Progress In Electromagnetics Research, PIER 51, 167–185, 2005

GUIDED WAVES IN UNIAXIAL WIRE MEDIUM SLAB

I. S. Nefedov

Institute of Radio Engineering and Electronics
Russian Academy of Sciences
410019 Zelyonaya 38, Saratov, Russia

A. J. Viitanen

Department of Electrical and Communications Engineering
Helsinki University of Technology
P.O. Box 3000, FI-02015 HUT, Finland

Abstract—Guided waves in a wire medium slab are studied. The wire medium is considered as a continuous medium described in terms of uniaxial permittivity dyadic. Nonlocal model of the wire medium, taking into account spatial dispersion, is used in the analysis. Different cases of an arrangement of the wires are considered. Analytic expressions for the fields in unbounded media and numerical solutions for eigenmodes spectrum in wire medium slab are obtained. Comparison of the results given by old (local) and new model of wire medium is presented.

1 Introduction

2 Theory

3 Arbitrary Directed wires in x-y Plane
3.1 Guided Waves

4 Wires in x-z Plane
4.1 Dispersion

5 Wires in y-z Plane

6 Special Cases
6.1 Wires in x-direction
6.2 Wires in y-direction
6.3 Wires in z-direction

7 Conclusion

References

1. INTRODUCTION

The medium, formed by a regular lattice of ideally conducting wires with small radii compared to the lattice period and the wavelength is well known for a long time [1, 2] and is referred usually as a *wire medium* or *rodded medium*. Recently attention to the wire medium was increased due to realization of "left-handed" or "backward-wave" materials [3, 4]. The well-known model describes wire medium at low-frequency in terms of dyadic permittivity (assuming z-direction of the wires):

$$\overline{\overline{\epsilon}} = \epsilon_t(\mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y) + \epsilon_z\mathbf{u}_z\mathbf{u}_z \tag{1}$$

where ϵ_t is the permittivity of host medium and ϵ_z is given by plasma model formula:

$$\epsilon_z = \epsilon_t\left(1 - \frac{\omega_p^2}{\omega^2\epsilon_t}\right) = \epsilon_t\left(1 - \frac{k_p^2}{k^2}\right) \tag{2}$$

where $k = \omega/c\sqrt{\epsilon_r} = k_o\sqrt{\epsilon_r}$, c is the speed of light, $\epsilon_r = \epsilon_t/\epsilon_0$, ϵ_0 is the permittivity of vacuum, the constant ω_p (or corresponding k_p) is an equivalent "plasma frequency" that gives grounds to call wire medium as "artificial plasma". There exist different models of k_p and we will use below the model proposed in [5, 6]. Note, that the effective medium parameter (2) is obtained from full-wave analysis of two-dimensional periodic lattice of the conductive wires, see [6, 7]. It can be used for solution of the boundary-value problems at low frequencies (for dense lattices, when a period of wires is much smaller than wavelength). The model (1,2) takes into account the direction of wires as well as their density and radius. Application of the continuous medium model in solution of the boundary-value problems gives a possibility to display the most general electromagnetic features of the considered structures, which can be studied later on more rigorously.

It was shown in [8], that three-dimensional cubic lattice, formed of three mutually perpendicular wire arrays, can be considered as isotropic artificial plasma, whose scalar permittivity is expressed by formula (2). Recently we have found, that the model (2) becomes incorrect, if the wavevector in wire medium has a component γ parallel to the wires [7]. In this case application of Eq. (2) leads to unphysical results for the problems of wave propagation in a waveguide filled with a wire medium. The failure of old *local* model (2) is caused by *nonlocal* nature of infinitely long wires, where effects of spatial dispersion take place even at very low frequency. New simple model, taking into account spatial dispersion, was proposed in [7]:

$$\epsilon_z = \epsilon_t \left(1 - \frac{k_p^2}{k^2 - \gamma^2}\right). \tag{3}$$

Thus, the wire medium, considered as continuous one, is a unique artificial material, exhibiting strong spatial dispersion at very small frequencies and its waveguiding properties were never investigated.

In this paper we discuss the guided modes propagation for the different cases of the wires arrangement and compare results obtained using new and old models of the wire medium. Strong difference between the results, caused by spatial dispersion, is demonstrated.

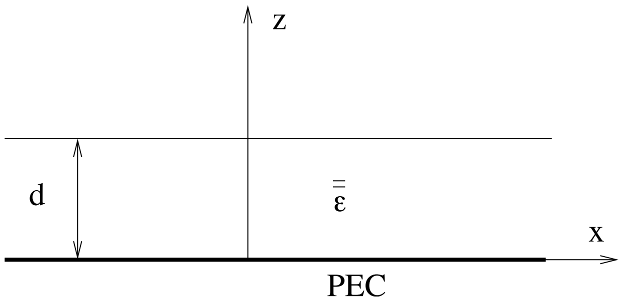

Figure 1. Geometry of wire medium slab problem.

2. THEORY

Let us study guided waves propagating in y direction on a metal plane covered by wire medium layer of thickness d, as it is illustrated in Figure 1. Time harmonic fields ($e^{j\omega t}$) are considered. To obtain the natural modes propagating in the structure, we solve the source free Maxwell's equations. Because the structure extends to infinity in x direction, the fields in the wire medium are constant with respect to x, and depend on y and z as $e^{-jk_y y - j\beta z}$. The constitutive relations of wire medium, where the direction of wires is in the plane perpendicular to a coordinate axis (these special cases are studied here), are expressed as

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E}, \qquad \mathbf{B} = \mu_o \mathbf{H}. \tag{4}$$

Here the permittivity in (x, y, z) coordinate system is expressed through transformation

$$\bar{\bar{\epsilon}} = A \bar{\bar{\epsilon}}' A^T, \tag{5}$$

where A is the matrix of rotation about the respective axis and A^T is the transposed matrix. Dyadic $\overline{\overline{\epsilon}}'$ is a diagonal dyadic in primed coordinate system.

The source free Maxwell equations are written for plane wave fields using the constitutive relations for wire medium

$$(k_y\mathbf{u}_y + \beta\mathbf{u}_z) \times \mathbf{E} = \omega\mu_o\mathbf{H} \tag{6}$$

$$(k_y\mathbf{u}_y + \beta\mathbf{u}_z) \times \mathbf{H} = -\omega A\overline{\overline{\epsilon}}' A^T \cdot \mathbf{E} \tag{7}$$

After eliminating the magnetic field, the equation for electric field (eigenvalue equation) is obtained in the form

$$\left[(k_y^2 + \beta^2)\overline{\overline{I}} - \omega^2\mu_o\overline{\overline{\epsilon}}' - A^T(k_y\mathbf{u}_y + \beta\mathbf{u}_z)(k_y\mathbf{u}_y + \beta\mathbf{u}_z)A\right] \cdot \left(A^T\mathbf{E}\right) = 0 \tag{8}$$

In this form the eigenvalue equation is in a form, which is applicable for the wires positioned in x-y, x-z or y-z planes. The eigenvalues and eigenwaves of (8) are propagation constants and electric field components of plane waves, respectively, propagating in z-direction under fixed k_y. They are evaluated in three cases of the wires arrangement and the guided waves are studied for conductor backed slab structure.

3. ARBITRARY DIRECTED WIRES IN x-y PLANE

Consider arbitrary arrangement of the wires in x-y plane, when the angle between wires and y-axis is equal ϕ. The permittivity dyadic in coordinate system (x', y', z'), where y'-axis is directed along the wires, $z' = z$, looks as:

$$\overline{\overline{\epsilon}}' = \begin{pmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_w & 0 \\ 0 & 0 & \epsilon_t \end{pmatrix}, \tag{9}$$

where

$$\epsilon_w = \epsilon_t \left(1 - \frac{k_p^2}{k^2 - (k_y \cos\phi)^2}\right) \tag{10}$$

The permittivity dyadic in (x, y, z) coordinate system is expressed as (5) where A is the matrix of rotation by the angle ϕ about z-axis

$$A = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

The notation of parameters used in the text are $\epsilon_t = \epsilon_r\epsilon_o$ and $k = k_o\sqrt{\epsilon_r}$ where $k_o = \omega\sqrt{\mu_o\epsilon_o}$.

Introducing parameters s and p such that

$$s = k_y^2 + \beta^2 - k^2, \quad p = k_y^2 + \beta^2 - k^2 \frac{\epsilon_w}{\epsilon_t} \tag{12}$$

and writing $\mathbf{E}' = A^T \mathbf{E}$, the eigenvalue equation (8) is in matrix form as

$$\begin{bmatrix} k_y^2 \sin^2\phi - s & k_y^2 \sin\phi\cos\phi & k_y\beta\sin\phi \\ k_y^2 \sin\phi\cos\phi & k_y^2\cos^2\phi - p & k_y\beta\cos\phi \\ k_y\beta\sin\phi & k_y\beta\cos\phi & \beta^2 - s \end{bmatrix} \begin{bmatrix} E'_x \\ E'_y \\ E'_z \end{bmatrix} = 0. \tag{13}$$

Equating the determinant equal to zero results in eigenvalue equation

$$s[ps - p(\beta^2 + k_y^2\sin^2\phi) - sk_y^2\cos^2\phi] = 0 \tag{14}$$

which is solved. The separate solutions are:

$$s = 0 \quad \Leftrightarrow \quad \beta^2 = \beta_o^2 = k^2 - k_y^2 \tag{15}$$

$$ps - p(\beta^2 + k_y^2\sin^2\phi) - sk_y^2\cos^2\phi = 0 \quad \Leftrightarrow \quad \beta^2 = \beta_e^2 = k^2 - k_y^2 - k_p^2 \tag{16}$$

There are two eigenwaves, ordinary wave (β_o) and extraordinary wave (β_e) propagating in the wire medium. One can show from the Maxwell equations, that it is impossible to separate waves into TE and TM parts for arbitrary ϕ values. Thus, the fields inside the wire medium slab are expressed by four waves (transmitted and reflected) of unbounded wire medium, whose wavevector components (β) are equal to that of ordinary and extraordinary waves which do not explicitly depend on ϕ. The eigenvectors for electric field are obtained from the equation (13) using the transformation $\mathbf{E} = A\mathbf{E}'$. We study the guided waves in a metal backed wire medium slab. To determine the propagation constant of these waves, usually, the field inside the slab are presented in terms of eigenvectors. Considering the continuity of tangential field components at the interface between the wire medium and the free space we arrive at the eigenvalue equation from which the propagation constant k_y is found. In this study, however, instead of using eigenvector presentation a transfer matrix method given in [9] is used.

3.1. Guided Waves

One way to calculate the propagation constant k_y of guided modes is to use 4×4 matrix equation for tangential field components. Calculation of the mode spectrum can be carried out using 4×4 transfer matrix of the wire slab, which connects tangential (x and y) components of electric and magnetic field at the slab boundaries $z = 0$ and $z = d$. Transfer matrix $[B]$ for (bi)anisotropic medium can be obtained from the Maxwell equations. Let us denote 4-vector of tangential field components as $X = (E_x, E_y, H_x, H_y)^T$, where '$T$' is the transposition symbol. Assuming $\partial/\partial x = 0$, $\partial/\partial y = -jk_y$, the field components E_z and H_z can be expressed in terms of four tangential components. Thus, the source-free Maxwell equations can be written as a system of four ordinary differential equations (ODE), which looks in matrix notation as

$$\frac{d}{dz}X = [M]X. \tag{17}$$

The elements of matrix $[M]$ for the most general case of bianisotropic medium, including particular case of the wire medium, are given in [9]. Nonzero matrix elements for the wire medium, arranged in x-y plane are given below:

$$\begin{aligned} M_{14} &= -jk_o\mu_r\eta_o \\ M_{23} &= j\eta(k_o^2\epsilon_{zz}\mu_r - k_y^2)/(k_o\epsilon_{zz}) \\ M_{31} &= jk_o\epsilon_{yx}/\eta_o \\ M_{32} &= jk_o\epsilon_{yy}/\eta_o \\ M_{41} &= -j(k_o^2\epsilon_{xx}\mu_r - k_y^2)/(k_o\mu_r\eta_o) \\ M_{42} &= -jk_o\epsilon_{xy}/\eta_o \end{aligned} \tag{18}$$

where μ_r and ϵ_{kl}, $k, l = x, y, z$ are relative permeability and permittivity, respectively. $\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}}$ is the wave impedance in free space. Here $\mu_r = 1$ and the components of the permittivity dyadic in this particular case are $\epsilon_{xx} = (\epsilon_t \cos^2\phi + \epsilon_w \sin^2\phi)/\epsilon_o$, $\epsilon_{xy} = \epsilon_{yx} = (\epsilon_t - \epsilon_w)\sin\phi\cos\phi/\epsilon_o$, $\epsilon_{yy} = (\epsilon_t \sin^2\phi + \epsilon_w \cos^2\phi)/\epsilon_o$ and $\epsilon_{zz} = \epsilon_t/\epsilon_o$. The eigennumbers of the matrix $[M]$ are $\pm j\beta_o$ and $\pm j\beta_e$ which correspond to the eigenvalues for ordinary and extraordinary waves. Despite the angle between wires and y-axis does not enter explicitly in (16) for extraordinary wave, the angle dependency exists there implicitly via k_y because the angle enters through the elements of the matrix (18). Thus, both eigenvalues and guided wave spectrum depend on the angle ϕ and spatial dispersion affects the eigenvalues.

Solution of the Cauchy problem for ODE (17) has form

$$X(z) = [B]X(0), \tag{19}$$

where matrix exponent

$$[B] = \exp([M]z) \tag{20}$$

is the transfer matrix of the layer having thickness z. Transfer matrix technique is very convenient for solution of the boundary-value problems for layered media because it formalizes the process of satisfying the boundary conditions at the interfaces of the layers reducing it to multiplication of the transfer matrices of separate layers.

If $[B]$ is the transfer matrix of the wire medium slab, following relation takes place:

$$(E_x(d), E_y(d), H_x(d), H_y(d))_{col} = [B](E_x(0), E_y(0), H_x(0), H_y(0))_{col} \tag{21}$$

where '*col*' denotes a column-vector. For the air, the same components can be expressed via E_z and H_z:

$$\begin{aligned} E_x^a &= -\frac{k_o\eta_o}{k_y} H_z^a = -p_1 H_z^a, \quad & E_y^a &= -\frac{k_o^2 - k_y^2}{k_y\beta} E_z^a = -p_2 E_z^a, \\ H_x^a &= \frac{k_o}{k_y\eta_o} E_z^a = -p_3 E_z^a, \quad & H_y^a &= -\frac{k_o^2 - k_y^2}{k_y\beta} H_z^a = -p_4 H_z^a, \end{aligned} \tag{22}$$

where $\mathrm{Im}(\beta) < 0$. Writing boundary conditions for the tangential field components at $z = d$, the system of equations, linking tangential field components at $z = d$, looks as

$$\begin{aligned} B_{13}H_x^0 + B_{14}H_y^0 + p_1 H_z^a &= 0 \\ B_{23}H_x^0 + B_{24}H_y^0 + p_2 E_z^a &= 0 \\ B_{33}H_x^0 + B_{34}H_y^0 + p_3 E_z^a &= 0 \\ B_{43}H_x^0 + B_{44}H_y^0 + p_4 H_z^a &= 0, \end{aligned} \tag{23}$$

where H_x^0 and H_y^0 are unknown components of magnetic field at the metal $z = 0$. Non-trivial solutions of this system exist if its determinant is equal to zero, that gives dispersion equation.

The results of the mode spectrum calculations for the different angles ϕ are given at Figure 2. The wire lattice is inserted to isotropic host medium with $\epsilon_r = 3$. Parameters of the wires are taken by such a manner, that plasma wavevector $k_p d = 1.9$. We will consider only two lowest modes in each case. For the slab without wires the dominant TM^z or TM^y mode is shown by solid curve 1. Second-order mode in this case is TE^z or TE^y one which is illustrated by the dashed curve 0. The dominant TM^z (TM^y) mode has no low-frequency cut-off. The cases $\phi = 0$ and $\phi = \pi/2$ correspond to the wires in y-direction and x-direction, respectively.

If the wires are perpendicular to propagation direction, $\phi = \pi/2$, dispersion curve for the ordinary wave (TM mode) coincides with the same for the case of absence of wires, see solid curve 1. The wires do

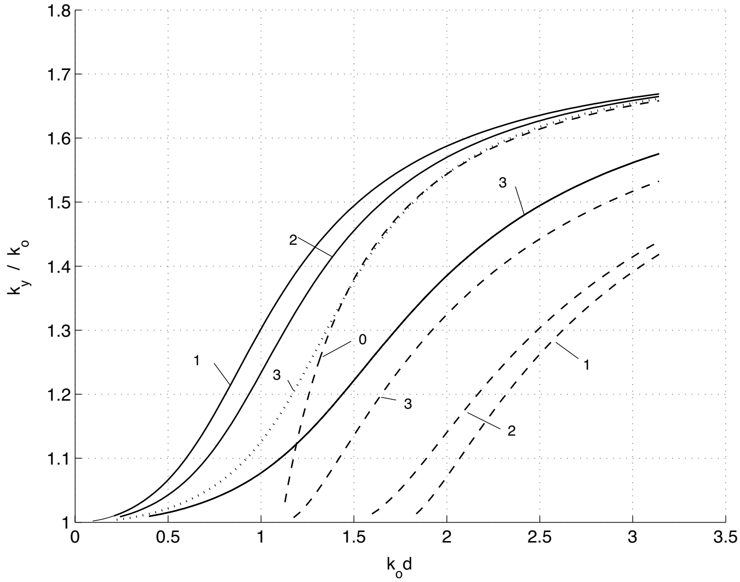

Figure 2. Slow-wave factor k_y/k_o versus k_od, calculated for different angles ϕ between y-axis and the wires in x-y plane. Permittivity of the host medium $\epsilon_r = 3$. Solid and dashed curves show dispersion of the lowest and second-order modes, respectively. The curves 1 correspond to the case $\phi = \pi/2$, the curves $2 - \phi = \pi/4$, the curves $3 - \phi = 0$. The dashed curve 0 corresponds to TE mode, propagating in the absence of wire filling. The dotted curve illustrates dispersion, given by old model for extraordinary wave with $\phi = 0$.

not influence on this mode. Extraordinary wave (TE mode) shifts to high-frequency region, that illustrates the dashed curve 1.

For $\phi = 0$, TM mode corresponds to the extraordinary wave (solid curve 3). It shifts to high-frequency region in comparison with the case of absence of wires, but has no low-frequency cut-off. Thus, the dominant mode has no low-frequency cut-off if the wires are parallel to a ground plane despite electric field has nonzero projection on the wires. The guided (non-radiative) waves in open structure should be slow-waves ($k_y/k_o > 1$) and it takes place only if $k_od > 0$. As a comparison, the result, given by old model for extraordinary wave ($\phi = 0$), is presented by dotted curve. Spatial dispersion does not affect the ordinary TE mode and both of models give the same result (see dashed curve 3).

In order to better understand this, consider wire media slab,

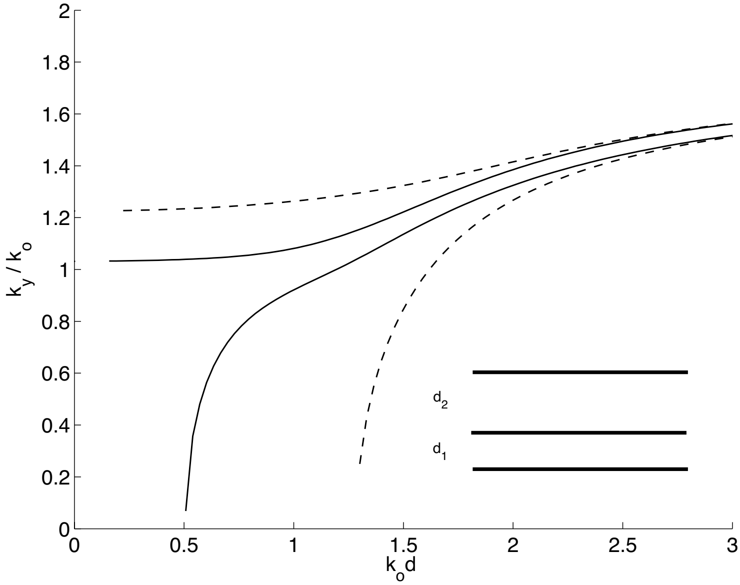

Figure 3. Slow-wave factor k_y/k_o versus $k_o d$, calculated for $d_2/d_1 = 5$ (solid curves) and $d_2/d_1 = 0.5$ (dashed curves). The wires are directed along y-axis. Parameters of the wire slab are the same as in Figure 2. Dispersion of the two lowest-order modes is shown.

bounded by two metal planes (see insertion to the Figure 3). Using transfer matrix $[B]$ of two-layered structure, bounded by the planes at $z = 0$ and $z = d_1 + d_2$, dispersion equation can be obtained:

$$\det \begin{pmatrix} B_{13} & B_{14} \\ B_{23} & B_{24} \end{pmatrix} = 0. \tag{24}$$

Consideration of the bounded waveguide allows to see dispersion at $k_o d = 0$. Obviously, if $d_2 = 0$, the wires in x-y plane do not influence the fields of TEM mode, which remains dispersionless one. Removal of a top metal plane causes appearing y-component of electric field and the wave dispersion. It is illustrated by the Figure 3, where the wave dispersion is presented for different distances of top metal plane from the wires.

4. WIRES IN x-z PLANE

Let us next take the direction of wires in $x - z$-plane, then

$$\overline{\overline{\epsilon}}' = \begin{pmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_t & 0 \\ 0 & 0 & \epsilon_w \end{pmatrix}, \qquad \epsilon_w = \epsilon_t \left(1 - \frac{k_p^2}{k^2 - (\beta \cos\vartheta)^2} \right). \tag{25}$$

The permittivity dyadic in (x, y, z) coordinate system is expressed through the rotation matrix A by an angle ϑ about y-axis

$$\overline{\overline{\epsilon}} = A \overline{\overline{\epsilon}}' A^T, \qquad A = \begin{pmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \tag{26}$$

Denoting the parameters p and s as in the previous section and writing $\mathbf{E}' = A^T \mathbf{E}$, the eigenvalue equation (8) is in matrix form as

$$\begin{bmatrix} \beta^2 \sin^2\vartheta - s & k_y \beta \sin\vartheta & \beta^2 \sin\vartheta \cos\vartheta \\ k_y \beta \sin\vartheta & k_y^2 - s & k_y \beta \cos\vartheta \\ \beta^2 \sin\vartheta \cos\vartheta & k_y \beta \cos\vartheta & \beta^2 \cos^2\vartheta - p \end{bmatrix} \begin{bmatrix} E'_x \\ E'_y \\ E'_z \end{bmatrix} = 0. \tag{27}$$

Putting the determinant equal to zero results in eigenvalue equation

$$s[ps - p(\beta^2 \sin^2\vartheta + k_y^2) - s\beta^2 \cos^2\vartheta] = 0, \tag{28}$$

which gives us the solutions for ordinary and extraordinary waves, $\beta_o^2 = k^2 - k_y^2$ and $\beta_e^2 = k^2 - k_y^2 - k_p^2$, respectively.

As in the previous section, we consider the fields at PEC boundary and at the interface of wire medium and free space. The PEC boundary couples the ordinary and extraordinary waves, except in the special cases $\vartheta = 0$ or $\vartheta = \pi/2$. Nonzero elements of matrix $[M]$ for x-z case are following:

$$\begin{aligned}
M_{14} &= -jk_o \mu_r \eta_o \\
M_{21} &= jk_y \epsilon_{zx}/\epsilon_{zz} \\
M_{23} &= j\eta(k_o^2 \epsilon_{zz} \mu_r - k_y^2)/(k_o \epsilon_{zz}) \\
M_{32} &= jk_o \epsilon_{yy}/\eta_o \\
M_{41} &= -j((k_o^2 \epsilon_{xx} \mu_r - k_y^2)/(k_o \mu_r) - k_o \epsilon_{xz} \epsilon_{zx}/\epsilon_{zz})/\eta_o \\
M_{43} &= -jk_y \epsilon_{xz}/\epsilon_{zx}.
\end{aligned} \tag{29}$$

In this case $\epsilon_{xx} = (\epsilon_t \cos^2\vartheta + \epsilon_w \sin^2\vartheta)/\epsilon_o$, $\epsilon_{xz} = \epsilon_{zx} = (\epsilon_t - \epsilon_w) \sin\vartheta \cos\vartheta/\epsilon_o$, $\epsilon_{yy} = \epsilon_t/\epsilon_o$ and $\epsilon_{zz} = (\epsilon_t \sin^2\vartheta + \epsilon_w \cos^2\vartheta)/\epsilon_o$. The eigennumbers of $[M]$ correspond to those of ordinary and extraordinary waves. Solving the ordinary differential equation (17) and using the transfer matrix [B] for a layer of wire medium, the relation linking tangential field components at $z = 0$ (PEC boundary) and at $z = d$ is obtained. The eigenvalue equation is solved numerically.

4.1. Dispersion

Figure 4 shows dispersion of the waves in x-z case for different angles ϑ between the wires and z-axis. Parameters of the wire and host media are the same as in the previous cases. Curve '0' corresponds to $\vartheta = \pi/2$ (wires in x-direction), that is TM_0 mode, the same as in absence of the wire filling because there is no component of electric field along the wires. We show only two lowest modes.

For the last case it is denoted, that for extraordinary wave the

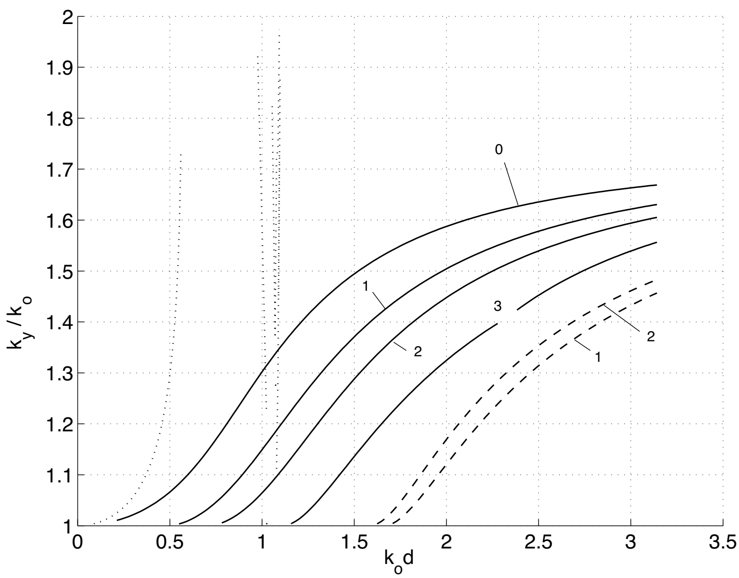

Figure 4. Slow-wave factor k_y/k_o versus k_od, the wires lie on x-z plane. The solid and dashed curves correspond to the dispersion of the lowest and second-order modes, respectively. Dotted curves show characteristics, calculated using conventional model of wire medium. The curve 0 shows dispersion of TM_0 mode in the case $\vartheta = \pi/2$, the curves 1 and 2 relate to $\vartheta = \pi/4$ and $\vartheta = \pi/6$, respectively and the curve 3 corresponds to the wires directed along z axis, i.e, $\vartheta = 0$.

propagation factor in z direction is equal to $\beta = \sqrt{\omega^2\mu_o\epsilon_r\epsilon_o - k_y^2 - k_p^2}$ and the cut-off number for the guided waves is $k_c = k_p/\sqrt{\epsilon_r - 1}$. Note, that it relates to the dominant mode, which has no low-frequency cut-off for an isotropic slab, $\epsilon_r > 0$ (see curve '0'). Electric and magnetic fields are described by trigonometric functions beyond cut-off within the slab and by exponent functions in air for the slow-waves $k_y/k_o > 1$.

In difference with the new model, the old model (2) gives physically meaningless results as illustrated in Figure 4. It becomes even more clear, if we consider the wire medium slab (the wires are parallel to z-axis), bounded by electric walls at $z = 0$ and $z = d$. Then the propagation constant for TM modes is expressed as

$$k_y = \sqrt{k_o^2\epsilon - \beta_m\epsilon/\epsilon_r} \tag{30}$$

where $\epsilon < 0$ is determined by formula

$$\epsilon = \epsilon_r\left(1 - \frac{k_p^2}{k^2}\right) \tag{31}$$

and $\beta_m = m\pi/d$, m is an integer. Evidently, it follows from formula (30), that the waves can propagate at any k_o if m is enough large and $\epsilon < 0$, which is impossible. If to remove the top metal plane, k_y is found from equation (30) and there also exists countable spectrum of modes with limiting point $k_o = k_p - 0$. Some lowest modes for open structure are shown by dotted curves at Figure 4. Thus, if for the wires in x-y plane we observe some quantitative difference in the results given by old and new models, here the old model displays total inconsistence.

5. WIRES IN y-z PLANE

In this section we consider the wires in y-z plane, i.e., in the same plane as the propagation vector. The permittivity in coordinate system with y' axis, directed along the wires, is expressed as

$$\overline{\overline{\epsilon}}' = \begin{pmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_w & 0 \\ 0 & 0 & \epsilon_t \end{pmatrix}, \quad \epsilon_w = \epsilon_t\left(1 - \frac{k_p^2}{k^2 - (k_y\cos\psi + \beta\sin\psi)^2}\right) \tag{32}$$

The permittivity in (x, y, z) coordinate system is obtained through the transformation

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}, \tag{33}$$

where A is the rotation matrix (by an angle ψ) about x-axis. The eigenvalue equation is derived straightforwardly inserting the permittivity dyadic in (7) and eliminating either electric or magnetic field in (6) and (7). Here, as in the previous cases, we use the parameters p and s and write the eigenvalue equation (8) in matrix form

$$\begin{bmatrix} -s & 0 & 0 \\ 0 & (k_y\cos\psi+\beta\sin\psi)^2-p & -(k_y\cos\psi+\beta\sin\psi)(k_y\sin\psi-\beta\cos\psi) \\ 0 & -(k_y\cos\psi+\beta\sin\psi)(k_y\sin\psi-\beta\cos\psi) & (k_y\sin\psi-\beta\cos\psi)^2-s \end{bmatrix} \begin{bmatrix} E'_x \\ E'_y \\ E'_z \end{bmatrix} = 0 \qquad (34)$$

Equating the determinant to zero results in eigenvalue equation

$$s[ps-(k_y\cos\psi+\beta\sin\psi)^2 s-(k_y\sin\psi-\beta\cos\psi)^2 p]=0, \qquad (35)$$

the solutions of which are propagation factors for ordinary and extraordinary waves. The propagation factor for ordinary waves results from

$$s=0 \quad \Leftrightarrow \quad \beta_o^2=k^2-k_y^2 \qquad (36)$$

and the corresponding eigenvector is

$$\mathbf{E}_o=E_o\mathbf{u}_x\, e^{-jk_y y}e^{-j\beta_o z}. \qquad (37)$$

The magnetic field is

$$\mathbf{H}_o=\frac{E_o}{\eta}\left[\frac{\beta_o}{k}\mathbf{u}_y-\frac{k_y}{k}\mathbf{u}_z\right]e^{-jk_y y}e^{-j\beta_o z}. \qquad (38)$$

For extraordinary waves the propagation factor is obtained from the equation

$$ps-(k_y\cos\psi+\beta\sin\psi)^2 s-(k_y\sin\psi-\beta\cos\psi)^2 p=0 \;\Leftrightarrow\; \beta_e^2=k^2-k_y^2-k_p^2. \qquad (39)$$

Surprisingly enough, the value of the propagation factor β_e is the same for the waves propagating in $-z$-direction. This can be shown when, in the matrix equation as well as in the expression for ϵ_w the propagation factor $\beta\rightarrow-\beta$. The eigenvector is

$$\mathbf{H}_e=H_e\mathbf{u}_x\, e^{-jk_y y}e^{-j\beta_e z}. \qquad (40)$$

Denoting $\gamma_\pm = k_y\cos\psi\pm\beta_e\sin\psi$ and using (7), finally, the corresponding electric field is obtained for the waves traveling in $+z$-direction (γ_+) and in $-z$-direction (γ_-):

$$\mathbf{E}_{e\pm}=-\eta H_{e\pm}\left[\left[\pm\frac{\beta_e}{k}\frac{k^2-\gamma_\pm^2-k_p^2\sin^2\psi}{k^2-\gamma_\pm^2-k_p^2}-\frac{k_y}{k}\frac{k_p^2\sin\psi\cos\psi}{k^2-\gamma_\pm^2-k_p^2}\right]\mathbf{u}_y\right.$$

$$+\left[\pm\frac{\beta_e}{k}\frac{k_p^2\sin\psi\cos\psi}{k^2-\gamma_\pm^2-k_p^2}-\frac{k_y}{k}\frac{k^2-\gamma_\pm^2-k_p^2\cos^2\psi}{k^2-\gamma_\pm^2-k_p^2}\right]\mathbf{u}_z\Bigg]\ e^{-jk_yy}e^{\mp j\beta_e z} \tag{41}$$

It is interesting to notice, that the fields inside the wire medium in this special case are decomposed into TE and TM parts with any ψ values. The ordinary wave is TE and extraordinary wave is TM. This was not possible in the two previous cases. This fact can be used for determining the guided waves in the slab problem.

Outside the slab, in free space, the transverse electric and magnetic fields are given in terms of z-components of the fields

$$\mathbf{e}^a(z) = \frac{1}{\omega^2\mu_o\epsilon_o-\beta^2}[-\beta k_yE_z\mathbf{u}_y-\omega\mu_ok_yH_z\mathbf{u}_x]e^{-j\beta z} \tag{42}$$

$$\mathbf{h}^a(z) = \frac{1}{\omega^2\mu_o\epsilon_o-\beta^2}[-\beta k_yH_z\mathbf{u}_y+\omega\epsilon_ok_yE_z\mathbf{u}_x]e^{-j\beta z} \tag{43}$$

where $\beta = \sqrt{k_o^2-k_y^2}$. For guided waves the propagation factor is imaginary $\beta=-j\sqrt{k_y^2-k_o^2}=-jq$ ($-$ sign for decaying fields) and the transverse fields in the region $z\geq d$ are

$$\mathbf{e}^a(z) = \left[\frac{jq}{k_y}E_z\mathbf{u}_y-\frac{\omega\mu_o}{k_y}H_z\mathbf{u}_x\right]e^{-qz} \tag{44}$$

$$\mathbf{h}^a(z) = \left[\frac{jq}{k_y}H_z\mathbf{u}_y+\frac{\omega\epsilon_o}{k_y}E_z\mathbf{u}_x\right]e^{-qz} \tag{45}$$

The continuity of the transverse fields at the interface $z=d$ gives us the eigenvalue equation for guided modes.

For ordinary wave the condition for guided modes is as for a PEC backed dielectric slab. There is no effects caused by wires. The guided waves are obtained by solving the equation

$$-\beta_o\cot\beta_od=q \tag{46}$$

For extraordinary wave (TM mode) the effects of wires are present. Denoting the relation between transverse fields as $\mathbf{e}_\pm=Z_\pm(-\mathbf{u}_z\times\mathbf{h}_\pm)$, the wave impedance of extraordinary wave propagating in $+z$-direction is

$$Z_+=\eta\left[\frac{\beta_e}{k}\frac{k^2-\gamma_+^2-k_p^2\sin^2\psi}{k^2-\gamma_+^2-k_p^2}-\frac{k_y}{k}\frac{k_p^2\sin\psi\cos\psi}{k^2-\gamma_+^2-k_p^2}\right] \tag{47}$$

and for wave propagating in $-z$-direction

$$Z_-=\eta\left[-\frac{\beta_e}{k}\frac{k^2-\gamma_-^2-k_p^2\sin^2\psi}{k^2-\gamma_-^2-k_p^2}-\frac{k_y}{k}\frac{k_p^2\sin\psi\cos\psi}{k^2-\gamma_-^2-k_p^2}\right] \tag{48}$$

It can be shown (made numerically) that $Z_+ = -Z_-$ which simplify field expressions. This is also related to the reciprocity of wire medium. Taking into account the PEC boundary condition, the transverse fields inside the slab can be written as

$$\begin{aligned} \mathbf{e}(z) &= 2jZ_+H_{e+} \sin\beta_e z \, \mathbf{u}_y, \\ \mathbf{h}(z) &= H_{e+}\left[e^{-j\beta_e z} - \frac{Z_+}{Z_-}e^{j\beta_e z}\right]\mathbf{u}_x = 2H_{e+}\cos\beta_e d\mathbf{u}_x \end{aligned} \tag{49}$$

At the interface $z = d$, the continuity of the tangential fields straightforwardly results in the condition for the guided waves for TM modes

$$Z_+ \tan\beta_e d = \frac{q\eta_o}{k_o} \tag{50}$$

which has real solutions for k_y with any ψ values.

6. SPECIAL CASES

The special cases where the direction of wires coincides with the direction of coordinate axes are considered. In these cases the expressions are simplified remarkably, and it is found that in these particular cases the decomposition into TE and TM waves takes place. The ordinary and extraordinary waves correspond to TE or TM waves. Generally, the decomposition into TE and TM parts is not possible except for the wire medium in y-z plane. The eigenvalue equation with the respective dispersion relations

$$q^2d^2 + \beta_o^2 d^2 = (\epsilon_r - 1)k_o^2 d^2 \tag{51}$$

for ordinary waves and

$$q^2d^2 + \beta_e^2 d^2 = (\epsilon_r - 1)k_o^2 d^2 - k_p^2 d^2 \tag{52}$$

for extraordinary waves form a system of two equations from which we can find the slow wave factor k_y/k_o.

6.1. Wires in x-direction

This special case is obtained by taking $\phi = \pi/2$ or $\vartheta = \pi/2$. The eigenvalue equation (23) reduces to

$$\left[\frac{qk}{k_y\beta_o}\cos\beta_o d - \frac{k_o\eta}{k_y\eta_o}\sin\beta_o d\right]\left[\frac{qk\eta}{k_y\beta_e\eta_o}\sin\beta_e d + \frac{k_o}{k_y}\cos\beta_e d\right] = 0, \tag{53}$$

which leads to separate solutions: for ordinary waves (TM modes)

$$\frac{\beta_o \tan \beta_o d}{\epsilon_r} = q \tag{54}$$

and for extraordinary waves (TE modes)

$$-\beta_e \cot \beta_e d = q. \tag{55}$$

The lowest mode is ordinary wave (TM_0 mode), and the eigenvalue equation is like for a normal metal backed dielectric slab problem, there is no low frequency cut-off. For extraordinary wave (TE mode) there is a shifted cut-off frequency.

6.2. Wires in y-direction

The special case when the wires are in the same direction, as the propagation direction, is obtained at $\phi = 0$ or $\psi = 0$. Then the eigenvalue equation (23) reduces to the form:

$$\left[\frac{q\eta}{k_o\eta_o}\sin\beta_o d + \frac{\beta_o}{k}\cos\beta_o d\right]\left[\frac{k_o q}{k_y^2}\cos\beta_e d - \frac{k_o^2\eta}{k_y^2\eta_o}\left(\frac{k^2 - k_y^2}{k\beta_e}\right)\sin\beta_e d\right] = 0. \tag{56}$$

The eigenvalue equation is split into two parts, for TM modes ($E_z \neq 0, H_z = 0$) and for TE modes ($H_z \neq 0, E_z = 0$), respectively. These equations are

$$\frac{(k^2 - k_y^2)\tan\beta_e d}{\epsilon_r \beta_e} = q \tag{57}$$

and

$$-\beta_o \cot \beta_o d = q. \tag{58}$$

It can be noticed, that for the lowest extraordinary wave (TM_0 mode) there is no cut-off at low frequency. For ordinary wave (TE mode) there is a normal cut-off since it is not the lowest mode.

6.3. Wires in z-direction

The special case, $\vartheta = 0$ or $\psi = \pi/2$, is considered in this subsection. For extraordinary waves (TM modes) $Z_+ = -Z_- = \frac{\eta\beta_e}{k}$ and the eigenvalue equation (50) takes the form

$$\frac{\beta_e \tan \beta_e d}{\epsilon_r} = q. \tag{59}$$

Because for the guided waves q should be a positive real number in lossless case, also β_e should be a real number. Considering the dispersion relation (52), the relevant solutions for q and β_e exist when $k_o > \frac{k_p}{\sqrt{\epsilon_r - 1}}$. Thus, there is a low frequency cut-off for the lowest mode. This effect is not present for normal dielectric slab structure. Also for x- or y-directed wire medium there is no cut-off effects for the lowest modes.

The eigenvalue equation for ordinary waves (TE modes) is (46) which is like for a normal metal backed dielectric slab. This is not the lowest mode and there is a normal cut-off effect.

7. CONCLUSION

In this paper we considered the guided waves in wire media slab, restricting ourselves with the cases of the wires positioned in x-y plane, in x-z plane and in y-z plane. We have shown the difference between the waveguide mode spectra giving by the conventional model [2] and the new [7] model of wire medium in which the spatial dispersion is taken into account. The mode spectra both for old (local) and new model depend on the angle between wires and coordinate axes. Comparison of the results using old and new models of wire medium is shown in Figures 2 and 4. We have found, that the dominant mode without low-frequency cut-off can propagate for any cases of the wires direction in x-y plane and it does not exist if an angle between wires and z-axis is not equal to $\pi/2$. In the medium with the wires in y-z plane, the fields are decomposed in TE (ordinary wave) and TM (extraordinary wave) cases. Especially, when the wires are in z direction, there exists low-frequency cut-off for the lowest mode. This effect can be used for preventing undesirable field propagation and coupling through a surface at low frequency.

In our analysis we assume the wire array as a continuous medium and do not consider a set of TEM modes, which can propagate along the wires. Namely, in the set of N wires above a metal plane there exist N TEM modes, having the same propagation constants $k_o\sqrt{\epsilon_r}$ and different field distribution determining their wave impedances.

REFERENCES

1. Brown, J., "Artificial dielectrics," *Progress in Dielectrics*, Vol. 2, 195–225, 1960.
2. King, R. J., D. V. Thiel, and K. S. Park, "The synthesis of surface

reactance using an artificial dielectric," *IEEE Trans. Antennas and Propagat.*, Vol. 31, 471–476, 1983.

3. Veselago, V. G., "The electrodynamics of substances with simultaneously negative values of ϵ and μ," *Soviet Physics Uspekhi*, Vol. 10, 509–514, 1968 (originally published in *Uspekhi Fizicheskikh Nauk*, Vol. 92, 517–526, 1967 (in Russian).
4. Smith, D. R., W. J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, "Composite medium with simultaneously negative permeability and permittivity," *Phys. Rev. Lett.*, Vol. 84, 4184–4187, 2000.
5. Maslovski, S. I., S. A. Tretyakov, and P. A. Belov, "Wire media with negative effective permittivity: a quasi-static model," *Microwave and Optical Technology Letters*, Vol. 35, No. 1, 47–51, October 2002.
6. Belov, P. A., S. A. Tretyakov, and A. J. Viitanen, "Dispersion and reflection properties of artificial media formed by regular lattices of ideally conducting wires," *Journal of Electrom. Waves and Applic.*, Vol. 16, No. 8, 1153–1170, 2002.
7. Belov, P. A., R. Marques, S. I. Maslovski, I. S. Nefedov, M. Silveirinha, C. R. Simovski, and S. A. Tretyakov, "Strong spatial dispersion in wire media in the very large wavelength limit," *Phys. Rev. B*, Vol. 63, 113103-1–4, 2003.
8. Pendry, J. B., A. J. Holden, W. J. Stewart, and I. Youngs, "Extremely low frequency plasmons in metallic mesostructures," *Phys. Rev. Lett.*, Vol. 76, No. 25, 4773–4776, 1996.
9. Nefedov, I. S., "Microstrip slow-wave structures on the bianisotropic substrate," *Electromagnetics*, Vol. 17, No. 4, 343–360, 1997.

Igor S. Nefedov received Dipl. Physicist, Ph.D., and Doctor of Sciences Degrees (all in radiophysics) from the Saratov State University (SSU) (Russia), in 1972, 1981, and 1998, respectively. From 1975 to 1992 he was with the research Institute of Mechanics and Physics at SSU, USSR. Since 1992 he has been with the Institute of Radio Engineering and Electronics, Russian Academy of Science, Saratov, Russia. Presently, he is Leading Researcher at this Institute. He has been a visiting Professor at the Radio Laboratory of the Helsinki University of Technology (2001–2002). Since 2003 he is Professor at SSU, chair of Laser and Computer physics. His main scientific interests are electromagnetic field theory, complex media electromagnetics and optics.

Ari J. Viitanen received the Dipl.Eng., Lic.Tech., and Dr.Tech. degrees in electrical engineering from the Helsinki University of Technology, Finland in 1984, 1989, and 1991, respectively. From 1985 to 1989 he was a Research Engineer with the Nokia Research Center. Since 1990 he has been with the Electromagnetics Laboratory, Helsinki University of Technology. His main research interests are electromagnetic field theory, complex media and microwave engineering.