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# DIFFRACTION BY AN ANISOTROPIC IMPEDANCE HALF PLANE — REVISED SOLUTION

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**Abstract**—In a recent solution of this problem there is a subtle error that shows up in the coefficient of reflection off the lower surface of the half plane for oblique incidence and that is attributable to an unacceptable normalization of the spectra to produce the incident field. The correction of this error requires a substantial modification to the original analysis, but this has been carried out and new data are presented.

## 1. BACKGROUND

In a recent paper [1] Maliuzhinets' method [2] was used to solve the problem of a plane electromagnetic wave incident at an oblique (skew) angle on a half plane subject to first order impedance boundary conditions on the faces. In terms of the cylindrical polar coordinates $(\rho, \phi, z)$ the upper and lower faces of the half plane are $\phi = \pi$ and $\phi = -\pi$ respectively where the boundary conditions are

$$E_\rho = \mp\eta_2 Z H_z, \qquad E_z = \pm\eta_1 Z H_\rho \tag{1}$$

on $\phi = \pm\pi$, and

$$\overline{\overline{\eta}} = \eta_1\hat{\rho}\hat{\rho} + \eta_2\hat{z}\hat{z}$$

is the normalized surface impedance, the same on both faces. The half plane is illuminated by a plane wave whose $z$ components are

$$E_z^{\mathrm{i}} = e_z e^{-jk\hat{\imath}\cdot\mathbf{r}}, \qquad ZH_z^{\mathrm{i}} = h_z e^{-jk\hat{\imath}\cdot\mathbf{r}} \tag{2}$$

where

$$\hat{\imath} = -\hat{x}\cos\phi_0\sin\beta - \hat{y}\sin\phi_0\sin\beta + \hat{z}\cos\beta$$

is the direction of incidence and a time factor $e^{j\omega t}$ has been suppressed. For normal incidence $\beta = \pi/2$.

Following Maliuzhinets [2] the total field components $E_z$ and $ZH_z$ are written as

$$\begin{aligned} E_z(\rho,\phi,z) &= \frac{e^{-jkz\cos\beta}}{2\pi j}\int_\gamma e^{jk\rho\sin\beta\cos\alpha} s_{\mathrm{e}}(\alpha+\phi)d\alpha \\ ZH_z(\rho,\phi,z) &= \frac{e^{-jkz\cos\beta}}{2\pi j}\int_\gamma e^{jk\rho\sin\beta\cos\alpha} s_{\mathrm{h}}(\alpha+\phi)d\alpha \end{aligned} \tag{3}$$

where $\gamma$ is the Sommerfeld double loop contour. The spectra must be $O(1)$ as $|\mathrm{Im}\cdot\alpha| \to \infty$, and apart from factors incorporating the optics pole at $\alpha = \phi_0$ (factors which can be inserted in the later stages of the analysis), $s_{\mathrm{e}}(\alpha)$ and $s_{\mathrm{h}}(\alpha)$ must be free of poles in $|\mathrm{Re}\cdot\alpha| \le 2\pi$ and free of zeros in $|\mathrm{Re}\cdot\alpha| \le \pi$. Application of the boundary conditions leads to four equations involving $s_{\mathrm{e}}(\alpha)$ and $s_{\mathrm{h}}(\alpha)$, and to decouple the equations resulting from the conditions at the upper surface, new spectra are defined as (see† (1.7))

$$\begin{aligned} t_{\mathrm{e}}(\alpha) &= \left(\sin\alpha + \frac{1}{\eta_1}\sin\beta\right) s_{\mathrm{e}}(\alpha+\pi) - \cos\alpha\cos\beta s_{\mathrm{h}}(\alpha+\pi) \\ t_{\mathrm{h}}(\alpha) &= \cos\alpha\cos\beta s_{\mathrm{e}}(\alpha+\pi) + (\sin\alpha + \eta_2\sin\beta)s_{\mathrm{h}}(\alpha+\pi), \end{aligned} \tag{4}$$

† Equations in [1] are cited as $1\cdot n$.

implying

$$\begin{aligned} s_{\mathrm{e}}(\alpha+\pi) &= \frac{1}{\Gamma^{++}}\{(\sin\alpha+\eta_2\sin\beta)t_{\mathrm{e}}(\alpha)+\cos\alpha\cos\beta t_{\mathrm{h}}(\alpha)\} \\ s_{\mathrm{h}}(\alpha+\pi) &= \frac{1}{\Gamma^{++}}\left\{-\cos\alpha\cos\beta t_{\mathrm{e}}(\alpha)+\left(\sin\alpha+\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{h}}(\alpha)\right\} \end{aligned} \tag{5}$$

where the notation is as in [1].

The boundary conditions at $\phi = \pi$ then show

$$t_{\mathrm{e,h}}(-\alpha) = t_{\mathrm{e,h}}(\alpha) \tag{6}$$

and from the boundary conditions at $\phi = -\pi$ we obtain the second order difference equation (1.12)

$$(\Gamma^{++})^2 t(\alpha+6\pi) - \left\{2\Gamma^{++}\Gamma^{--} + \left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta\right]^2\right\} t(\alpha+2\pi) + (\Gamma^{--})^2 t(\alpha-2\pi) = 0 \tag{7}$$

where

$$\Gamma^{++}, \Gamma^{--} = \sin^2\beta(\sin\alpha \pm \sin\theta_1)(\sin\alpha \pm \sin\theta_2) \tag{8}$$

with

$$\sin\theta_{1,2} = \frac{1}{2\sin\beta}\left\{\frac{1}{\eta_1}+\eta_2 \pm \sqrt{\left(\frac{1}{\eta_1}-\eta_2\right)^2 + \left(\frac{\eta_2}{\eta_1}-1\right)\cos^2\beta}\right\}. \tag{9}$$

It is assumed that $0 < \mathrm{Re.}\ \theta_{1,2} \leq \pi/2$ and the square root is such that when $\beta = \pi/2$

$$\sin\theta_{1,2} = \frac{1}{2}\left\{\frac{1}{\eta_1}+\eta_2 \pm \left(\frac{1}{\eta_1}-\eta_2\right)\right\} = \frac{1}{\eta_1}, \quad \eta_2.$$

For isotropic impedances $\eta_1 = \eta_2 = \eta$ and

$$\sin\theta_1 = \frac{1}{\eta\sin\beta}, \qquad \sin\theta_2 = \frac{\eta}{\sin\beta},$$

for all $\beta$.

Equation (7) is satisfied by both $t_{\mathrm{e}}(\alpha)$ and $t_{\mathrm{h}}(\alpha)$ and was the focus of the work in [1]. Solutions were obtained which are free of poles and zeros in the appropriate regions and of the required order, and from these, expressions for $t_{\mathrm{e}}(\alpha)$ and $t_{\mathrm{h}}(\alpha)$ were constructed which, when normalized, reproduced the incident field. The resulting $s_{\mathrm{e}}(\alpha)$

and $s_{\rm h}(\alpha)$ then appeared to satisfy all of the required conditions and, in particular, agreed with the known expressions when $\beta = \pi/2$ corresponding to normal incidence.

In seeking to extend the method to a wedge of arbitrary angle a fallacy was discovered. Though the coefficients of reflection off the upper surface agree with their optics values for all $\beta$, those for the lower surface do so only if $\beta = \pi/2$, and because of this, the solution cannot be correct for arbitrary $\beta$. This is true also for the approximate solution [3] and since, for a half plane, it is sufficient to restrict the incident angle $\phi_0$ to $\leq \phi_0 \leq \pi$, the lower surface reflection coefficient was not checked. The error can be traced to the boundary conditions at the lower surface. What was not realized is that these force a connection between $t_{\rm e}(\alpha)$ and $t_{\rm h}(\alpha)$ that no longer permits the normalization that was carried out to produce the incident field. Fortunately the error can be corrected, albeit with some difficulty, and we now present the revised solution.

## 2. MODIFIED SPECTRA

From the boundary conditions at the lower face we obtain (see equation following (1.11))

$$\begin{aligned}\frac{4}{\eta_1}\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\cos\alpha\sin^2\beta\cos\beta t_{\rm h}(\alpha) &= (\Gamma^{++})^2 t_{\rm e}(\alpha+4\pi)\\ &\quad -\left\{\Gamma^{++}\Gamma^{--}-2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta\Gamma^{-+}\right\}t_{\rm e}(\alpha) \qquad (10)\end{aligned}$$

and by duality

$$\begin{aligned}4\eta_2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\cos\alpha\sin^2\beta\cos\beta t_{\rm e}(\alpha) &= (\Gamma^{++})^2 t_{\rm h}(\alpha+4\pi)\\ &\quad -\left\{\Gamma^{++}\Gamma^{--}+2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta\Gamma^{+-}\right\}t_{\rm h}(\alpha). \qquad (11)\end{aligned}$$

On inserting (10) into (5)

$$\begin{aligned}s_{\rm e}(\alpha+\pi) &= \frac{1}{\frac{4}{\eta_1}\left(\frac{1}{\eta_1}-\eta_2\right)\sin^2\beta\sin\alpha}\\ &\quad\cdot\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta-\Gamma^{--}\right]t_{\rm e}(\alpha)+\Gamma^{++}t_{\rm e}(\alpha+4\pi)\right\}\\ s_{\rm h}(\alpha+\pi) &= \frac{1}{\frac{4}{\eta_1}\left(\frac{1}{\eta_1}-\eta_2\right)\sin^2\beta\sin\alpha\cos\alpha\cos\beta}\end{aligned}$$

$$\cdot\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta\left(\sin\alpha-\frac{1}{\eta_1}\sin\beta\right)\right.\right.\tag{12}$$
$$\left.-\left(\sin\alpha+\frac{1}{\eta_1}\sin\beta\right)\Gamma^{--}\right]t_\mathrm{e}(\alpha)$$
$$\left.+\left(\sin\alpha+\frac{1}{\eta_1}\sin\beta\right)\Gamma^{++}t_\mathrm{e}(\alpha+4\pi)\right\}$$

which depend on $t_\mathrm{e}(\alpha)$ alone. In contrast to (5), these are no longer explicitly singular at the zeros of $\Gamma^{++}$, and in spite of the factor $\cos\beta$ in the denominator of the expression for $s_\mathrm{h}(\alpha+\pi)$, the expression vanishes when $\beta=\pi/2$.

Similarly, by inserting (11) into (5)

$$s_\mathrm{e}(\alpha+\pi)=-\frac{1}{4\eta_2\left(\frac{1}{\eta_1}-\eta_2\right)\sin^2\beta\sin\alpha\cos\alpha\cos\beta}$$
$$\cdot\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta(\sin\alpha-\eta_2\sin\beta)\right.\right.$$
$$\left.\left.+(\sin\alpha+\eta_2\sin\beta)\Gamma^{--}\right]-(\sin\alpha+\eta_2\sin\beta)\Gamma^{++}t_\mathrm{h}(\alpha+4\pi)\right\}\tag{13}$$

$$s_\mathrm{h}(\alpha+\pi)=\frac{1}{4\eta_2\left(\frac{1}{\eta_1}-\eta_2\right)\sin^2\beta\sin\alpha}$$
$$\cdot\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta+\Gamma^{--}\right]t_\mathrm{h}(\alpha)-\Gamma^{++}t_\mathrm{h}(\alpha+4\pi)\right\}.$$

These are two sets of representations for $s_\mathrm{e}(\alpha+\pi)$ and $s_\mathrm{h}(\alpha+\pi)$, each consistent with the boundary conditions at the upper and lower faces of the half plane provided $t_\mathrm{e}(\alpha)$ and $t_\mathrm{h}(\alpha)$ satisfy (7). The most general expressions for $s_\mathrm{e}(\alpha+\pi)$ and $s_\mathrm{h}(\alpha+\pi)$ are then

$$s_\mathrm{e}(\alpha+\pi)=\frac{C_\mathrm{e}(\alpha)\cos\alpha}{2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta}\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta-\Gamma^{--}\right]\right.$$
$$\left.\cdot t_\mathrm{e}(\alpha)+\Gamma^{++}t_\mathrm{e}(\alpha+4\pi)\right\}-\frac{C_\mathrm{h}(\alpha)}{2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta\cos\beta}$$
$$\cdot\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta(\sin\alpha-\eta_2\sin\beta)\right.\right.$$
$$\left.\left.+(\sin\alpha+\eta_2\sin\beta)\Gamma^{--}\right]t_\mathrm{h}(\alpha)-(\sin\alpha+\eta_2\sin\beta)\Gamma^{++}t_\mathrm{h}(\alpha+4\pi)\right\}$$

$$\begin{aligned}
s_{\rm h}(\alpha+\pi) = & \frac{C_{\rm e}(\alpha)}{2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta\cos\beta} \\
& \cdot\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta\left(\sin\alpha-\frac{1}{\eta_1}\sin\beta\right)\right.\right. \\
& \left. -\left(\sin\alpha+\frac{1}{\eta_1}\sin\beta\right)\Gamma^{--}\right]t_{\rm e}(\alpha) \\
& \left. +\left(\sin\alpha+\frac{1}{\eta_1}\sin\beta\right)\Gamma^{++}t_{\rm e}(\alpha+4\pi)\right\} \\
& +\frac{C_{\rm h}(\alpha)\cos\alpha}{2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta} \\
& \cdot\left\{\left[2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta+\Gamma^{++}\right]t_{\rm h}(\alpha)-\Gamma^{++}t_{\rm h}(\alpha+4\pi)\right\}
\end{aligned} \tag{14}$$

where $C_{\rm e}(\alpha)$ and $C_{\rm h}(\alpha)$ are even $4\pi$ periodic functions and $t_{\rm e}(\alpha)$ and $t_{\rm h}(\alpha)$ are independent even particular solutions of (7).

## 3. REDUCED PERIOD EQUATIONS

Instead of the second order difference equation (7) whose period is $4\pi$, consider the reduced period equations

$$\Gamma^{++}t(\alpha+2\pi)\mp 2\left(\frac{1}{\eta_1}-\eta_2\right)\sin\alpha\sin\beta t(\alpha)-\Gamma^{--}t(\alpha-2\pi)=0. \tag{15}$$

Any solution of (15) is a solution of (7) and the most general solution of (7) is a linear combination of particular solutions of (15) with $4\pi$ periodic multipliers.

Consider first the case of normal incidence for which $\beta = \pi/2$. Apart from a $4\pi$ periodic factor incorporating the optics pole, the known expression for $s_{\rm e}(\alpha)$ is [2]

$$s_{\rm e}(\alpha)=\frac{1}{2}\Psi(\alpha-\pi,\theta_1)\Psi(\alpha+\pi,\theta_1)$$

where

$$\Psi(\alpha,\theta)=\psi_\pi\left(\alpha+\frac{\pi}{2}-\theta\right)\psi_\pi\left(\alpha-\frac{\pi}{2}+\theta\right) \tag{16}$$

and $\psi_\pi(\alpha)$ is the Maliuzhinets half plane function [2]. Since

$$\Psi(\alpha+2\pi,\theta)=\frac{2\left(\cos\frac{\alpha}{2}-\cos\frac{\theta}{2}\right)\left(\cos\frac{\alpha}{2}-\sin\frac{\theta}{2}\right)}{\sin\alpha+\sin\theta}\Psi(\alpha,\theta)$$

it follows that

$$s_{\mathrm{e}}(\alpha+\pi) = \frac{\left(\cos\frac{\alpha}{2} - \cos\frac{\theta_1}{2}\right)\left(\cos\frac{\alpha}{2} - \sin\frac{\theta_1}{2}\right)}{\sin\alpha + \sin\theta_1}\{\Psi(\alpha,\theta_1)\}^2$$

and hence, from (5),

$$t_{\mathrm{e}}(\alpha) = \left(\cos\frac{\alpha}{2} - \frac{\cos\theta_1}{2}\right)\left(\cos\frac{\alpha}{2} - \sin\frac{\theta_1}{2}\right)\{\Psi(\alpha,\theta_1)\}^2 \tag{17}$$

which is $O\{\exp(\frac{3}{2}|\mathrm{Im}\cdot\alpha|)\}$ as $|\mathrm{Im}\cdot\alpha| \to \infty$ and satisfies (15) with the upper sign. Similarly

$$t_{\mathrm{h}}(\alpha) = \left(\cos\frac{\alpha}{2} - \cos\frac{\theta_2}{2}\right)\left(\cos\frac{\alpha}{2} - \sin\frac{\theta_2}{2}\right)\{\Psi(\alpha,\theta_2)\}^2 \tag{18}$$

and satisfies (15) with the lower sign. These are also the solutions of (15) in the special case of isotropic impedances for all $\beta$. Writing

$$t_{\mathrm{e,h}}(\alpha) = D(\alpha,\theta_{1,2})t_{1,2}(\alpha) \tag{19}$$

where $D(\alpha,\theta)$ is the $4\pi$ periodic function

$$D(\alpha,\theta) = \left(\cos\frac{\alpha}{2} - \cos\frac{\theta}{2}\right)\left(\cos\frac{\alpha}{2} - \sin\frac{\theta}{2}\right), \tag{20}$$

then

$$t_{1,2}(\alpha) = \{\Psi(\alpha,\theta_{1,2})\}^2 \tag{21}$$

which are the even solutions of

$$\Gamma^{++}t_{1,2}(\alpha+2\pi) \mp 2\left(\frac{1}{\eta_1} - \eta_2\right)\sin\alpha\sin\beta\frac{D(\alpha,\theta_{1,2})}{D(\alpha+2\pi,\theta_{1,2})}t_{1,2}(\alpha) - \Gamma^{--}t_{1,2}(\alpha-2\pi) = 0 \tag{22}$$

that are $O\{\exp(\frac{1}{2}|\mathrm{Im}\cdot\alpha|\}$ as $|\mathrm{Im}\cdot\alpha| \to \infty$ and free of poles and zeros in $|\mathrm{Re}\cdot\alpha| \leq 3\pi$ and $|\mathrm{Re}\cdot\alpha| \leq 2\pi$ respectively.

In general, if (15) are the equations satisfied by $t_{\mathrm{e}}(\alpha)$ and $t_{\mathrm{h}}(\alpha)$, substitution into (14) gives

$$\begin{aligned} s_{\mathrm{e}}(\alpha+\pi) &= C_{\mathrm{e}}(\alpha)\cos\alpha\{t_{\mathrm{e}}(\alpha) + t_{\mathrm{e}}(\alpha+2\pi)\} \\ &\quad - \frac{C_{\mathrm{h}}(\alpha)}{\cos\beta}\{(\sin\alpha - \eta_2\sin\beta)t_{\mathrm{h}}(\alpha) + (\sin\alpha + \eta_2\sin\beta)t_{\mathrm{h}}(\alpha+2\pi)\} \\ s_{\mathrm{h}}(\alpha+\pi) &= \frac{C_{\mathrm{e}}(\alpha)}{\cos\beta}\left\{\left(\sin\alpha - \frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\alpha) + \left(\sin\alpha + \frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\alpha+2\pi)\right\} \\ &\quad + C_{\mathrm{h}}(\alpha)\cos\alpha\{t_{\mathrm{h}}(\alpha) + t_{\mathrm{h}}(\alpha+2\pi)\}. \end{aligned} \tag{23}$$

Reverting to the special cases cited above for which $t_{\mathrm{e}}(\alpha)$ and $t_{\mathrm{h}}(\alpha)$ are given in (17) and (18) respectively, the properties of $\psi_\pi(\alpha)$ show

$$t_{\mathrm{e}}(\alpha)+t_{\mathrm{e}}(\alpha+2\pi)=\frac{2\sin\theta_1}{\sin\alpha+\sin\theta_1}t_{\mathrm{e}}(\alpha)$$

and therefore

$$\frac{t_{\mathrm{e}}(\alpha+2\pi)}{t_{\mathrm{e}}(\alpha)}=-1+O\{\exp(-|\mathrm{Im}\cdot\alpha|)\} \tag{24}$$

as $|\mathrm{Im}\cdot\alpha|\to\infty$. We note in passing that

$$\frac{t_{\mathrm{e}}(\alpha)}{\sin\alpha+\sin\theta_1}=\Psi(\alpha,\theta_1)\Psi(\alpha+2\pi,\theta_1) \tag{25}$$

which is free of poles and zeros in $-2\pi\leq\mathrm{Re}\cdot\alpha\leq 0$, and there are similar relations satisfied by $t_{\mathrm{h}}(\alpha)$. Also

$$\frac{1}{\cos\beta}\left\{\left(\sin\alpha-\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\alpha)+\left(\sin\alpha+\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\alpha+2\pi)\right\}$$
$$=\sin\alpha\cos\beta\frac{2\sin\theta_1}{\sin\alpha+\sin\theta_1}t_{\mathrm{e}}(\alpha)$$

$$\frac{1}{\cos\beta}\{(\sin\alpha-\eta_2\sin\beta)t_{\mathrm{h}}(\alpha)+(\sin\alpha+\eta_2\sin\beta)t_{\mathrm{h}}(\alpha+2\pi)\}$$
$$=\sin\alpha\cos\beta\frac{2\sin\theta_2}{\sin\alpha+\sin\theta_2}t_{\mathrm{h}}(\alpha)$$

so that

$$\begin{aligned} s_{\mathrm{e}}(\alpha+\pi) &= \cos\alpha C_{\mathrm{e}}(\alpha)\frac{2\sin\theta_1}{\sin\alpha+\sin\theta_1}t_{\mathrm{e}}(\alpha) \\ &\quad -\sin\alpha\cos\beta C_{\mathrm{h}}(\alpha)\frac{2\sin\theta_2}{\sin\alpha+\sin\theta_2}t_{\mathrm{h}}(\alpha) \\ s_{\mathrm{h}}(\alpha+\pi) &= \sin\alpha\cos\beta C_{\mathrm{e}}(\alpha)\frac{2\sin\theta_1}{\sin\alpha+\sin\theta_1}t_{\mathrm{e}}(\alpha) \\ &\quad +\cos\alpha C_{\mathrm{h}}(\alpha)\frac{2\sin\theta_2}{\sin\alpha+\sin\theta_2}t_{\mathrm{h}}(\alpha). \end{aligned} \tag{26}$$

Since the coefficients of $C_{\mathrm{e}}(\alpha)$ and $C_{\mathrm{h}}(\alpha)$ are $O\{\exp(\frac{3}{2}|\mathrm{Im}\cdot\alpha|\}$ as $|\mathrm{Im}\cdot\alpha|\to\infty$, $C_{\mathrm{e}}(\alpha)$ and $C_{\mathrm{h}}(\alpha)$ must be $O\{\exp(-\frac{3}{2}|\mathrm{Im}\cdot\alpha|)\}$ and free of poles and zeros in $-2\pi\leq\mathrm{Re}\cdot\alpha\leq 0$ apart from the optics pole at $\alpha=\phi_0-\pi$. The resulting expressions for $s_{\mathrm{e}}(\alpha+\pi)$ and $s_{\mathrm{h}}(\alpha+\pi)$ are in agreement with the known solutions for anisotropic impedances with $\beta=\pi/2$ [2] and for isotropic impedances for all $\beta$ [4].

## 4. DETERMINATION OF $t_{\mathrm{e}}(\alpha)$

In general, with (19) as a guide, write

$$t_{\mathrm{e,h}}(\alpha) = D(\alpha, \theta'_{1,2}) t_{1,2}(\alpha) \tag{27}$$

for some $\theta'_{1,2}$ where $D(\alpha, \theta)$ is given in (20) and $t_{1,2}(\alpha)$ satisfy (22) with $\theta_{1,2}$ replaced by $\theta'_{1,2}$. If

$$t_1(\alpha) = p(\alpha)\bar{t}_1(\alpha) \tag{28}$$

where

$$\frac{p(\alpha+2\pi)}{p(\alpha-2\pi)} = \frac{(\sin\alpha - \sin\theta_1)(\sin\alpha - \sin\theta_2)}{(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)} \tag{29}$$

substitution into (22) gives

$$\bar{t}_1(\alpha+2\pi) + c_1(\alpha, \theta'_1)\bar{t}_1(\alpha) - \bar{t}_1(\alpha - 2\pi) = 0 \tag{30}$$

with

$$c_1(\alpha, \theta'_1) = -\frac{2\left(\dfrac{1}{\eta_1} - \eta_2\right)\dfrac{\sin\alpha}{\sin\beta}}{(\sin\alpha + \sin\theta_1)(\sin\alpha + \sin\theta_2)} \frac{p(\alpha)}{p(\alpha+2\pi)} \frac{D(\alpha, \theta'_1)}{D(\alpha+2\pi, \theta'_1)}.$$

The solution of (29) that is free of poles and zeros in $|\mathrm{Re}\cdot\alpha| \leq 2\pi$ is the even function

$$p(\alpha) = \Psi(\alpha, \theta_1)\Psi(\alpha, \theta_2) \tag{31}$$

and this is $O\{\exp(\frac{1}{2}|\mathrm{Im}\cdot\alpha|\}$ as $|\mathrm{Im}\cdot\alpha| \to \infty$. From the properties of $\psi_\pi(\alpha)$ we have

$$\frac{p(\alpha+2\pi)}{p(\alpha)} = \frac{\cos\dfrac{1}{2}\left(\alpha - \dfrac{\pi}{2}\right) - \cos\dfrac{1}{2}\left(\theta_1 - \dfrac{\pi}{2}\right)}{\sin\dfrac{1}{2}\left(\alpha - \dfrac{\pi}{2}\right) - \cos\dfrac{1}{2}\left(\theta_1 - \dfrac{\pi}{2}\right)} \cdot \frac{\cos\dfrac{1}{2}\left(\alpha - \dfrac{\pi}{2}\right) - \cos\dfrac{1}{2}\left(\theta_2 - \dfrac{\pi}{2}\right)}{\sin\dfrac{1}{2}\left(\alpha - \dfrac{\pi}{2}\right) - \cos\dfrac{1}{2}\left(\theta_2 - \dfrac{\pi}{2}\right)}, \tag{32}$$

and after some manipulation we obtain

$$c_1(\alpha, \theta'_1) = -\frac{\dfrac{1}{2}\left(\dfrac{1}{\eta_1} - \eta_2\right)\dfrac{\sin\alpha}{\sin\beta}}{D(\alpha, \theta_1)D(\alpha, \theta_2)} \cdot \frac{D(\alpha, \theta'_1)}{D(\alpha+2\pi, \theta'_1)} \tag{33}$$

which is an odd function of $\alpha$, vanishing as $|\mathrm{Im}\cdot\alpha| \to \infty$. For any $\theta_1'$ (30) has an even solution which is free of poles and zeros in $|\mathrm{Re}\cdot\alpha| \leq 2\pi$ and $O(1)$. This is the one we seek, and in the special cases (for which $\theta_1' = \theta_1$) the solution is

$$\bar{t}_1(\alpha) = \frac{\Psi(\alpha, \theta_1)}{\Psi(\alpha, \theta_2)}. \tag{34}$$

Though this has poles at $\alpha = \pm(2\pi + \theta_2)$, $\pm(3\pi - \theta_2)$, these are cancelled by the zeros of $p(\alpha)$, with the result that $t_1(\alpha)$ is free of poles in $|\mathrm{Re}\cdot\alpha| \leq 3\pi$.

Equation (30) is similar to the one treated in [1], and for any given $\theta_1'$ the solution can be obtained in the manner described there. Application of a modified Fourier transform gives

$$\bar{t}_1(\alpha) - \bar{t}_1(j\infty) = -\frac{j}{4\pi}\int_0^{j\infty} c_1(\alpha', \theta_1')\bar{t}_1(\alpha')\frac{\sin\dfrac{\alpha'}{2}}{\cos\dfrac{\alpha}{2} + \cos\dfrac{\alpha'}{2}}d\alpha' \tag{35}$$

where we have used the fact that $\bar{t}_1(\alpha)$ is an even function, and this is an inhomogeneous integral equation for $\bar{t}_1(\alpha)$ for positive pure imaginary $\alpha$. We can map the path of integration on to the real $x$ axis from 0 to 1 by writing $x = -j\tan(\alpha'/4)$, and if $y = -j\tan(\alpha/4)$, then

$$f_1(y) - f_1(1) = \frac{2}{\pi}\left(\frac{1}{\eta_1} - \eta_2\right)\frac{1}{\sin\beta}\int_0^1 x^2(1 - x^4)g_1(x)f_1(x)\frac{1 - y^2}{1 - x^2y^2}dx \tag{36}$$

where $f_1(y) = f_1(-j\tan\frac{\alpha}{4}) = \bar{t}_1(\alpha)$ and

$$\begin{aligned} g_1(x) = {} & \frac{\left\{x^2\left(1+\cos\dfrac{\theta_1'}{2}\right)+1-\cos\dfrac{\theta_1'}{2}\right\}\left\{x^2\left(1+\sin\dfrac{\theta_1'}{2}\right)+1-\sin\dfrac{\theta_1'}{2}\right\}}{\left\{x^2\left(1-\cos\dfrac{\theta_1'}{2}\right)+1+\cos\dfrac{\theta_1'}{2}\right\}\left\{x^2\left(1-\sin\dfrac{\theta_1'}{2}\right)+1+\sin\dfrac{\theta_1'}{2}\right\}} \\ & \cdot\left[\left\{x^2\left(1+\cos\frac{\theta_1}{2}\right)+1-\cos\frac{\theta_1}{2}\right\}\left\{x^2\left(1+\sin\frac{\theta_1}{2}\right)+1-\sin\frac{\theta_1}{2}\right\}\right. \\ & \left.\cdot\left\{x^2\left(1+\cos\frac{\theta_2}{2}\right)+1-\cos\frac{\theta_2}{2}\right\}\left\{x^2\left(1+\sin\frac{\theta_2}{2}\right)+1-\sin\frac{\theta_2}{2}\right\}\right]^{-1}. \end{aligned} \tag{37}$$

For any $\theta_1'$ the solution can be found by representing $f_1(y)$ as a Taylor series in $y^2$, and only a few terms are necessary for good accuracy. Having computed $f_1(y)$ for real $y$, $0 \leq y \leq 1$, (36) can be used to determine $\bar{t}_1(\alpha)$ throughout the strip $0 \leq \mathrm{Re}\cdot\alpha \leq 2\pi$. However, as $\mathrm{Re}\cdot\alpha$ gets close to $2\pi$, the integrand has a pole which approaches the path of integration and actually lies on it if $\alpha = 2\pi$. A more convenient representation is then

$$\begin{aligned} f_1(y)-f_1(1) = \frac{2}{\pi}\left(\frac{1}{\eta_1}-\eta_2\right)\frac{1}{\sin\beta}\Bigg[\int_0^1 \Bigg\{&(1-x^4)g_1(x)f_1(x) \\ &-\left(1-\frac{1}{y^4}\right)g_1\left(\frac{1}{y}\right)f_1\left(\frac{1}{y}\right)\Bigg\}\frac{x^2(1-y^2)}{1-x^2y^2}dx \\ &+\left(1-\frac{1}{y^4}\right)\left(1-\frac{1}{y^2}\right)g_1\left(\frac{1}{y}\right)f_1\left(\frac{1}{y}\right)\left(1+\frac{j\alpha}{4y}\right)\Bigg]. \end{aligned} \tag{38}$$

Outside $|\mathrm{Re}\cdot\alpha| \leq 2\pi$ the analytic continuation is provided by the difference equation (30).

We now turn to the specification of $\theta_1'$. For large $|\mathrm{Im}\cdot\alpha|$ (36) shows

$$f_1(y) \sim f_1(1) + \bar{k}_1(1-y^2)$$

where

$$\bar{k}_1 = \frac{2}{\pi}\left(\frac{1}{\eta_1}-\eta_2\right)\frac{1}{\sin\beta}\int_0^1 x^2(1+x^2)g_1(x)f_1(x)dx, \tag{39}$$

and since $1-y^2 = 2(\cos\frac{\alpha}{2}+1)^{-1}$, it follows that

$$\bar{t}_1(\alpha) \sim f_1(1)\left\{1-\frac{k_1}{2\cos\dfrac{\alpha}{2}}\right\}$$

with

$$k_1 = -\frac{2\bar{k}_1}{f_1(1)}. \tag{40}$$

Hence

$$\frac{\bar{t}_1(\alpha+2\pi)}{\bar{t}_1(\alpha)} \sim 1+\frac{k_1}{\cos\dfrac{\alpha}{2}}$$

for large $|\mathrm{Im}\cdot\alpha|$. But from (32)

$$\frac{p(\alpha+2\pi)}{p(\alpha)} \sim -\left\{1-\frac{1}{\cos\dfrac{\alpha}{2}}\left(\cos\frac{\theta_1}{2}+\sin\frac{\theta_1}{2}+\cos\frac{\theta_2}{2}+\sin\frac{\theta_2}{2}\right)\right\}$$

and therefore

$$\frac{t_{\rm e}(\alpha+2\pi)}{t_{\rm e}(\alpha)} \sim -\left\{1+\frac{k_1-\ell_1}{\cos\dfrac{\alpha}{2}}\right\} \tag{41}$$

where

$$\ell_1 = \cos\frac{\theta_1}{2}+\sin\frac{\theta_1}{2}+\cos\frac{\theta_2}{2}+\sin\frac{\theta_2}{2}-2\left(\cos\frac{\theta_1'}{2}+\sin\frac{\theta_1'}{2}\right). \tag{42}$$

The angle $\theta_1'$ is uniquely specified by the requirement that $k_1 = \ell_1$ so that $t_{\rm e}(\alpha+2\pi)/t_{\rm e}(\alpha)$ has the same asymptotic behavior (24) as in the special cases.

In practive it is easy to determine $\theta_1'$ numerically. For anisotropic impedances with $\beta = \pi/2$, $\theta_1' = \theta_1$. If, for example, $\eta_1 = 2$ and $\eta_2 = 4$, $k_1 = \ell_1 = 1.0113$, but as $\beta$ decreases with $\theta_1' = \theta_1$, $k_1$ and $\ell_1$ both increase, the latter more rapidly than the former, and it is necessary to increase $\theta_1'$ beyond $\theta_1$ to maintain $k_1 = \ell_1$. The resulting $\sin\theta_1'$ for selected $\beta$ are shown in Table 1.

## 5. DETERMINATION OF $t_{\rm h}(\alpha)$

The procedure is similar to that for $t_{\rm e}(\alpha)$. From (27) with

$$t_2(\alpha) = p(\alpha)\bar{t}_2(\alpha), \tag{43}$$

the equation satisfied by $\bar{t}_2(\alpha)$ is

$$\bar{t}_2(\alpha+2\pi)+c_2(\alpha,\theta_2')\bar{t}_2(\alpha)-\bar{t}_2(\alpha-2\pi)=0 \tag{44}$$

with (see (33))

$$\begin{aligned} c_2(\alpha,\theta_2') &= -c_1(\alpha,\theta_2') \\ &= \frac{\dfrac{1}{2}\left(\dfrac{1}{\eta_1}-\eta_2\right)\dfrac{\sin\alpha}{\sin\beta}}{D(\alpha,\theta_1)D(\alpha,\theta_2)}\cdot\frac{D(\alpha,\theta_2')}{D(\alpha+2\pi,\theta_2')}. \end{aligned} \tag{45}$$

Writing $f_2(y) = f_2(-j\tan\frac{\alpha}{4}) = \bar{t}_2(\alpha)$ and

$$g_2(x) = \frac{\left\{x^2\left(1+\cos\dfrac{\theta_2'}{2}\right)+1-\cos\dfrac{\theta_2'}{2}\right\}\left\{x^2\left(1+\sin\dfrac{\theta_2'}{2}\right)+1-\sin\dfrac{\theta_2'}{2}\right\}}{\left\{x^2\left(1-\cos\dfrac{\theta_2'}{2}\right)+1+\cos\dfrac{\theta_2'}{2}\right\}\left\{x^2\left(1-\sin\dfrac{\theta_2'}{2}\right)+1+\sin\dfrac{\theta_2'}{2}\right\}}$$
$$\cdot\left[\left\{x^2\left(1+\cos\frac{\theta_1}{2}\right)+1-\cos\frac{\theta_1}{2}\right\}\left\{x^2\left(1+\sin\frac{\theta_1}{2}\right)+1-\sin\frac{\theta_1}{2}\right\}\right.$$

$$\cdot\left\{x^2\left(1+\cos\frac{\theta_2}{2}\right)+1-\cos\frac{\theta_2}{2}\right\}\left\{x^2\left(1+\sin\frac{\theta_2}{2}\right)+1-\sin\frac{\theta_2}{2}\right\}\Bigg]^{-1}. \tag{46}$$

we have

$$f_2(y)-f_2(1) = -\frac{2}{\pi}\left(\frac{1}{\eta_1}-\eta_2\right)\frac{1}{\sin\beta}\int_0^1 x^2(1-x^4)g_2(x)f_2(x)\frac{1-y^2}{1-x^2y^2}dx, \tag{47}$$

and for any given $\theta_2'$ this can be solved as before. For large $|\mathrm{Im}\cdot\alpha|$

$$\frac{\bar{t}_2(\alpha+2\pi)}{t_2(\alpha)} \sim 1+\frac{k_2}{\cos\dfrac{\alpha}{2}}$$

where

$$k_2 = -\frac{2\bar{k}_2}{f_2(1)} \tag{48}$$

with

$$\bar{k}_2 = -\frac{2}{\pi}\left(\frac{1}{\eta_1}-\eta_2\right)\frac{1}{\sin\beta}\int_0^1 x^2(1+x^2)g_2(x)f_2(x)dx. \tag{49}$$

Hence

$$\frac{t_{\mathrm{h}}(\alpha+2\pi)}{t_{\mathrm{h}}(\alpha)} \sim -\left\{1+\frac{k_2-\ell_2}{\cos\dfrac{\alpha}{2}}\right\} \tag{50}$$

where

$$\ell_2 = \cos\frac{\theta_1}{2}+\sin\frac{\theta_1}{2}+\cos\frac{\theta_2}{2}+\sin\frac{\theta_2}{2}-2\left(\cos\frac{\theta_2'}{2}+\sin\frac{\theta_2'}{2}\right), \tag{51}$$

and $\theta_2'$ must be such that $k_2 = \ell_2$. The resulting values of $\sin\theta_2'$ are included in Table 1. We observe that as $\beta$ decreases from $\pi/2$, $\sin\theta_2'$ becomes less than $\sin\theta_2$ and $\sin\theta_1'$ greater than $\sin\theta_1$, but with $\sin\theta_1'+\sin\theta_2'$ only a little less than $\sin\theta_1+\sin\theta_2$.

## 6. COEFFICIENTS $C_{\mathrm{e}}(\alpha)$ AND $C_{\mathrm{h}}(\alpha)$

Having determined $t_{\mathrm{e}}(\alpha)$ and $t_{\mathrm{h}}(\alpha)$ the expressions for $s_{\mathrm{e}}(\alpha+\pi)$ and $s_{\mathrm{h}}(\alpha+\pi)$ are given in (23). The determinant of the matrix of the coefficients $C_{\mathrm{e}}(\alpha)$ and $C_{\mathrm{h}}(\alpha)$ is $Q(\alpha+\pi)$ where

$$\begin{aligned} Q(\alpha) &= \cos^2\alpha\{t_{\mathrm{e}}(\alpha-\pi)+t_{\mathrm{e}}(\alpha+\pi)\}\{t_{\mathrm{h}}(\alpha-\pi)+t_{\mathrm{h}}(\alpha+\pi)\} \\ &+\frac{1}{\cos^2\beta}\left\{\left(\sin\alpha+\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\alpha-\pi)+\left(\sin\alpha-\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\alpha+\pi)\right\} \\ &\cdot\{(\sin\alpha+\eta_2\sin\beta)t_{\mathrm{h}}(\alpha-\pi)+(\sin\alpha-\eta_2\sin\beta)t_{\mathrm{h}}(\alpha+\pi)\}, \end{aligned} \tag{52}$$

**Table 1.** Computed $\sin\theta_1'$ and $\sin\theta_2'$ for $\eta_1 = 2,\ \eta_2 = 4$.

| $\beta$ | $\sin\theta_1'$ | $\sin\theta_2'$ | $\sin\theta_1$ | $\sin\theta_2$ |
|---|---|---|---|---|
| $\pi/2$ | 0.5000 | 4.0000 | 0.5000 | 4.0000 |
| $\pi/3$ | 0.5386 | 4.5329 | 0.4969 | 4.7002 |
| $\pi/6$ | 0.8201 | 8.1419 | 0.5950 | 8.4056 |
| $\pi/12$ | 1.4882 | 15.1606 | 0.9701 | 16.4158 |

and since $t_\mathrm{e}(\alpha)$ and $t_\mathrm{h}(\alpha)$ are even functions of $\alpha$, so is $Q(\alpha)$. In the special cases (25) and (26) show

$$\begin{aligned} Q(\alpha) \ = \ & 4\sin\theta_1\sin\theta_2\sin^2\beta(\cos^2\alpha+\cot^2\beta)\Psi(\alpha-\pi,\theta_1) \\ & \Psi(\alpha+\pi,\theta_1)\Psi(\alpha-\pi,\theta_2)\Psi(\alpha+\pi,\theta_2) \end{aligned}$$

whose only zeros in $|\mathrm{Re}\cdot\alpha| \le \pi$ are at $\alpha = \frac{\pi}{2}\pm\gamma, -\frac{\pi}{2}\pm\gamma$ where $\gamma = j\ln(\tan\frac{\beta}{2})$. In general the zeros $\alpha = \pm\alpha_{1,2}$ of (52) can be found numerically by searching the strip $O \le \mathrm{Re}\cdot\alpha \le \pi$ in the vicinity of $\mathrm{Re}\cdot\alpha = \pi/2$. For $\eta_1 = 2$ and $\eta_2 = 4$ the zeros are listed in Table 2, and for real $\eta_1$ and $\eta_2$, $\mathrm{Re}\cdot\alpha_1 = \mathrm{Re}\cdot\alpha_2$.

**Table 2.** Computed zeros for $\eta_1 = 2,\ \eta_2 = 4$.

| $\beta$ | $\mathrm{Re}\cdot\alpha_1$ | $\mathrm{Im}\cdot\alpha_1$ | $\mathrm{Re}\cdot\alpha_2$ | $\mathrm{Im}\cdot\alpha_2$ |
|---|---|---|---|---|
| $\pi/2$ | 1.5684 | −0.6046 | 1.5684 | −0.5746 |
| $\pi/3$ | 1.4521 | −0.5150 | 1.4521 | −0.7502 |
| $\pi/6$ | 1.4003 | 0.0695 | 1.4003 | −1.3835 |
| $\pi/12$ | 1.3745 | 2.4915 | 1.3745 | −2.0197 |

The factors multiplying $C_\mathrm{e}(\alpha)$ and $C_\mathrm{h}(\alpha)$ in (23) are $O\{\exp(\frac{3}{2}|\mathrm{Im}\cdot\alpha|)\}$ as $|\mathrm{Im}\cdot\alpha| \to \infty$, and to have $s_\mathrm{e}(\alpha+\pi)$ and $s_\mathrm{h}(\alpha+\pi)$ be $O(1)$ and free of poles and zeros in $-2\pi \le \mathrm{Re}\cdot\alpha \le 0$ apart from the optics pole at $\alpha = \phi_0 - \pi$ we now write

$$C_{\mathrm{e,h}}(\alpha) = \frac{A_{\mathrm{e,h}}}{(\cos\alpha+\cos\alpha_1)(\cos\alpha+\cos\alpha_2)}\left\{\sigma(\alpha+\pi)+a_{\mathrm{e,h}}+b_{\mathrm{e,h}}\cos\frac{\alpha}{2}\right\} \tag{53}$$

where

$$\sigma(\alpha) = \frac{\frac{1}{2}\cos\frac{\phi_0}{2}}{\sin\frac{\alpha}{2}-\sin\frac{\phi_0}{2}} \tag{54}$$

and $A_{\mathrm{e,h}}$, $a_{\mathrm{e,h}}$ and $b_{\mathrm{e,h}}$ are constants to be determined. To reproduce the incident field (2) we require that $s_{\mathrm{e}}(\phi_0) = e_z$, $s_{\mathrm{h}}(\phi_0) = h_z$, and hence, from (23),

$$\begin{aligned} e_z &= -\Big[A_{\mathrm{e}}\cos\phi_0\{t_{\mathrm{e}}(\pi-\phi_0)+t_{\mathrm{e}}(\pi+\phi_0)\} \\ &\quad -\frac{A_{\mathrm{h}}}{\cos\beta}\{(\sin\phi_0+\eta_2\sin\beta)t_{\mathrm{h}}(\pi-\phi_0) \\ &\quad +(\sin\phi_0-\eta_2\sin\beta)t_{\mathrm{h}}(\pi+\phi_0)\}\Big] \\ &\quad \{(\cos\phi_0-\cos\alpha_1)(\cos\phi_0-\cos\alpha_2)\}^{-1} \\ h_z &= -\left[\frac{A_{\mathrm{e}}}{\cos\beta}\left\{\left(\sin\phi_0+\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\pi-\phi_0)\right.\right. \\ &\quad \left.+\left(\sin\phi_0-\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\pi+\phi_0)\right\} \\ &\quad \left.+A_{\mathrm{h}}\cos\phi_0\{t_{\mathrm{h}}(\pi-\phi_0)+t_{\mathrm{h}}(\pi+\phi_0)\}\right] \\ &\quad \cdot\{(\cos\phi_0-\cos\alpha_1)(\cos\phi_0-\cos\alpha_2)\}^{-1} \end{aligned}$$

where we have used the fact that $t_1(\alpha)$ and $t_2(\alpha)$ are even functions. It follows that

$$\begin{aligned} A_{\mathrm{e}} &= -\frac{(\cos\phi_0-\cos\alpha_1)(\cos\phi_0-\cos\alpha_2)}{Q(\phi_0)} \\ &\quad \cdot\Big[\cos\phi_0\{t_{\mathrm{h}}(\pi-\phi_0)+t_{\mathrm{h}}(\pi+\phi_0)\}e_z \\ &\quad +\frac{1}{\cos\beta}\{(\sin\phi_0+\eta_2\sin\beta)t_{\mathrm{h}}(\pi-\phi_0) \\ &\quad +(\sin\phi_0-\eta_2\sin\beta)t_{\mathrm{h}}(\pi+\phi_0)\}h_z\Big] \\ A_{\mathrm{h}} &= -\frac{(\cos\phi_0-\cos\alpha_1)(\cos\phi_0-\cos\alpha_2)}{Q(\phi_0)} \\ &\quad \cdot\left[\frac{1}{\cos\beta}\left\{\left(\sin\phi_0+\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\pi-\phi_0)\right.\right. \\ &\quad \left.+\left(\sin\phi_0-\frac{1}{\eta_1}\sin\beta\right)t_{\mathrm{e}}(\pi+\phi_0)\right\}e_z \\ &\quad \left.-\cos\phi_0\{t_{\mathrm{e}}(\pi-\phi_0)+t_{\mathrm{e}}(\pi+\phi_0)\}h_z\right]. \end{aligned} \tag{55}$$

The remaining constants $a_{\mathrm{e,h}}$ and $b_{\mathrm{e,h}}$ are needed to eliminate the poles of $s_{\rm e}(\alpha+\pi)$ and $s_{\rm h}(\alpha+\pi)$ at $\alpha=-\pi\pm\alpha_{1,2}$. Based on the expression for $s_{\rm e}(\alpha+\pi)$ we have

$$A_{\rm e}\cos\alpha_{1,2}\left\{\sigma(\pm\alpha_{1,2})+a_{\rm e}\pm b_{\rm e}\sin\frac{\alpha_{1,2}}{2}\right\}\{t_{\rm e}(\pi-\alpha_{1,2})+t_{\rm e}(\pi+\alpha_{1,2})\}$$
$$=\pm\frac{A_{\rm h}}{\cos\beta}\left\{\sigma(\pm\alpha_{1,2})+a_{\rm h}\pm b_{\rm h}\sin\frac{\alpha_{1,2}}{2}\right\}\{(\sin\alpha_{1,2}+\eta_2\sin\beta)t_{\rm h}(\pi-\alpha_{1,2})\}$$
$$+(\sin\alpha_{1,2}-\eta_2\sin\beta)t_{\rm h}(\pi+\alpha_{1,2})\} \qquad (56)$$

which are four equations from which $a_{\rm e,h}$ and $b_{\rm e,h}$ can be found. By virtue of the definitions of $\alpha_{1,2}$ it is easily verified that the poles of $s_{\rm h}(\alpha+\pi)$ are also eliminated. The expressions for $s_{\rm e}(\alpha)$ and $s_{\rm h}(\alpha)$ are now complete and, as required, are $O(1)$ as $|\mathrm{Im}\cdot\alpha|\to\infty$.

## 7. TOTAL FIELD

From the Maliuzhinets representation (3) for the total field we obtain (see (1.45))

$$E_z(\rho,\phi,z) = 2\pi j\sum \mathrm{Res.}+\frac{e^{-jkz\cos\beta}}{2\pi j} \qquad (57)$$
$$\cdot\int_{-j\infty-\pi/2}^{-j\infty+\pi/2} e^{-jk\rho\sin\beta\cos\alpha}\{s_{\rm e}(\alpha+\phi-\pi)-s_{\rm e}(\alpha+\phi+\pi)\}d\alpha \qquad (58)$$

with a similar expression for $ZH_z$. The residues are those of the optics pole, giving rise to the incident field if $\phi_0-\pi<\phi<\phi_0+\pi$, the wave reflected off the upper surface if $\pi-\phi_0<\phi<3\pi-\phi_0$, and the wave reflected off the lower surface if $-3\pi-\phi_0<\phi<-\pi-\phi_0$, plus those of any surface wave poles that may be captured in the closure of the path of integration. The surface wave poles are those of $p(\alpha)$ that are closest to the strip $|\mathrm{Re}\cdot\alpha|\le 2\pi$. They are $\alpha=\pm(3\pi+\theta_{1,2})$, implying poles of $s_{\rm e,h}(\alpha)$ at $\alpha=\pm(2\pi+\theta_{1,2})$, and it can be verified that the expressions for $\sin\theta_{1,2}$ given in (9) are consistent with the surface waves specified in [15]. With $A_1$ and $A_2$ given in (55), the upper surface reflection coefficient is found to be

$$R = \left\{-1+\frac{2\sin\phi_0(\sin\phi_0+\eta_2\sin\beta)}{\sin^2\beta(\sin\phi_0+\sin\theta_1)(\sin\phi_0+\sin\theta_2)}\right\}e_z$$
$$-\frac{2\sin\phi_0\cos\phi_0\cos\beta}{\sin^2\beta(\sin\phi_0+\sin\theta_1)(\sin\phi_0+\sin\theta_2)}h_z$$

and the lower surface reflection coefficient is

$$R = \left\{-1 + \frac{2\sin\phi_0(\sin\phi_0 - \eta_2\sin\beta)}{\sin^2\beta(\sin\phi_0 - \sin\theta_1)(\sin\phi_0 - \sin\theta_2)}\right\} e_z + \frac{2\sin\phi_0\cos\phi_0\cos\beta}{\sin^2\beta(\sin\phi_0 - \sin\theta_1)(\sin\phi_0 - \sin\theta_2)} h_z.$$

These can be derived using only (55) and the fact that $t_{\mathrm{e}}(\alpha)$ and $t_{\mathrm{h}}(\alpha)$ are even functions satisfying their respective difference equations, and in contrast to the previous solution [1], both reflection coefficients agree with their optics values.

Apart from the residue contributions, the diffracted field for $k\rho \gg 1$ is

$$E_z^{\mathrm{d}}(\rho, \phi, z) = \frac{e^{-jk(\rho\sin\beta + z\cos\beta) - j\pi/4}}{\sqrt{2\pi k\rho\sin\beta}}\{s_{\mathrm{e}}(\phi - \pi) - s_{\mathrm{e}}(\phi + \pi)\}. \quad (59)$$

In Figure 1 the backscattered ($\phi = \phi_0$) far field amplitude based on this non-uniform representation is plotted as a function of $\phi_0$, $0 \leq \phi_0 \leq \pi$, for $\eta_1 = 2$, $\eta_2 = 4$ and a variety of $\beta$. The result for $\beta = \pi/2$ (normal incidence) agrees with the known solution [2] and for other values of $\beta$ the differences from the previous (erroneous) solution [1] are most noticeable when either $\phi_0$ or $\pi - \phi_0$ is small.

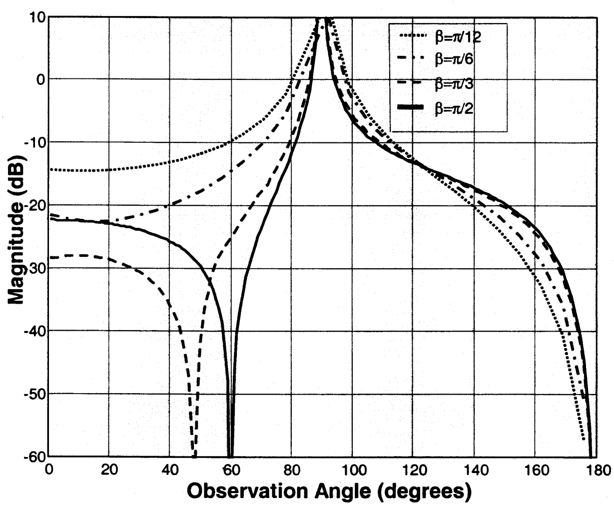


**Figure 1.** Backscattered far field amplitude for $\eta_1 = 2$, $\eta_2 = 4$ and various $\beta$.

## 8. CONCLUDING REMARKS

For the problem of a plane wave incident at an oblique (skew) angle on a half plane subject to anisotropic impedance boundary conditions, Maluizhinets' method [2] leads to a second order difference equation of period $4\pi$ satisfied by linear combinations of the spectra representing the total field components $E_z$ and $Z_0 H_z$. When the impedances are the same on both faces of the half plane, the equation is equivalent to two second order equations having the reduced period $2\pi$, and by application of a modified Fourier transform, these can be converted to inhomogeneous integral equations which are easily solved. In contrast to the work in [1], the expression of the original spectra $s_{\mathrm{e}}(\alpha)$ and $s_{\mathrm{h}}(\alpha)$ in terms of these solutions is more complicated, but as we have shown, the analysis can still be carried out. The result is believed to be the first exact solution of the problem and satisfies all the tests that have been applied to it. In the special cases of isotropic impedances for all $\beta$ and anisotropic impedances for $\beta = \pi/2$ the expressions for the spectra are identical to the known results.

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