equation (66) is now valid for $1 \leq l \leq N$. For such a pulse form given by (129), the integrals of motion are

$$E = \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left|q^{(l)}\right|^2 dt = \sum_{l=1}^{N} \frac{A_l^2}{B_l} I_{0,2,0}^{(l)} \quad (137)$$

$$M = \frac{i}{2} D(z) \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left(q^{(l)} q_t^{(l)*} - q^{(l)*} q_t^{(l)}\right) dt = \sum_{l=1}^{N} \kappa_l \frac{A_l^2}{B_l} I_{0,2,0}^{(l)} \quad (138)$$

while the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2} \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left\{ D(z) \left|q_t^{(l)}\right|^2 - g(z) \sum_{m \neq l}^{N} \alpha_{lm} \left|q^{(l)}\right|^2 \left|q^{(m)}\right|^2 \right\} dt \\ &= \frac{D(z)}{2} \sum_{l=1}^{N} \frac{A_l^2}{B_l^3} \left\{ B_l^4 I_{0,0,2}^{(l)} + 4C_l^2 I_{2,2,0}^{(l)} + \kappa_l^2 B_l^2 I_{0,2,0}^{(l)} \right\} \\ &\quad - \frac{g(z)}{2} \sum_{l=1}^{N} \sum_{m \neq l}^{N} \alpha_{lm} A_l^2 A_m^2 J_{l,m} \end{aligned} \quad (139)$$

6.2. Variational Formulation

For solitons in multiple channels, governed by (129), the Lagrangian is given by

$$L = \frac{1}{2} \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left[i\left(q^{(l)*} q_z^{(l)} - q^{(l)} q_z^{(l)*}\right) \right.$$

$$-D(z)\left|q_t^{(l)}\right|^2+g(z)\left|q^{(l)}\right|^4+2g(z)\sum_{m\neq l}^{N}\alpha_{lm}\left|q^{(l)}\right|^2\left|q^{(m)}\right|^2\Bigg]dt \quad (140)$$

Now, using (71) and (129), the Lagrangian reduces to

$$L=\sum_{l=1}^{N}\left[-D(z)\frac{A_l^2}{2B_l^3}\left\{B_l^4 I_{0,0,2}^{(l)}+4C_l^2 I_{2,2,0}^{(l)}+\kappa_l^2 B_l^2 I_{0,2,0}^{(l)}\right\}-\frac{g(z)}{2}\frac{A_l^4}{B_l}I_{0,4,0}^{(l)}\right.$$
$$\left.-\frac{A_l^2}{B_l^3}\frac{dC_l}{dz}I_{2,2,0}^{(l)}+\frac{A_l^2}{B_l}\left(t_l\frac{d\kappa_l}{dz}-\frac{d\theta_l}{dz}\right)I_{0,2,0}^{(l)}+g(z)A_l^2\sum_{m\neq l}^{N}\alpha_{lm}A_m^2 J_{l,m}\right] \quad (141)$$

Substituting A_l, B_l, C_l, κ_l, t_l and θ_l for r in (18) yields the following set of equations

$$\frac{dA_l}{dz} = -D(z)A_lC_l \quad (142)$$
$$\frac{dB_l}{dz} = -2D(z)B_lC_l \quad (143)$$
$$\frac{dC_l}{dz} = \frac{D(z)}{2}\left\{B_l^4\frac{I_{0,0,2}^{(l)}}{I_{2,2,0}^{(l)}}-4C_l^2\right\}-\frac{g(z)}{4}A_l^2B_l^2\frac{I_{0,4,0}^{(l)}}{I_{2,2,0}^{(l)}}$$
$$-\frac{g(z)}{2I_{2,2,0}^{(l)}}B_l^3\sum_{m\neq l}^{N}\alpha_{lm}A_m^2J_{l,m} \quad (144)$$
$$\frac{d\kappa_l}{dz} = 0 \quad (145)$$
$$\frac{dt_l}{dz} = -D(z)\kappa_l \quad (146)$$
$$\frac{d\theta_l}{dz} = \frac{D(z)}{2}\left\{\kappa_l^2-2B_l^2\frac{I_{0,0,2}^{(l)}}{I_{0,2,0}^{(l)}}\right\}+\frac{5}{4}g(z)A_l^2\frac{I_{0,4,0}^{(l)}}{I_{0,2,0}^{(l)}}$$
$$+\frac{3}{2}\frac{g(z)}{I_{0,2,0}^{(l)}}B_l\sum_{m\neq l}^{N}\alpha_{lm}A_m^2J_{l,m} \quad (147)$$

Now, from (142) and (143) it can be concluded that $A_l = K_l\sqrt{B_l}$ where the constant K_l is proportional to the square root of the energy of the pulse in the lth channel as seen from (137). So, the number of parameters reduce by N. Thus, (142)–(147), respectively, modify to

$$\frac{dB_l}{dz} = -2D(z)B_lC_l \quad (148)$$

$$\begin{aligned}
\frac{dC_l}{dz} &= \frac{D(z)}{2}\left\{B_l^4\frac{I_{0,0,2}^{(l)}}{I_{2,2,0}^{(l)}} - 4C_l^2\right\} - \frac{g(z)}{4}K_l^2 B_l^3\frac{I_{0,4,0}^{(l)}}{I_{2,2,0}^{(l)}} \\
&\quad -\frac{g(z)}{2}\frac{B_l^3}{I_{2,2,0}^{(l)}}\sum_{m\neq l}^{N}\alpha_{lm}K_m^2 B_m J_{l,m} \qquad (149) \\
\frac{d\kappa_l}{dz} &= 0 \qquad (150) \\
\frac{dt_l}{dz} &= -D(z)\kappa_l \qquad (151) \\
\frac{d\theta_l}{dz} &= \frac{D(z)}{2}\left\{\kappa_l^2 - 2B_l^2\frac{I_{0,0,2}^{(l)}}{I_{0,2,0}^{(l)}}\right\} + \frac{5}{4}g(z)K_l^2 B_l\frac{I_{0,4,0}^{(l)}}{I_{0,2,0}^{(l)}} \\
&\quad +\frac{3}{2}g(z)\frac{B_l}{I_{0,2,0}^{(l)}}\sum_{m\neq l}^{N}\alpha_{lm}K_m^2 B_m J_{l,m} \qquad (152)
\end{aligned}$$

6.2.1. Gaussian Pulses

For a pulse of Gaussian type, $f(\tau) = e^{-\frac{1}{2}\tau^2}$. Thus, the conserved quantities respectively reduce to

$$E = \sum_{l=1}^{N}\int_{-\infty}^{\infty}\left|q^{(l)}\right|^2 dt = \sqrt{\frac{\pi}{2}}\sum_{l=1}^{N}\frac{A_l^2}{B_l} = \sqrt{\frac{\pi}{2}}\sum_{l=1}^{N}K_l^2 \qquad (153)$$

$$\begin{aligned}
M &= \frac{i}{2}D(z)\sum_{l=1}^{N}\int_{-\infty}^{\infty}\left(q^{(l)}q_t^{(l)*} - q^{(l)*}q_t^{(l)}\right)dt \\
&= D(z)\sqrt{\frac{\pi}{2}}\sum_{l=1}^{N}\kappa_l\frac{A_l^2}{B_l} = D(z)\sqrt{\frac{\pi}{2}}\sum_{l=1}^{N}\kappa_l K_l^2 \qquad (154)
\end{aligned}$$

while the Hamiltonian is

$$\begin{aligned}
H &= \frac{1}{2}\sum_{l=1}^{N}\int_{-\infty}^{\infty}\left\{D(z)\left|q_t^{(l)}\right|^2 - g(z)\sum_{m\neq l}^{N}\alpha_{lm}\left|q^{(l)}\right|^2\left|q^{(m)}\right|^2\right\}dt \\
&= \frac{D(z)}{2}\sqrt{\frac{\pi}{2}}\sum_{l=1}^{N}K_l^2\left(B_l^2 + \frac{C_l^2}{B_l^2} + \kappa_l^2\right) \\
&\quad -\frac{g(z)}{2}\sum_{l=1}^{N}\sum_{m\neq l}^{N}\alpha_{lm}K_l^2K_m^2B_lB_m\sqrt{\frac{2\pi}{B_l^2+B_m^2}}
\end{aligned}$$

$$\cdot \exp\left\{-\frac{B_l^2 B_m^2}{2\left(B_l^2+B_m^2\right)}\left(t_l-t_m\right)^2\right\} \tag{155}$$

Also, the parameter dynamics given by (148)–(152) respectively are

$$\frac{dB_l}{dz} = -2D(z)B_lC_l \tag{156}$$

$$\begin{aligned}\frac{dC_l}{dz} &= 2D(z)\left(B_l^4-C_l^2\right)-\frac{\sqrt{2}}{2}g(z)K_l^2B_l^3\\ &\quad -\sqrt{2}g(z)B_l^3\sum_{m\neq l}^{N}\frac{\alpha_{lm}K_m^2B_m}{\sqrt{B_l^2+B_m^2}}\\ &\quad \cdot \exp\left\{-\frac{B_l^2 B_m^2}{2\left(B_l^2+B_m^2\right)}\left(t_l-t_m\right)^2\right\}\end{aligned} \tag{157}$$

$$\frac{d\kappa_l}{dz} = 0 \tag{158}$$

$$\frac{dt_l}{dz} = -D(z)\kappa_l \tag{159}$$

$$\begin{aligned}\frac{d\theta_l}{dz} &= \frac{D(z)}{2}\left(\kappa_l^2-B_l^2\right)+\frac{5\sqrt{2}}{8}g(z)K_l^2B_l\\ &\quad +\frac{3\sqrt{2}}{2}g(z)B_l\sum_{m\neq l}^{N}\frac{\alpha_{lm}K_m^2B_m}{\sqrt{B_l^2+B_m^2}}\\ &\quad \cdot \exp\left\{-\frac{B_l^2 B_m^2}{2\left(B_l^2+B_m^2\right)}\left(t_l-t_m\right)^2\right\}\end{aligned} \tag{160}$$

6.2.2. Super-Gaussian Pulses

For a SG pulse, as before, $f(\tau) = e^{-\tau^{2p}/2}$ for $p \geq 1$. The integrals of motion, in this case, respectively are

$$\begin{aligned}E &= \sum_{l=1}^{N}\int_{-\infty}^{\infty}\left|q^{(l)}\right|^2 dt = \frac{1}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right)\sum_{l=1}^{N}\frac{A_l^2}{B_l}\\ &= \frac{1}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right)\sum_{l=1}^{N}K_l^2\end{aligned} \tag{161}$$

$$M = \frac{i}{2}D(z)\sum_{l=1}^{N}\int_{-\infty}^{\infty}\left(q^{(l)}q_t^{(l)*}-q^{(l)*}q_t^{(l)}\right)dt$$

$$= \frac{D(z)}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right)\sum_{l=1}^{N}\kappa_l\frac{A_l^2}{B_l} = \frac{D(z)}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right)\sum_{l=1}^{N}\kappa_l K_l^2 \qquad (162)$$

while the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}\sum_{l=1}^{N}\int_{-\infty}^{\infty}\left\{D(z)\left|q_t^{(l)}\right|^2 - g(z)\sum_{m\neq l}^{N}\alpha_{lm}\left|q^{(l)}\right|^2\left|q^{(m)}\right|^2\right\}dt \\ &= \sum_{l=1}^{N}\left[D(z)\frac{K_l}{4pB_l^3}\left\{pB_l^5\Gamma\left(\frac{2p-1}{2p}\right) + 8B_lC_l^2\Gamma\left(\frac{3}{2p}\right)\right.\right. \\ &\quad \left. +2\kappa_l^2A_l^2B_l^3\Gamma\left(\frac{1}{2p}\right)\right\} - \frac{g(z)}{2}K_lB_l\sum_{m\neq l}^{N}\alpha_{lm}K_mB_m \\ &\quad \left.\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left\{B_l^{2p}\left(t-t_l\right)^{2p} + B_m^{2p}\left(t-t_m\right)^{2p}\right\}\right]dt\right] \qquad (163) \end{aligned}$$

Also, the evolution equations for the pulse parameters (148)–(152) respectively reduce to

$$\frac{dB_l}{dz} = -2D(z)B_lC_l \qquad (164)$$

$$\begin{aligned} \frac{dC_l}{dz} &= \frac{D(z)}{8}\left\{B_l^4p(2p-1)\frac{\Gamma\left(\frac{2p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)} - 8C_l^2\right\} - \frac{g(z)}{2^{\frac{4p+1}{2p}}}K_l^2B_l^3\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)} \\ &\quad -\frac{pg(z)}{\Gamma\left(\frac{3}{2p}\right)}B_l^3\sum_{m\neq l}^{N}\alpha_{lm}K_m^2B_m \\ &\quad \int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left\{B_l^{2p}\left(t-t_l\right)^{2p} + B_m^{2p}\left(t-t_m\right)^{2p}\right\}\right]dt \qquad (165) \end{aligned}$$

$$\frac{d\kappa_l}{dz} = 0 \qquad (166)$$

$$\frac{dt_l}{dz} = -D(z)\kappa_l \qquad (167)$$

$$\begin{aligned} \frac{d\theta_l}{dz} &= \frac{D(z)}{2}\left\{\kappa_l^2 - B_l^2p(2p-1)\frac{\Gamma\left(\frac{2p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)}\right\} + \frac{5}{2^{\frac{4p+1}{2p}}}g(z)K_l^2B_l \\ &\quad +\frac{3}{2}\frac{p}{\Gamma\left(\frac{1}{2p}\right)}g(z)B_l\sum_{m\neq l}^{N}\alpha_{lm}K_m^2B_m \\ &\quad \int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left\{B_l^{2p}\left(t-t_l\right)^{2p} + B_m^{2p}\left(t-t_m\right)^{2p}\right\}\right]dt \qquad (168) \end{aligned}$$

7. PERTURBATION TERMS

In this section, the DMNLSE in presence of the perturbation terms will be considered. In optical solitons perturbation terms do arise and cannot be avoided. The typical perturbation terms that are studied in optics are the higher order dispersion terms, nonlinear damping, Raman scattering, nonlinear dispersion, saturable amplifiers, self-steepening and many more. In this section, the adiabatic dynamics of the parameters of the perturbed pulses will be obtained. The study will be split into three sections namely polarization preserving fibers, birefringent fibers and multiple channels.

7.1. Polarization Preserving Fibers

The perturbed DMNLSE is given by

$$iq_z + \frac{D(z)}{2} q_{tt} + g(z)|q|^2 q = i\epsilon R[q, q^*] \tag{169}$$

where the perturbation parameter ϵ, called the relative width of the spectrum, arises due to quasi-monochromaticity [8, 16] so that $0 < \epsilon \ll 1$. Moreover, in (165), R represents the perturbation terms of the DMNLSE. In presence of the perturbation terms, the EL equation modify to [10, 11]

$$\frac{\partial L}{\partial r} - \frac{d}{dz}\left(\frac{\partial L}{\partial r_z}\right) = i\epsilon \int_{-\infty}^{\infty} \left(R\frac{\partial q^*}{\partial r} - R^*\frac{\partial q}{\partial r}\right) dt \tag{170}$$

where r represents the six soliton parameters. Once again, substituting A, B, C, κ, $\bar{t}$ and θ for r in (170) yields the following adiabatic evolution equations.

$$\frac{dA}{dz} = -ACD(z) - \frac{\epsilon}{2}\int_{-\infty}^{\infty} \Re[Re^{-i\phi}]\left(\frac{\tau^2}{I_{2,2,0}} - \frac{3}{I_{0,2,0}}\right) f(\tau) d\tau \tag{171}$$

$$\frac{dB}{dz} = -2BCD(z) - \epsilon\frac{B}{A}\int_{-\infty}^{\infty} \Re[Re^{-i\phi}]\left(\frac{\tau^2}{I_{2,2,0}} - \frac{1}{I_{0,2,0}}\right) f(\tau) d\tau \tag{172}$$

$$\begin{aligned}\frac{dC}{dz} &= \left(\frac{B^4}{2}\frac{I_{0,0,2}}{I_{2,2,0}} - 2C^2\right) D(z) - \frac{g(z)A^2B^2}{4}\frac{I_{0,4,0}}{I_{2,2,0}} \\ &\quad -\frac{\epsilon B^2}{2AI_{2,2,0}}\int_{-\infty}^{\infty} \Im[Re^{-i\phi}]\left(f(\tau) + 2\tau\frac{df}{d\tau}\right) d\tau \end{aligned} \tag{173}$$

$$\frac{d\kappa}{dz} = \frac{2\epsilon}{ABI_{0,2,0}}\int_{-\infty}^{\infty} \left\{B^2\Im[Re^{-i\phi}]\frac{df}{d\tau} - 2C\Re[Re^{-i\phi}]\tau f(\tau)\right\} d\tau \tag{174}$$

$$\frac{d\bar{t}}{dz} = -\kappa D(z) + \frac{2\epsilon}{ABI_{0,2,0}}\int_{-\infty}^{\infty}\Re[Re^{-i\phi}]\tau f(\tau)d\tau \tag{175}$$

$$\begin{aligned}\frac{d\theta}{dz} &= \left(\frac{\kappa^2}{2}-\frac{I_{0,0,2}}{I_{0,2,0}}B^2\right)D(z)+\frac{5g(z)A^2}{4}\frac{I_{0,4,0}}{I_{0,2,0}}\\ &\quad+\frac{\epsilon}{2ABI_{0,2,0}}\int_{-\infty}^{\infty}\left\{B\Im[Re^{-i\phi}]\left(3f(\tau)+2\tau\frac{df}{d\tau}\right)\right.\\ &\quad\left.+\,4\kappa\Re[Re^{-i\phi}]\tau f(\tau)\right\}d\tau \end{aligned}\tag{176}$$

where τ is given by (24) while

$$\phi = C(z)\left\{t-\bar{t}(z)\right\}^2 - \kappa(z)\left\{t-\bar{t}(z)\right\} + \theta(z) \tag{177}$$

Also, $\Re$ and $\Im$ represent the real and imaginary parts respectively. Equations (30)–(35) are, now, special cases of (171)–(176) respectively for $\epsilon = 0$.

7.1.1. Gaussian Pulses

Now, substituting the integrals $I_{a,b,c}$ for the indicated values of a, b and c in (171)–(176) and the Gaussian pulse the following equations are obtained

$$\frac{dA}{dz} = -ACD(z) - \frac{\epsilon}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Re[Re^{-i\phi}]\left(4\tau^2-3\right)e^{-\tau^2}d\tau \tag{178}$$

$$\frac{dB}{dz} = -2BCD(z) - \epsilon\sqrt{\frac{2}{\pi}}\frac{B}{A}\int_{-\infty}^{\infty}\Re[Re^{-i\phi}]\left(4\tau^2-1\right)e^{-\tau^2}d\tau \tag{179}$$

$$\begin{aligned}\frac{dC}{dz} &= 2D(z)\left(B^4-C^2\right)\\ &\quad-\frac{g(z)A^2B^2}{\sqrt{2}} - 2\epsilon\sqrt{\frac{2}{\pi}}\frac{B^2}{A}\int_{-\infty}^{\infty}\Im[Re^{-i\phi}]\left(1-4\tau^2\right)e^{-\tau^2}d\tau\end{aligned} \tag{180}$$

$$\frac{d\kappa}{dz} = -\epsilon\sqrt{\frac{2}{\pi}}\frac{4}{AB}\int_{-\infty}^{\infty}\left\{\tau B^2\Im[Re^{-i\phi}]+\tau C\Re[Re^{-i\phi}]\right\}e^{-\tau^2}d\tau \tag{181}$$

$$\frac{d\bar{t}}{dz} = -\kappa D(z) + \epsilon\frac{\sqrt{2\pi}}{AB}\int_{-\infty}^{\infty}\Re[Re^{-i\phi}]\tau e^{-\tau^2}d\tau \tag{182}$$

$$\begin{aligned}\frac{d\theta}{dz} &= \frac{D(z)}{2}\left(\kappa^2-B^2\right)+\frac{5\sqrt{2}}{8}g(z)A^2\\ &\quad+\frac{\epsilon}{\sqrt{2\pi}}\frac{1}{AB}\int_{-\infty}^{\infty}\left\{B\Im[Re^{-i\phi}]\left(3-4\tau^2\right)+4\tau\kappa\Re[Re^{-i\phi}]\tau\right\}e^{-\tau^2}d\tau\end{aligned} \tag{183}$$

These equations now represent the evolution equations for the parameters of a Gaussian pulse propagating through an optical fiber in presence of the perturbation terms.

7.1.2. Super-Gaussian Pulses

For the perturbation terms of a SG pulse substituting the integrals $I_{a,b,c}^{(j)}$ for $j = 1$, 2 and the form of $f(\tau)$ in (171)–(176) leads to

$$\begin{aligned}\frac{dA}{dz} &= -ACD(z)\\ &\quad -\epsilon\frac{p^2 2^{\frac{p+2}{p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)}\int_{-\infty}^{\infty}\Re[Re^{-i\phi}]\left\{\frac{\tau^2}{p2^{\frac{1}{p}}}\Gamma\left(\frac{1}{2p}\right)-\frac{3}{p2^{\frac{3}{2p}}}\Gamma\left(\frac{3}{2p}\right)\right\}e^{-\tau^{2p}}d\tau\end{aligned} \tag{184}$$

$$\begin{aligned}\frac{dB}{dz} &= -2BCD(z)\\ &\quad -\epsilon\frac{B}{A}\frac{p^2 2^{\frac{2}{p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)}\int_{-\infty}^{\infty}\Re[Re^{-i\phi}]\left\{\frac{\tau^2}{p2^{\frac{1}{2p}}}\Gamma\left(\frac{1}{2p}\right)-\frac{1}{p2^{\frac{3}{2p}}}\Gamma\left(\frac{3}{2p}\right)\right\}e^{-\tau^{2p}}d\tau\end{aligned} \tag{185}$$

$$\begin{aligned}\frac{dC}{dz} &= \left\{B^4\frac{p^2}{2^{\frac{p-2}{p}}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}-2C^2\right\}D(z)-g(z)A^2B^2\frac{1}{2^{\frac{4p+1}{2p}}}\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}\\ &\quad -\epsilon\frac{B^2}{A}\frac{p2^{\frac{2p+3}{2p}}}{\Gamma\left(\frac{3}{2p}\right)}\int_{-\infty}^{\infty}\Im[Re^{-i\phi}]\left(1-4p\tau^{2p}\right)e^{-\tau^{2p}}d\tau\end{aligned} \tag{186}$$

$$\frac{d\kappa}{dz} = -\epsilon\frac{1}{AB}\frac{p2^{\frac{2p-1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}\int_{-\infty}^{\infty}\left\{2p\tau^{2p-1}B^2\Im[Re^{-i\phi}]+2\tau C\Re[Re^{-i\phi}]\right\}e^{-\tau^{2p}}d\tau \tag{187}$$

$$\frac{d\bar{t}}{dz} = -\kappa D(z)+\epsilon\frac{1}{AB}\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}\int_{-\infty}^{\infty}\Re[Re^{-i\phi}]\tau e^{-\tau^{2p}}d\tau \tag{188}$$

$$\frac{d\theta}{dz} = \left\{\frac{\kappa^2}{2}-p^2 2^{\frac{1}{p}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)}B^2\right\}D(z)+5g(z)A^2\frac{1}{2^{\frac{4p+1}{2p}}}$$

$$+\epsilon\frac{1}{AB}\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}\int_{-\infty}^{\infty}\left\{B\Im[Re^{-i\phi}]\left(3-4p\tau^{2p}\right)+4\kappa\Re[Re^{-i\phi}]\tau\right\}e^{-\tau^{2p}}d\tau \tag{189}$$

7.2. Birefringent Fibers

The perturbed DM-VNLSE is given by

$$iu_z+\frac{D(z)}{2}u_{tt}+g(z)\left(|u|^2+\alpha|v|^2\right)u\ =\ i\epsilon R_1[u,u^*;v,v^*] \tag{190}$$

$$iv_z+\frac{D(z)}{2}v_{tt}+g(z)\left(|v|^2+\alpha|u|^2\right)v\ =\ i\epsilon R_2[v,v^*;u,u^*] \tag{191}$$

Here, R_1 and R_2 represent the perturbation terms and ϵ is the perturbation parameter as before. In presence of the perturbation terms, the EL equations modify to

$$\frac{\partial L}{\partial r}-\frac{d}{dz}\left(\frac{\partial L}{\partial r_z}\right)=i\epsilon\int_{-\infty}^{\infty}\left(R_1\frac{\partial u^*}{\partial r}-R_1^*\frac{\partial u}{\partial r}\right)dt \tag{192}$$

and

$$\frac{\partial L}{\partial r}-\frac{d}{dz}\left(\frac{\partial L}{\partial r_z}\right)=i\epsilon\int_{-\infty}^{\infty}\left(R_2\frac{\partial v^*}{\partial r}-R_2^*\frac{\partial v}{\partial r}\right)dt \tag{193}$$

where r represents twelve soliton parameters. Once again, substituting A_j, B_j, C_j, κ_j, t_j and θ_j where $j=1,2$ for r in (192) and (193), the following adiabatic evolution equations are obtained

$$\frac{dA_1}{dz}=-D(z)A_1C_1$$
$$-\frac{\epsilon}{2}\int_{-\infty}^{\infty}\Re[R_1e^{-i\phi_1}]\left\{\frac{\tau_1^2}{I_{2,2,0}^{(1)}}-\frac{3}{I_{0,2,0}^{(1)}}\right\}f(\tau_1)d\tau_1 \tag{194}$$

$$\frac{dB_1}{dz}=-2D(z)B_1C_1-\epsilon\frac{B_1}{A_1}\int_{-\infty}^{\infty}\Re[R_1e^{-i\phi_1}]\left\{\frac{\tau_1^2}{I_{2,2,0}^{(1)}}-\frac{1}{I_{0,2,0}^{(1)}}\right\}f(\tau_1)d\tau_1 \tag{195}$$

$$\frac{dC_1}{dz}=D(z)\left\{\frac{B_1^4}{2}\frac{I_{0,0,2}^{(1)}}{I_{2,2,0}^{(1)}}-2C_1^2\right\}-\frac{g(z)A_1^2B_1^2}{4}\frac{I_{0,4,0}^{(1)}}{I_{2,2,0}^{(1)}}-\frac{\alpha g(z)}{2}A_2^2B_1^3\frac{J_{2,2,0,0}^{0,0}}{I_{2,2,0}^{(1)}}$$
$$-\frac{\epsilon B_1^2}{2A_1I_{2,2,0}^{(1)}}\int_{-\infty}^{\infty}\Im[R_1e^{-i\phi_1}]\left\{f(\tau_1)+2\tau_1\frac{df}{d\tau_1}\right\}d\tau_1 \tag{196}$$

$$\frac{d\kappa_1}{dz} = \frac{2\epsilon}{A_1 B_1 I^{(1)}_{0,2,0}} \int_{-\infty}^{\infty} \left\{ B_1^2 \Im[R_1 e^{-i\phi_1}] \frac{df}{d\tau_1} - 2C_1 \Re[R_1 e^{-i\phi_1}] \tau_1 f(\tau_1) \right\} d\tau_1 \tag{197}$$

$$\frac{dt_1}{dz} = -\kappa_1 D(z) + \frac{2\epsilon}{A_1 B_1 I^{(1)}_{0,2,0}} \int_{-\infty}^{\infty} \Re[R_1 e^{-i\phi_1}] \tau_1 f(\tau_1) d\tau_1 \tag{198}$$

$$\begin{aligned} \frac{d\theta_1}{dz} = {} & D(z) \left\{ \frac{\kappa_1^2}{2} - \frac{I^{(1)}_{0,0,2}}{I^{(1)}_{0,2,0}} B_1^2 \right\} + \frac{5g(z)A_1^2}{4} \frac{I^{(1)}_{0,4,0}}{I^{(1)}_{0,2,0}} + \frac{3}{2} \alpha g(z) A_2^2 B_1 \frac{J^{0,0}_{2,2,0,0}}{I^{(1)}_{0,2,0}} \\ & + \frac{\epsilon}{2A_1 B_1 I^{(1)}_{0,2,0}} \int_{-\infty}^{\infty} \left\{ B_1 \Im[R_1 e^{-i\phi_1}] \left(3f(\tau_1) + 2\tau_1 \frac{df}{d\tau_1} \right) \right. \\ & \left. + 4\kappa_1 \Re[R_1 e^{-i\phi_1}] \tau_1 f(\tau_1) \right\} d\tau_1 \end{aligned} \tag{199}$$

$$\frac{dA_2}{dz} = -D(z) A_2 C_2 - \frac{\epsilon}{2} \int_{-\infty}^{\infty} \Re[R_2 e^{-i\phi_2}] \left\{ \frac{\tau_2^2}{I^{(2)}_{2,2,0}} - \frac{3}{I^{(2)}_{0,2,0}} \right\} f(\tau_2) d\tau_2 \tag{200}$$

$$\frac{dB_2}{dz} = -2D(z) B_2 C_2 - \epsilon \frac{B_2}{A_2} \int_{-\infty}^{\infty} \Re[R_2 e^{-i\phi_2}] \left\{ \frac{\tau_2^2}{I^{(2)}_{2,2,0}} - \frac{1}{I^{(2)}_{0,2,0}} \right\} f(\tau_2) d\tau_2 \tag{201}$$

$$\begin{aligned} \frac{dC_2}{dz} = {} & D(z) \left\{ \frac{B_2^4}{2} \frac{I^{(2)}_{0,0,2}}{I^{(2)}_{2,2,0}} - 2C_2^2 \right\} - \frac{g(z) A_2^2 B_2^2}{4} \frac{I^{(2)}_{0,4,0}}{I^{(2)}_{2,2,0}} - \frac{\alpha g}{2} A_1^2 B_2^3 \frac{J^{0,0}_{2,2,0,0}}{I^{(2)}_{2,2,0}} \\ & - \frac{\epsilon B_2^2}{2A_2 I^{(2)}_{2,2,0}} \int_{-\infty}^{\infty} \Im[R_2 e^{-i\phi_2}] \left\{ f(\tau_2) + 2\tau_2 \frac{df}{d\tau_2} \right\} d\tau_2 \end{aligned} \tag{202}$$

$$\frac{d\kappa_2}{dz} = \frac{2\epsilon}{A_2 B_2 I^{(2)}_{0,2,0}} \int_{-\infty}^{\infty} \left\{ B_2^2 \Im[R_2 e^{-i\phi_2}] \frac{df}{d\tau_2} - 2C_2 \Re[R_2 e^{-i\phi_2}] \tau_2 f(\tau_2) \right\} d\tau_2 \tag{203}$$

$$\frac{dt_2}{dz} = -\kappa_2 D(z) + \frac{2\epsilon}{A_2 B_2 I^{(2)}_{0,2,0}} \int_{-\infty}^{\infty} \Re[R_2 e^{-i\phi_2}] \tau_2 f(\tau_2) d\tau_2 \tag{204}$$

$$\begin{aligned} \frac{d\theta_2}{dz} = {} & D(z) \left\{ \frac{\kappa_2^2}{2} - \frac{I^{(2)}_{0,0,2}}{I^{(2)}_{0,2,0}} B_2^2 \right\} + \frac{5g(z)A_2^2}{4} \frac{I^{(2)}_{0,4,0}}{I^{(2)}_{0,2,0}} + \frac{3}{2} \alpha g(z) A_1^2 B_2 \frac{J^{0,0}_{2,2,0,0}}{I^{(2)}_{0,2,0}} \\ & + \frac{\epsilon}{2A_2 B_2 I^{(2)}_{0,2,0}} \int_{-\infty}^{\infty} \left\{ B_2 \Im[R_2 e^{-i\phi_2}] \left(3f(\tau_2) + 2\tau_2 \frac{df}{d\tau_2} \right) \right. \end{aligned}$$

$$+4\kappa_2\Re[R_2 e^{-i\phi_2}]\tau_2 f(\tau_2)\Big\}\, d\tau_2 \tag{205}$$

where τ_l and ϕ_l are given by (73) and (74) respectively. These relations will now be simplified to obtain the dynamics for the Gaussian and SG solitons in the following subsections.

7.2.1. Gaussian Pulses

Here, again $f(\tau_j) = e^{-\frac{1}{2}\tau_j^2}$ where $j = 1, 2$. Also using the integrals $I_{a,b,c}^{(j)}$ for $j = 1,\ 2$ in (194)-(205) the adiabatic parameter dynamics of perturbed Gaussian pulses are

$$\frac{dA_1}{dz} = -D(z)A_1C_1 - \frac{\epsilon}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Re[R_1 e^{-i\phi_1}]\left(4\tau_1^2 - 3\right)e^{-\tau_1^2}\, d\tau_1 \tag{206}$$

$$\frac{dB_1}{dz} = -2D(z)B_1C_1 - \epsilon\sqrt{\frac{2}{\pi}}\frac{B_1}{A_1}\int_{-\infty}^{\infty}\Re[R_1 e^{-i\phi_1}]\left(4\tau_1^2 - 1\right)e^{-\tau_1^2}\, d\tau_1 \tag{207}$$

$$\begin{aligned}\frac{dC_1}{dz} =& 2D(z)\left(B_1^4 - C_1^2\right) - \frac{1}{\sqrt{2}}g(z)A_1^2B_1^2 \\ &-2\alpha g(z)A_2^2B_1^3\sqrt{\frac{2}{B_1^2+B_2^2}}\, e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2} \\ &-2\epsilon\sqrt{\frac{2}{\pi}}\frac{B_1^2}{A_1}\int_{-\infty}^{\infty}\Im[R_1 e^{-i\phi_1}]\left(1 - 4\tau_1^2\right)e^{-\tau_1^2}\, d\tau_1\end{aligned} \tag{208}$$

$$\frac{d\kappa_1}{dz} = -\frac{2\epsilon}{A_1B_1}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left\{B_1^2\Im[R_1 e^{-i\phi_1}]2\tau_1 + 2C_1\Re[R_1 e^{-i\phi_1}]\tau_1\right\}e^{-\tau_1^2}\, d\tau_1 \tag{209}$$

$$\frac{dt_1}{dz} = -D(z)\kappa_1 + \frac{2\epsilon}{A_1B_1}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\Re[R_1 e^{-i\phi_1}]\tau_1 e^{-\tau_1^2}\, d\tau_1 \tag{210}$$

$$\begin{aligned}\frac{d\theta_1}{dz} =& \frac{D(z)}{2}\left(\kappa_1^2 - B_1^2\right) + \frac{5}{4\sqrt{2}}g(z)A_1^2 \\ &+\frac{3}{2}\alpha g(z)A_2^2B_1\sqrt{\frac{2}{B_1^2+B_2^2}}\, e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2} \\ &+\frac{\epsilon}{\sqrt{2\pi}}\frac{1}{A_1B_1}\int_{-\infty}^{\infty}\Big\{B_1\Im[R_1 e^{-i\phi_1}]\left(3 - 4\tau_1^2\right) \\ &+ 4\kappa_1\Re[R_1 e^{-i\phi_1}]\tau_1\Big\}e^{-\tau_1^2}\, d\tau_1\end{aligned} \tag{211}$$

$$\frac{dA_2}{dz} = -D(z)A_2C_2 - \frac{\epsilon}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Re[R_2e^{-i\phi_2}]\left(4\tau_2^2-3\right)e^{-\tau_2^2}d\tau_2 \quad (212)$$

$$\frac{dB_2}{dz} = -2D(z)B_2C_2 - \epsilon\sqrt{\frac{2}{\pi}}\frac{B_2}{A_2}\int_{-\infty}^{\infty}\Re[R_2e^{-i\phi_2}]\left(4\tau_2^2-1\right)e^{-\tau_2^2}d\tau_2 \quad (213)$$

$$\begin{aligned}\frac{dC_2}{dz} =& 2D(z)\left(B_2^4-C_2^2\right) - \frac{1}{\sqrt{2}}g(z)A_2^2B_2^2\\ &-2\alpha g(z)A_1^2B_2^3\sqrt{\frac{2}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2}\\ &-2\epsilon\sqrt{\frac{2}{\pi}}\frac{B_2^2}{A_2}\int_{-\infty}^{\infty}\Im[R_2e^{-i\phi_2}]\left(1-4\tau_2^2\right)e^{-\tau_2^2}d\tau_2 \quad (214)\end{aligned}$$

$$\frac{d\kappa_2}{dz} = -\frac{2\epsilon}{A_2B_2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left\{B_2^2\Im[R_2e^{-i\phi_2}]2\tau_2+2C_2\Re[R_2e^{-i\phi_2}]\tau_2\right\}e^{-\tau_2^2}d\tau_2 \quad (215)$$

$$\frac{dt_2}{dz} = -D(z)\kappa_2 + \frac{2\epsilon}{A_2B_2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\Re[R_2e^{-i\phi_2}]\tau_2e^{-\tau_2^2}d\tau_2 \quad (216)$$

$$\begin{aligned}\frac{d\theta_2}{dz} =& \frac{D(z)}{2}\left(\kappa_2^2-B_2^2\right) + \frac{5}{4\sqrt{2}}g(z)A_2^2\\ &+\frac{3}{2}\alpha g(z)A_1^2B_2\sqrt{\frac{2}{B_1^2+B_2^2}}e^{-\frac{B_1^2B_2^2}{B_1^2+B_2^2}(t_1-t_2)^2}\\ &+\frac{\epsilon}{\sqrt{2\pi}}\frac{1}{A_2B_2}\int_{-\infty}^{\infty}\left\{B_2\Im[R_2e^{-i\phi_2}]\left(3-4\tau_2^2\right)\right.\\ &\left.+\,4\kappa_2\Re[R_2e^{-i\phi_2}]\tau_2\right\}e^{-\tau_2^2}d\tau_2 \quad (217)\end{aligned}$$

7.2.2. Super-Gaussian Pulses

For the SG pulses, the integrals $I_{a,b,c}^{(j)}$ are used in (194)–(205) and the form of the SG soliton for $f(\tau_j)$ is considered, to finally obtain

$$\begin{aligned}\frac{dA_1}{dz} =& -D(z)A_1C_1 - \epsilon\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)}\\ &\int_{-\infty}^{\infty}\Re[R_1e^{-i\phi_1}]\left\{\tau_1^2 2^{\frac{1}{2p}}\Gamma\left(\frac{1}{2p}\right)-3\Gamma\left(\frac{3}{2p}\right)\right\}e^{-\tau_1^{2p}}d\tau_1 \quad (218)\end{aligned}$$

$$\frac{dB_1}{dz} = -2D(z)B_1C_1 - \epsilon\frac{B_1}{A_1}\frac{p2^{\frac{1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)}$$

$$\int_{-\infty}^{\infty} \Re[R_1 e^{-i\phi_1}] \left\{\tau_1^2 2^{\frac{1}{p}} \Gamma\left(\frac{1}{2p}\right) - \Gamma\left(\frac{3}{2p}\right)\right\} e^{-\tau_1^{2p}} d\tau_1 \tag{219}$$

$$\begin{aligned}
\frac{dC_1}{dz} &= D(z)\left\{B_1^4 \frac{p^2}{2^{\frac{p-2}{p}}} \frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)} - 2C_1^2\right\} - g(z)A_1^2 B_1^2 \frac{1}{2^{\frac{4p+1}{2p}}} \frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)} \\
&\quad - \frac{p}{2^{\frac{2p-3}{2p}}} \frac{A_2^3 B_1^2}{\Gamma\left(\frac{3}{2p}\right)} \int_{-\infty}^{\infty} \exp\Big[-2\left\{B_1^{2p}(t-t_1)^{2p} + B_2^{2p}(t-t_2)^{2p}\right\}\Big] dt \\
&\quad - \epsilon \frac{B_1^2}{A_1} \frac{p 2^{\frac{2p+3}{2p}}}{\Gamma\left(\frac{3}{2p}\right)} \int_{-\infty}^{\infty} \Im[R_1 e^{-i\phi_1}] \left(1 - 4p\tau_1^{2p}\right) e^{-\tau_1^{2p}} d\tau_1
\end{aligned} \tag{220}$$

$$\begin{aligned}
\frac{d\kappa_1}{dz} &= -\epsilon \frac{1}{A_1 B_1} \frac{p 2^{\frac{2p-1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)} \int_{-\infty}^{\infty} \Big\{2p\tau_1^{2p-1} B_1^2 \Im[R_1 e^{-i\phi_1}] \\
&\quad + 2\tau C_1 \Re[R_1 e^{-i\phi_1}]\Big\} e^{-\tau_1^{2p}} d\tau_1
\end{aligned} \tag{221}$$

$$\frac{dt_1}{dz} = -D(z)\kappa_1 + \epsilon \frac{1}{A_1 B_1} \frac{p 2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)} \int_{-\infty}^{\infty} \Re[R_1 e^{-i\phi_1}] \tau_1 e^{-\tau_1^{2p}} d\tau_1 \tag{222}$$

$$\begin{aligned}
\frac{d\theta_1}{dz} &= D(z)\left\{\frac{\kappa_1^2}{2} - p^2 2^{\frac{1}{p}} \frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)} B_1^2\right\} + 5g(z)A_1^2 \frac{1}{2^{\frac{4p+1}{2p}}} \\
&\quad - \frac{3p}{2^{\frac{2p-1}{2p}}} \alpha g(z) \frac{A_2^2 B_1}{\Gamma\left(\frac{1}{2p}\right)} \int_{-\infty}^{\infty} \exp\Big[-2\left\{B_1^{2p}(t-t_1)^{2p}\right. \\
&\quad \left. + B_2^{2p}(t-t_2)^{2p}\right\}\Big] dt + \epsilon \frac{1}{A_1 B_1} \frac{p 2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)} \\
&\quad \int_{-\infty}^{\infty} \left\{B_1 \Im[R_1 e^{-i\phi_1}] \left(3 - 4p\tau_1^{2p}\right) + 4\kappa_1 \Re[R_1 e^{-i\phi_1}]\tau_1\right\} e^{-\tau_1^{2p}} d\tau_1
\end{aligned} \tag{223}$$

$$\begin{aligned}
\frac{dA_2}{dz} &= -D(z)A_2 C_2 - \epsilon \frac{p 2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)} \\
&\quad \int_{-\infty}^{\infty} \Re[R_2 e^{-i\phi_2}] \left\{\tau_2^2 2^{\frac{1}{2p}} \Gamma\left(\frac{1}{2p}\right) - 3\Gamma\left(\frac{3}{2p}\right)\right\} e^{-\tau_2^{2p}} d\tau_2
\end{aligned} \tag{224}$$

$$\frac{dB_2}{dz} = -2D(z)B_2 C_2$$

$$-\epsilon\frac{B_2}{A_2}\frac{p2^{\frac{1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)}$$

$$\int_{-\infty}^{\infty}\Re[R_2e^{-i\phi_2}]\left\{\tau_2^2 2^{\frac{1}{p}}\Gamma\left(\frac{1}{2p}\right)-\Gamma\left(\frac{3}{2p}\right)\right\}e^{-\tau_2^{2p}}d\tau_2 \qquad (225)$$

$$\frac{dC_2}{dz} = D(z)\left\{B_2^4\frac{p^2}{2^{\frac{p-2}{p}}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}-2C_2^2\right\}-g(z)A_2^2B_2^2\frac{1}{2^{\frac{4p+1}{2p}}}\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}$$

$$-\frac{p}{2^{\frac{2p-3}{2p}}}\frac{A_1^3B_2^2}{\Gamma\left(\frac{3}{2p}\right)}\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}(t-t_1)^{2p}+B_2^{2p}(t-t_2)^{2p}\right\}\right]dt$$

$$-\epsilon\frac{B_2^2}{A_2}\frac{p2^{\frac{2p+3}{2p}}}{\Gamma\left(\frac{3}{2p}\right)}\int_{-\infty}^{\infty}\Im[R_2e^{-i\phi_2}]\left(1-4p\tau_2^{2p}\right)e^{-\tau_2^{2p}}d\tau_2 \qquad (226)$$

$$\frac{d\kappa_2}{dz} = -\epsilon\frac{1}{A_2B_2}\frac{p2^{\frac{2p-1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}\int_{-\infty}^{\infty}\left\{2p\tau_2^{2p-1}B_2^2\Im[R_2e^{-i\phi_2}]\right.$$

$$\left.+2\tau_2C_2\Re[R_2e^{-i\phi_2}]\right\}e^{-\tau_2^{2p}}d\tau_2 \qquad (227)$$

$$\frac{dt_2}{dz} = -D(z)\kappa_2+\epsilon\frac{1}{A_2B_2}\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}\int_{-\infty}^{\infty}\Re[R_2e^{-i\phi_2}]\tau_2e^{-\tau_2^{2p}}d\tau \qquad (228)$$

$$\frac{d\theta_2}{dz} = D(z)\left\{\frac{\kappa_2^2}{2}-p^2 2^{\frac{1}{p}}\frac{\Gamma\left(\frac{4p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)}B_2^2\right\}+5g(z)A_2^2\frac{1}{2^{\frac{4p+1}{2p}}}$$

$$-\frac{3p}{2^{\frac{2p-1}{2p}}}\alpha g(z)\frac{A_1^2B_2}{\Gamma\left(\frac{1}{2p}\right)}$$

$$\int_{-\infty}^{\infty}\exp\left[-2\left\{B_1^{2p}(t-t_1)^{2p}+B_2^{2p}(t-t_2)^{2p}\right\}\right]dt$$

$$+\epsilon\frac{1}{A_2B_2}\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}$$

$$\int_{-\infty}^{\infty}\left\{B_2\Im[R_2e^{-i\phi_2}]\left(3-4p\tau_2^{2p}\right)+4\kappa_2\Re[R_2e^{-i\phi_2}]\tau_2\right\}e^{-\tau_2^{2p}}d\tau_2$$

$$(229)$$

These are the parameter dynamics of chirped Gaussian and SG solitons can be very useful to study the vector solitons in a birefringent media.

7.3. Multiple Channels

The perturbed WDM system is given by

$$iq_z^{(l)} + \frac{D(z)}{2}q_{tt}^{(l)} + g(z)\left\{\left|q^{(l)}\right|^2 + \sum_{m\neq l}^{N}\alpha_{lm}\left|q^{(m)}\right|^2\right\}q^{(l)} = i\epsilon R\left[q^{(l)}, q^{(l)*}; q^{(m)}, q^{(m)*}\right] \tag{230}$$

where, $1 \leq l \leq N$ while $m \neq l$. In (230), R represents the perturbation terms for a WDM system with N channels. In presence of perturbation terms, the EL equation modify to [6, 10]

$$\frac{\partial L}{\partial r} - \frac{d}{dz}\left(\frac{\partial L}{\partial r_z}\right) = i\epsilon\int_{-\infty}^{\infty}\left(R\frac{\partial q^{(l)*}}{\partial r} - R^*\frac{\partial q^{(l)}}{\partial r}\right)dt \tag{231}$$

where r represents the $6N$ soliton parameters. Once again, substituting A_l, B_l, C_l, κ_l, t_l and θ_l for r in (231), the following adiabatic evolution equations are obtained

$$\frac{dA_l}{dz} = -D(z)A_lC_l - \frac{\epsilon}{2}\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\left\{\frac{\tau_l^2}{I_{0,2,0}^{(l)}} - \frac{3}{I_{2,2,0}^{(l)}}\right\}f(\tau_l)d\tau_l \tag{232}$$

$$\frac{dB_l}{dz} = -2D(z)B_lC_l - \epsilon\frac{B_l}{A_l}\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\left\{\frac{\tau_l^2}{I_{2,2,0}^{(l)}} - \frac{1}{I_{0,2,0}^{(l)}}\right\}f(\tau_l)d\tau_l \tag{233}$$

$$\begin{aligned}\frac{dC_l}{dz} = &\frac{D(z)}{2}\left\{B_l\frac{I_{0,0,2}^{(l)}}{I_{2,2,0}^{(l)}} - 4C_l^2\right\} - \frac{g(z)}{4}A_l^2B_l^2\frac{I_{0,4,0}^{(l)}}{I_{2,2,0}^{(l)}}\\ &-\frac{g(z)}{2I_{2,2,0}^{(l)}}B_l^3\sum_{m\neq l}^{N}\alpha_{lm}A_m^2J_{l,m}\\ &-\frac{\epsilon B_l^2}{2A_lI_{2,2,0}^{(l)}}\int_{-\infty}^{\infty}\Im[Re^{-i\phi_l}]\left\{f(\tau_l) + 2\tau_l\frac{df}{d\tau_l}\right\}d\tau_l\end{aligned} \tag{234}$$

$$\frac{d\kappa_l}{dz} = \frac{2\epsilon}{A_lB_lI_{0,2,0}^{(l)}}\int_{-\infty}^{\infty}\left\{B_l^2\Im[Re^{-i\phi_l}]\frac{df}{d\tau_l} - 2C_l\Re[Re^{-i\phi_l}]\tau f(\tau_l)\right\}d\tau_l \tag{235}$$

$$\frac{dt_l}{dz} = -D(z)\kappa_l + \frac{2\epsilon}{A_lB_lI_{0,2,0}^{(l)}}\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\tau f(\tau_l)d\tau_l \tag{236}$$

$$\frac{d\theta_l}{dz} = \frac{D(z)}{2}\left\{\kappa_l^2 - 2B_l^2\frac{I_{0,0,2}^{(l)}}{I_{0,2,0}^{(l)}}\right\} + \frac{5}{4}g(z)A_l^2\frac{I_{0,4,0}^{(l)}}{I_{0,2,0}^{(l)}}$$
$$+\frac{3}{2}\frac{g(z)}{I_{0,2,0}^{(l)}}B_l\sum_{m\neq l}^{N}\alpha_{lm}A_m^2 J_{l,m} + \frac{\epsilon}{2A_l B_l I_{0,2,0}^{(l)}}$$
$$\int_{-\infty}^{\infty}\left\{B_l\Im[Re^{-i\phi_l}]\left(3f(\tau_l)+2\tau_l\frac{df}{d\tau_l}\right)+4\kappa\Re[Re^{-i\phi_l}]\tau f(\tau_l)\right\}d\tau_l \tag{237}$$

where τ_l and ϕ_l are defined in (73) and (74) respectively for $1 \leq l \leq N$.

7.3.1. Gaussian Pulses

Now, substituting the integrals $I_{a,b,c}^{(l)}$ for the indicated values of a, b and c in (232)–(237) and the Gaussian pulse the following equations are obtained

$$\frac{dA_l}{dz} = -D(z)A_lC_l - \frac{\epsilon}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\left(4\tau_l^2-3\right)e^{-\tau_l^2}d\tau_l \tag{238}$$

$$\frac{dB_l}{dz} = -2D(z)B_lC_l - \epsilon\sqrt{\frac{2}{\pi}}\frac{B_l}{A_l}\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\left(4\tau_l^2-1\right)e^{-\tau_l^2}d\tau_l \tag{239}$$

$$\frac{dC_l}{dz} = 2D(z)\left(B_l^4 - C_l^2\right) - \frac{\sqrt{2}}{2}g(z)A_l^4B_l$$
$$-\sqrt{2}g(z)B_l^3\sum_{m\neq l}^{N}\frac{\alpha_{lm}A_m^4}{B_m\sqrt{B_l^2+B_m^2}}\exp\left\{-\frac{B_l^2B_m^2}{2\left(B_l^2+B_m^2\right)}\left(t_l-t_m\right)^2\right\}$$
$$-2\epsilon\sqrt{\frac{2}{\pi}}\frac{B_l^2}{A_l}\int_{-\infty}^{\infty}\Im[Re^{-i\phi_l}]\left(1-4\tau_l^2\right)e^{-\tau_l^2}d\tau_l \tag{240}$$

$$\frac{d\kappa_l}{dz} = -\epsilon\sqrt{\frac{2}{\pi}}\frac{4}{A_lB_l}\int_{-\infty}^{\infty}\left\{\tau_lB_l^2\Im[Re^{-i\phi_l}]+\tau_lC_l\Re[Re^{-i\phi_l}]\right\}e^{-\tau_l^2}d\tau_l \tag{241}$$

$$\frac{dt_l}{dz} = -D(z)\kappa_l + \epsilon\frac{\sqrt{2\pi}}{A_lB_l}\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\tau_le^{-\tau_l^2}d\tau_l \tag{242}$$

$$\frac{d\theta_l}{dz} = \frac{D(z)}{2}\left(\kappa_l^2 - B_l^2\right) + \frac{5\sqrt{2}}{8}g(z)\frac{A_l^4}{B_l}$$
$$+\frac{3\sqrt{2}}{2}g(z)B_l\sum_{m\neq l}^{N}\frac{\alpha_{lm}A_m^4}{B_m\sqrt{B_l^2+B_m^2}}\exp\left\{-\frac{B_l^2B_m^2}{2\left(B_l^2+B_m^2\right)}\left(t_l-t_m\right)^2\right\}$$

$$+\frac{\epsilon}{\sqrt{2\pi}}\frac{1}{A_l B_l}\int_{-\infty}^{\infty}\left\{B_l\Im[Re^{-i\phi_l}]\left(3-4\tau_l^2\right)\right.$$
$$\left.+4\tau_l\kappa_l\Re[Re^{-i\phi_l}]\tau_l\right\}e^{-\tau_l^2}d\tau_l \tag{243}$$

These equations now represent the evolution equations for the parameters of a Gaussian pulse, for $1 \le l \le N$, propagating through a DWDM in presence of the perturbation terms.

7.3.2. Super-Gaussian Pulses

For the perturbation terms of a SG pulse substituting the integrals $I_{a,b,c}^{(l)}$ for $1 \le l \le N$ and the form of $f(\tau_l)$ in (232)–(237) that leads to

$$\frac{dA_l}{dz} = -D(z)A_l C_l - \epsilon\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)}$$
$$\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\left\{\tau_l^2 2^{\frac{1}{2p}}\Gamma\left(\frac{1}{2p}\right)-3\Gamma\left(\frac{3}{2p}\right)\right\}e^{-\tau_l^{2p}}d\tau_l \tag{244}$$

$$\frac{dB_l}{dz} = -2D(z)B_l C_l - \epsilon\frac{B_l}{A_l}\frac{p2^{\frac{1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{3}{2p}\right)}$$
$$\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\left\{\tau_l^2 2^{\frac{1}{p}}\Gamma\left(\frac{1}{2p}\right)-\Gamma\left(\frac{3}{2p}\right)\right\}e^{-\tau_l^{2p}}d\tau_l \tag{245}$$

$$\frac{dC_l}{dz} = \frac{D(z)}{8}\left\{B_l^4 p(2p-1)\frac{\Gamma\left(\frac{2p-1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}-8C_l^2\right\}-\frac{g(z)}{2^{\frac{4p+1}{2p}}}A_l^4 B_l\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{3}{2p}\right)}$$
$$-\frac{pg(z)}{\Gamma\left(\frac{3}{2p}\right)}B_l^3\sum_{m\neq l}^{N}\alpha_{lm}\frac{A_m^4}{B_m}$$
$$\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left\{B_l^{2p}\left(t-t_l\right)^{2p}+B_m^{2p}\left(t-t_m\right)^{2p}\right\}\right]dt$$
$$-\epsilon\frac{B_l^2}{A_l}\frac{p2^{\frac{2p+3}{2p}}}{\Gamma\left(\frac{3}{2p}\right)}\int_{-\infty}^{\infty}\Im[Re^{-i\phi_l}]\left(1-4p\tau_l^{2p}\right)e^{-\tau_l^{2p}}d\tau_l \tag{246}$$

$$\frac{d\kappa_l}{dz} = -\epsilon\frac{1}{A_l B_l}\frac{p2^{\frac{2p-1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}$$
$$\int_{-\infty}^{\infty}\left\{2p\tau_l^{2p-1}B_l^2\Im[Re^{-i\phi_l}]+2\tau_l C_l\Re[Re^{-i\phi_l}]\right\}e^{-\tau_l^{2p}}d\tau_l \tag{247}$$

$$\frac{dt_l}{dz} = -D(z)\kappa_l + \epsilon\frac{1}{A_l B_l}\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}\int_{-\infty}^{\infty}\Re[Re^{-i\phi_l}]\tau_l e^{-\tau_l^{2p}}d\tau_l \tag{248}$$

$$\frac{d\theta_l}{dz} = \frac{D(z)}{2}\left\{\kappa_l^2 - B_l^2 p(2p-1)\frac{\Gamma\left(\frac{2p-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}\right)}\right\} + \frac{5}{2^{\frac{4p+1}{2p}}}g(z)\frac{A_l^4}{B_l}$$
$$+\frac{3}{2}\frac{p}{\Gamma\left(\frac{1}{2p}\right)}g(z)B_l\sum_{m\neq l}^{N}\alpha_{lm}\frac{A_m^4}{B_m}$$
$$\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left\{B_l^{2p}\left(t-t_l\right)^{2p} + B_m^{2p}\left(t-t_m\right)^{2p}\right\}\right]dt$$
$$+\epsilon\frac{1}{A_l B_l}\frac{p2^{\frac{2p+1}{2p}}}{\Gamma\left(\frac{1}{2p}\right)}$$
$$\int_{-\infty}^{\infty}\left\{B_l\Im[Re^{-i\phi_l}]\left(3-4p\tau_l^{2p}\right)+4\kappa_l\Re[Re^{-i\phi_l}]\tau_l\right\}e^{-\tau_l^{2p}}d\tau_l \tag{249}$$

So, now, these are the adiabatic evolution of the soliton parameters for a SG pulse, for $1 \leq l \leq N$, in presence of the perturbation terms.

8. NUMERICAL SIMULATIONS

In this section, the numerical simulations are carried out for (12). The details of the plots that describe the parameter evolution of the DM solitons are discussed here.

Figure 5 is the variation of the width (B) of the pulse while Figure 6 is the variation of the chirp (C) of the pulse. In both figures 5 and 6, the solid lines shows the analytical results while the dotted lines represent direct numerical simulation of (12) with the same initial condition. Figure 7, now, represents the periodic orbit of Figures 4 and 5 in the B-C plane.

Figures 8 and 9 shows the Poincare section in the $B-C$ plane, namely a trajectory of the parameters at every period of the dispersion map for several different values of the initial condition. Fig. 8 shows the plots for different values of $B(0)$ and $C(0)$ with a fixed energy namely $E = E_0 = 3.629$ while Fig. 9 is obtained for different values of E_0 with fixed values, $B(0) = 1/\sqrt{2}$ and $C(0) = 0$. As shown in these figures, evolution of the parameters starting from the vicinity of the stationary state is a closed loop around the fixed point corresponding to the stationary state, namely $B(0) = 1/\sqrt{2}$, $C(0) = 0$ and $E_0 = 3.629$, and exhibits a long term quasi-periodicity. This is in contrast to the

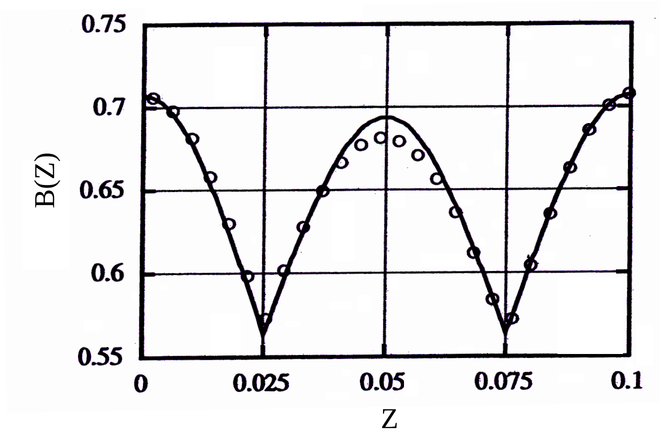

Figure 5. Evolution of the width of the pulse.

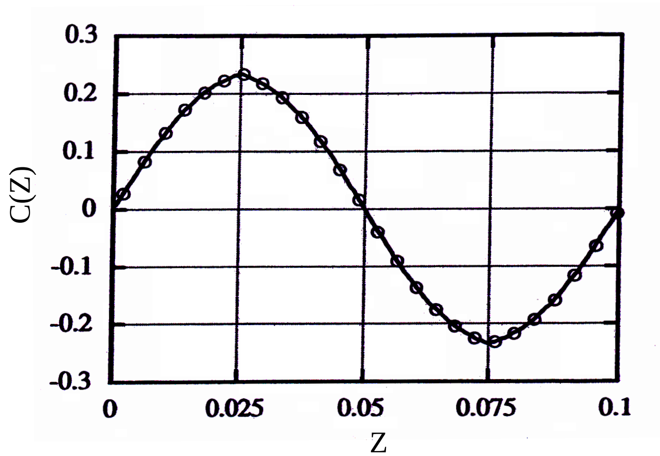

Figure 6. Evolution of the chirp of the pulse.

behaviour of classical or conventional solitons where the pulses having parameters deviate from the stationary state and generate radiation and approaches asymptotically to the stationary solution. Too much deviation from the stationary state, however, results in a collapse of the pulse, as shown in Fig. 8 for $E_0 = 1.0$ [50].

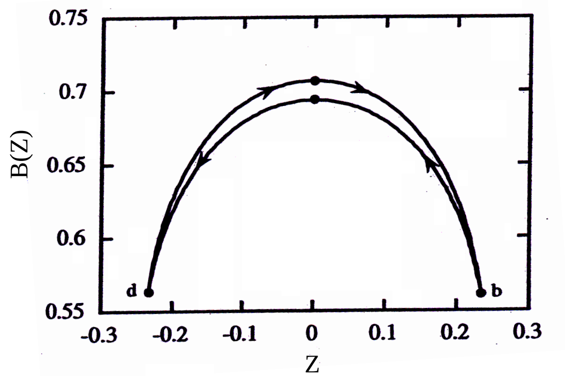

Figure 7. Periodic orbit of width versus chirp.

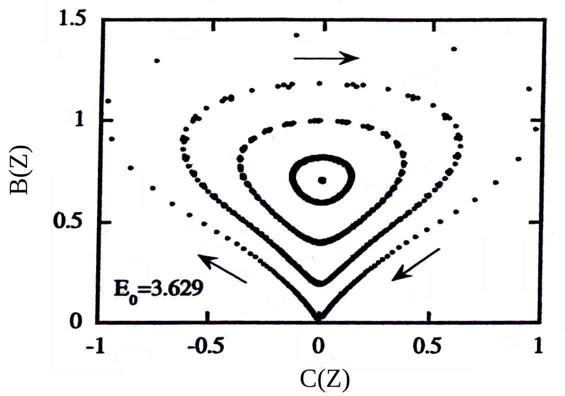

Figure 8. Periodic map for width versus chirp.

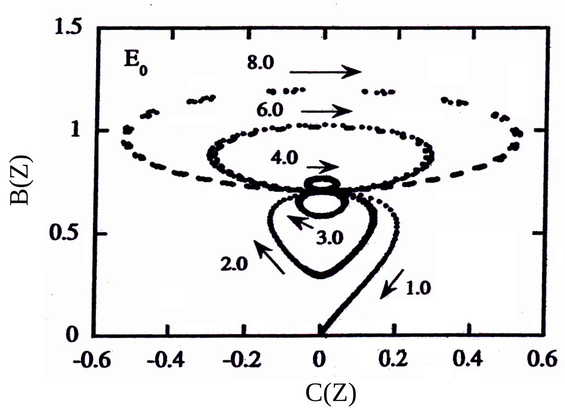

Figure 9. Periodic map for several value of E_0.

9. CONCLUSIONS

In this paper, the governing equations for the characteristic parameters for DM pulses were derived. The fundamental dynamics of DM pulses are characterized by their pulse width and their frequency chirp. Moreover, the adiabatic evolution of these parameters for the DM pulses under perturbations was obtained. Both the Gaussian and the super-Gaussian type pulses are considered in this paper. This study is then further extended to the case of multiple channels. Although only Gaussian and SG pulses were considered in this paper, one can use these results to obtain the parameter dynamics for other kinds of beams that are being lately considered in the field of Optics namely sinh-Gaussian, cosh-Gaussian, Hermite-Gaussian, super-sech just to name a few.

These Dynamical System of the pulse parameters can be used to study various aspects of soliton communications through long distance optical fibers. For example, in trans-oceanic distances there are many unwanted features that arise like the four-wave mixing, frequency and timing jitter, the amplitude jitter, the formation of ghost pulses [3] and many more. The parameter dynamics of the solitons obtained in this paper will serve as a basic necessity to study these aspects that serve as a hindrance in the optical soliton communication.

Moreover, in optical soliton communication the presence of perturbation term is eminent. As for example, terms like higher order dispersion (that arise in tran-oceanic soliton propagation, when the group velocity dispersion is small), Raman scattering, attenuation, nonlinear damping, two-photon absorption and other nonlocal perturbations do arise in practical situations. It is necessary to study the modified behaviour of the solitons in presence of these terms. The adiabatic parameter dynamics of solitons in presence of perturbation terms has been obtained in this paper to facilitate the study of the modified behaviour of solitons due to these particular type of perturbations.

Besides these deterministic perturbations, there also arises the issue of stochastic perturbations. Stochasticity arises with the chaotic nature of the initial pulse due to the partial coherence of the laser generated radiation. It also arises due to random nonuniformities in the optical fibers like the fluctuations in the dielectric constant the random variations of the fiber diameter and more. The chaotic field caused by a dynamic stochasticity might arise from a periodic modulation of the system parameters or when a periodic array of pulses propagate in a fiber optic resonator. Thus stochasticity is inevitable in optical soliton communications. The adiabatic parameter dynamics that are obtained in this paper for optical fibers can also be used to study optical solitons in presence of the stochastic perturbations by analysing the corresponding *Langevin equations*.

Moreover, there is also the factor of soliton radiation. These are the small amplitude dispersive waves that arises in the theory of soliton propagation. These radiation can be studied by using the variational principle and formulating the corresponding parameter dynamics of the solitons. These radiation do arise in polarization preserving fibers, birefringent fibers as well as multiple channels and thus the results of this paper can be used to study them.

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