INTRA-CHANNEL COLLISION OF KERR LAW OPTICAL SOLITONS

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Abstract—The intra-channel collision of optical solitons, with Kerr law nonlinearity, is studied in this paper by the aid of quasi-particle theory. The perturbation terms that are considered in this paper are the nonlinear gain and saturable amplifiers along with filters. The suppression of soliton-soliton interaction, in presence of these perturbation terms, is achieved. The numerical simulations support the quasi-particle theory.

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1. INTRODUCTION

The propagation of pulses through an optical fiber, for a Kerr nonlinear medium [1, 2], in an optical communication system, is governed by the Nonlinear Schrodinger’s Equation (NLSE). The NLSE is given as a result of balance, to the lowest order, in the Taylor series expansion of the wave number about its carrier frequency, between dispersion and nonlinearity. The dimensionless form of the NLSE is given by

\[ i \frac{\partial q}{\partial Z} + \frac{1}{2} \frac{\partial^2 q}{\partial T^2} + |q|^2 q = 0 \]  

Here, \( q \) is the normalized effective amplitude of the wave electric field while \( Z & T \) are the independent variables. Here, \( Z \) represents the distance along the fiber while \( T \) is the time.

Mathematically speaking, NLSE is a nonlinear partial differential equation (PDE). It belongs to a class of integrable PDEs called S-integrability. In this class all nonlinear PDEs are integrable by the method of Inverse Scattering Transform (IST). NLSE, being a member of this class, is also integrable by the method of IST.

In the anomalous dispersion regime, the particularly relevant solutions to (1) are called solitons, or nontopological solitons. In most cases, the interest is confined to a single pulse described by the 1-soliton solution of the NLSE. However, in this paper, the effects of the perturbation terms in NLSE on two soliton interaction will be studied. It is necessary to launch the solitons close to each other for enhancing the information carrying capacity of the fiber. If two solitons are placed close to each other then it can lead to their mutual interaction thus providing a very serious hindrance to the performance of the soliton transmission system. However, the presence of the perturbation terms of the NLSE can lead to the suppression of the two soliton interaction thus solving our problem.

The perturbed NLSE that is going to be studied in this paper for the SSI is

\[ i \frac{\partial q}{\partial Z} + \frac{1}{2} \frac{\partial^2 q}{\partial T^2} + |q|^2 q = iR[q,q^*] \]  

where

\[ R = \delta |q|^{2m} q + \sigma \int_{-\infty}^{T} |q|^2 \, dt + \beta \frac{\partial^2 q}{\partial T^2} \]  

Here, \( \delta \) is called the nonlinear gain, while \( \sigma \) is the coefficient of saturable amplifiers. Also, in (3), \( m \) could be 0, 1 or 2. For \( m = 0 \), there is linear gain, while for \( m = 1 \) it is called quadratic gain and for \( m = 2 \), it is quintic gain or gain saturation. The coefficient of \( \beta \) is called the bandpass filtering term.
The quasi-particle theory (QPT) of soliton-soliton interaction (SSI) has been investigated \[1, 3–9\] and it will be proved by virtue of it that the interaction can be suppressed due to the perturbation terms given in (3).

In (2) setting \(\epsilon = 0\), (1) is recovered which is the NLSE and is exactly integrable by IST \[2\]. The 1-soliton solution of (1) has the form \[10–15\]

\[
q(Z, T) = \frac{\eta}{\cosh[\eta(T - vZ - T_0)]} e^{(-\imath \kappa T + i\omega Z + i\sigma_0)}
\]

with \(\kappa = -v\)

and

\[
\omega = \frac{\eta^2 - \kappa^2}{2}
\]

Here \(\eta\) is the amplitude (or the inverse width) of the soliton, \(v\) is its velocity, \(\kappa\) is the soliton frequency while \(T_0\) and \(\sigma_0\) are the center of the soliton and the center of the soliton phase respectively.

Also the 2-soliton solution of the NLSE (1) takes the asymptotic form \[16–19\]

\[
q(Z, T) = \sum_{l=1}^{2} \frac{\eta_l}{\cosh[\eta_l(T - v_lZ - T_{0_l})]} e^{(-\imath \kappa_l T + i\omega_l Z + i\sigma_{0_l})}
\]

with \(\kappa_l = -v_l\)

and

\[
\omega_l = \frac{\eta_l^2 - \kappa_l^2}{2}
\]

where \(l = 1, 2\). In the study of SSI, the initial pulse waveform is taken to be \[20\]

\[
q(0, T) = \frac{A_1}{\cosh \left[ A_1 \left( T - \frac{T_0}{2} \right) \right]} e^{\imath \phi_1} + \frac{A_2}{\cosh \left[ A_2 \left( T + \frac{T_0}{2} \right) \right]} e^{\imath \phi_2}
\]

which represents the injection of 2-soliton like pulses into a fiber. Here \(T_0\) represents the initial separation of the solitons namely the center-to-center soliton separation. It is to be noted that for \(T_0 \to \infty\) \((10)\) represents exact soliton solutions, but for \(T_0 \sim O\ (1)\) it does not represent an exact 2-soliton solution. In this paper, the case of in-phase
injection of solitons with equal amplitudes will be studied, namely
\( A_1 = A_2, \phi_1 = \phi_2 \). Without any loss of generality, \( A_1 = A_2 = 1 \) and
\( \phi_1 = \phi_2 = 0 \) are chosen, so that (10) modifies to
\[
q(0, T) = \frac{1}{\cosh \left( T - \frac{T_0}{2} \right)} + \frac{1}{\cosh \left( T + \frac{T_0}{2} \right)}
\]  
(11)

2. QUASI-PARTICLE THEORY

The QPT dates back to 1981 since the appearance of the paper by Karpman & Solov’ev [7]. The mathematical approach to SSI will be studied using the QPT. Here, the solitons are treated as particles. If two pulses are separated and each of them is close to a soliton they can be written as the linear superposition of two soliton like pulses as [1]
\[
q(Z, T) = q_1(Z, T) + q_2(Z, T)
\]  
(12)

with
\[
q_l(Z, T) = \frac{A_l}{\cosh [A_l (T - T_l)]} e^{-iB_l(T - T_l) + i\delta_l}
\]  
(13)

where \( l = 1, 2 \) and \( A_l, B_l, T_l \) and \( \delta_l \) are functions of \( Z \). Here, \( A_l \) and \( B_l \) do not represent the amplitude and the frequency of the full wave form. However, they approach the amplitude and frequency respectively for large separation namely if \( \Delta T = T_1 - T_2 \to \infty \), then \( A_l \to \eta_l \) and \( B_l \to \kappa_l \). Since the waveform is assumed to remain in the form of two pulses, the method is called the quasi-particle approach. First, the equations for \( A_l, B_l, T_l \) and \( \delta_l \) using the soliton perturbation theory will be derived. Substituting (12) into (2) yields [1,2]
\[
i \frac{\partial q_l}{\partial Z} + \frac{1}{2} \frac{\partial^2 q_l}{\partial T^2} + |q_l|^2 q_l = i\epsilon R[q_l, q_l^*] - \left( q_l^2 q_l^* + 2|q_l|^2 q_l \right)
\]  
(14)

where \( l = 1, 2 \) and \( \tilde{l} = 3 - l \). Here, the separation
\[
|q|^2 q = \left( |q_1|^2 q_1 + q_1^2 q_2^* + 2|q_1|^2 q_2 \right) + \left( |q_2|^2 q_2 + q_2^2 q_1^* + 2|q_2|^2 q_1 \right)
\]  
(15)

was used based on the degree of overlapping. By the soliton perturbation theory, the evolution equations are
\[
\frac{dA_l}{dZ} = (-1)^{\tilde{l}+1} 4 A^3 e^{-A\Delta T} \sin(\Delta \phi) + \epsilon M_l
\]  
(16)
\[
\frac{dB_l}{dZ} = (-1)^{\tilde{l}+1} 4 A^3 e^{-A\Delta T} \cos(\Delta \phi) + \epsilon N_l
\]  
(17)
\frac{dT_l}{dZ} = -B_l - 2A e^{-A \Delta T_l} \sin(\Delta \phi_l) + \epsilon Q_l \tag{18}
\frac{d\delta_l}{dZ} = \frac{1}{2} \left( A_l^2 + B_l^2 \right) - 2AB e^{-A \Delta T_l} \sin(\Delta \phi_l) + 6A^2 e^{-A \Delta T_l} \cos(\Delta \phi_l) + \epsilon P_l \tag{19}

where

\begin{align*}
M_l &= \int_{-\infty}^{\infty} \Re \{ \hat{R}[q_l, q_l^*] e^{-i \phi_l} \} \frac{1}{\cosh \tau_l} d\tau_l \tag{20} \\
N_l &= -\int_{-\infty}^{\infty} \Im \{ \hat{R}[q_l, q_l^*] e^{-i \phi_l} \} \frac{\tanh \tau_l}{\cosh \tau_l} d\tau_l \tag{21} \\
Q_l &= \frac{1}{A_l^2} \int_{-\infty}^{\infty} \Re \{ \hat{R}[q_l, q_l^*] e^{-i \phi_l} \} \frac{\tau_l}{\cosh \tau_l} d\tau_l \tag{22} \\
P_l &= \frac{1}{A_l} \int_{-\infty}^{\infty} \Im \{ \hat{R}[q_l, q_l^*] e^{-i \phi_l} \} \left( 1 - \tau_l \frac{\tanh \tau_l}{\cosh \tau_l} \right) d\tau_l \tag{23}
\end{align*}

Here, in (20)–(23), \Re and \Im stands for the real and imaginary parts respectively. Also, the following notations are used

\hat{R}[q_l, q_l^*] = R[q_l, q_l^*] - i \left( q_l^2 q_l^* + 2|q_l|^2 q_l^* \right) \tag{24}

\begin{align*}
\tau_l &= A_l (T - T_l) \tag{25} \\
\phi_l &= B_l (T - T_l) - \delta_l \tag{26} \\
\Delta \phi &= B \Delta T + \Delta \delta \tag{27} \\
\Delta T &= T_1 - T_2 \tag{28} \\
\Delta \delta &= \delta_1 - \delta_2 \tag{29} \\
A &= \frac{1}{2} (A_1 + A_2) \tag{30} \\
B &= \frac{1}{2} (B_1 + B_2) \tag{31} \\
\Delta A &= A_1 - A_2 \tag{32} \\
\Delta B &= B_1 - B_2 \tag{33}
\end{align*}

Moreover, it was assumed that

\begin{align*}
|\Delta A| &\ll A \tag{34} \\
|\Delta B| &\ll 1 \tag{35} \\
A \Delta T &\gg 1 \tag{36} \\
|\Delta A| \Delta T &\ll 1 \tag{37}
\end{align*}
From (16) to (18) one can now obtain

\[
\begin{align*}
\frac{dA}{dZ} &= \epsilon M \\
\frac{dB}{dZ} &= \epsilon N
\end{align*}
\] (38) (39)

\[
\begin{align*}
\frac{d(\Delta A)}{dZ} &= 8A^3 e^{-A\Delta T} \sin(\Delta \phi) + \epsilon \Delta M \\
\frac{d(\Delta B)}{dZ} &= 8A^3 e^{-A\Delta T} \cos(\Delta \phi) + \epsilon \Delta N \\
\frac{d(\Delta T)}{dZ} &= -\Delta B + \epsilon \Delta Q \\
\frac{d(\Delta \phi)}{dZ} &= A \Delta A + \epsilon N \Delta T + \epsilon B \Delta Q + \epsilon \Delta P
\end{align*}
\] (40) (41) (42) (43)

where

\[
\begin{align*}
M &= \frac{1}{2} (M_1 + M_2) \\
N &= \frac{1}{2} (N_1 + N_2)
\end{align*}
\] (44) (45)

and \(\Delta M\), \(\Delta N\), \(\Delta Q\) and \(\Delta P\) are the variations of \(M\), \(N\), \(Q\) and \(P\) which are written as, for example

\[
\Delta M = \frac{\partial M}{\partial A} \Delta A + \frac{\partial M}{\partial B} \Delta B
\] (46)

assuming that they are functions of \(A\) and \(B\) only, which is, in fact, true for most of the cases of interest, otherwise, the equations for

\[
T = \frac{1}{2} (T_1 + T_2)
\] (47)

and

\[
\phi = \frac{1}{2} (\phi_1 + \phi_2)
\] (48)

would have been necessary. In presence of the perturbation terms, as given by, (3), the dynamical system of the soliton parameters, by virtue of soliton perturbation theory, are

\[
\frac{dA}{dZ} = \Gamma (1/2) \Gamma (m + 1) \frac{\Gamma (m + 1)}{\Gamma (m + 3/2)} \delta A^{2m+1} + 2 \sigma A^2 - \frac{2}{3} \beta A \left( A^2 + 3B^2 \right)
\] (49)
\[ \frac{dB}{dZ} = -\frac{4}{3} \beta A^2 B \]  

(50)

so that by virtue of (28), (29), (32) and (33)

\[ \frac{d(\Delta A)}{dZ} = 8 A^3 e^{-A \Delta T} \sin(\Delta \phi) + \frac{\delta}{2^{2m}} \frac{\Gamma(1/2) \Gamma(m + 1)}{\Gamma(m + 3/2)} \sum_{r=0}^{2m+1} \left( \frac{2m + 1}{2r + 1} \right) (2A)^{2r+1} (\Delta A)^{2m-2r} + 4\sigma A \Delta A - 2\beta (A^2 + B^2) \Delta A - 4\beta A B \Delta B \]  

(51)

\[ \frac{d(\Delta B)}{dZ} = 8 A^3 e^{-A \Delta T} \cos(\Delta \phi) - \frac{8}{3} \beta A B \Delta A - \frac{4}{3} \beta A^2 \Delta B \]  

(52)

\[ \frac{d(\Delta T)}{dZ} = -\Delta B + \sigma \]  

(53)

\[ \frac{d(\Delta \phi)}{dZ} = A \Delta A - \frac{4}{3} \beta A^2 B \Delta T \]  

(54)

where in (51)

\[ \binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{r(r-1) \cdots 3.2.1} \]  

(55)

By virtue of (51)–(54), one has the coupled system of equations for \( \Delta \phi \), the phase difference, and \( \Delta T \), the soliton separation, with the fixed point \( A = 1 \) and \( B = 0 \) as follows:

\[ \frac{d^2(\Delta T)}{dZ^2} + 4 \beta \frac{d(\Delta T)}{dZ} + 8 e^{-A \Delta T} \cos(\Delta \phi) = 0 \]  

(56)

\[ \frac{d^2(\Delta \phi)}{dZ^2} + (2\beta - 4\sigma) \frac{d(\Delta \phi)}{dZ} - \frac{\delta}{2^{2m}} \frac{\Gamma(1/2) \Gamma(m + 1)}{\Gamma(m + 3/2)} \sum_{r=0}^{2m+1} \left( \frac{2m + 1}{2r + 1} \right) (\Delta \phi)^{2m-2r} - 8 e^{-A \Delta T} \sin(\Delta \phi) = 0 \]  

(57)

Equations (56) and (57) implies that inserting filters produces a damping in both pulse separation and phase difference. In particular, if the pulses are initially in phase, namely \( \phi_0 = 0 \), then \( \Delta \phi \) remains zero (using \( \phi_0 = 0 \) and (54) with \( B = 0 \)), and the filtering simply gives a reduction in the attractive forces between the pulses.

For in-phase injection of solitons with equal amplitudes, the initial conditions, corresponding to the initial waveform (12), are

\[ A = 1, \quad B = 0, \quad \Delta A_0 = 0, \quad \Delta B_0 = 0, \quad \Delta T_0 = T_0 \quad \& \quad \Delta \phi_0 = 0 \]
3. OBSERVATIONS

The Mathematical set up, given by (49) to (57), will be used to study the various situations of the perturbed NLSE (2) to observe how SSI can be suppressed. In all cases, the initial center-to-center separation of the solitons is $T_0 = 9$. The following special cases are studied here:

3.1. $\sigma \neq 0, \delta \neq 0, \beta \neq 0, m = 0$

In this case, choosing,

$$\beta = 3(\sigma + \delta)$$

yields the stable fixed point of (49) and (50). The system given by (56)–(57) is subsequently further reduced to

$$\frac{d^2(\Delta T)}{dZ^2} + 4(\delta + \sigma)\frac{d(\Delta T)}{dZ} + 8e^{-\Delta T} \cos(\Delta \phi) = 0 \quad (59)$$

$$\frac{d^2(\Delta \phi)}{dZ^2} + 2(\sigma + 2\delta)\frac{d(\Delta \phi)}{dZ} - 8e^{-\Delta T} \sin(\Delta \phi) = 0 \quad (60)$$

Equations (59) and (60) show that there exists damping in pulse separation and phase difference. This is demonstrated in Figure 1(a) where $\sigma = \delta = 0.005$.

Figure 1a. $m = 0, \beta = 3(\sigma + \delta); \sigma = 0.005, \delta = 0.005.$
3.2. $\sigma \neq 0$, $\delta \neq 0$, $\beta \neq 0$, $m = 1$

In this case, choosing,

$$\beta = 3\sigma + 2\delta$$  \hspace{1cm} (61)

gives a stable fixed point of (49) and (50). The system given by (56)–(57) is subsequently further reduced to

$$\frac{d^2(\Delta T)}{dZ^2} + \frac{4}{3}(2\delta + 3\sigma)\frac{d(\Delta T)}{dZ} + 8e^{-\Delta T}\cos(\Delta \phi) = 0$$  \hspace{1cm} (62)

$$\frac{d^2(\Delta \phi)}{dZ^2} + 2\sigma\frac{d(\Delta \phi)}{dZ} - 8e^{-\Delta T}\sin(\Delta \phi) = 0$$  \hspace{1cm} (63)

Equations (62) and (63) show that there is damping in pulse separation and phase difference. This is demonstrated in Figure 1(b) where $\sigma = \delta = 0.005$.

3.3. $\sigma \neq 0$, $\delta \neq 0$, $\beta \neq 0$, $m = 2$

In this case, choosing,

$$\beta = 3\sigma + \frac{8}{5}\delta$$  \hspace{1cm} (64)

Figure 1b. $m = 1$, $\beta = 3\sigma + 2\delta$; $\sigma = 0.005$, $\delta = 0.005$, $\beta = 0.025$. 
one gets a stable fixed point of (49) and (50). The system given by (56)–(57) is subsequently further reduced to

\[
\frac{d^2(\Delta T)}{dZ^2} + \frac{4}{15}(8\delta + 15\sigma) \frac{d(\Delta T)}{dZ} + 8e^{-\Delta T} \cos(\Delta \phi) = 0 \quad (65)
\]

\[
\frac{d^2(\Delta \phi)}{dZ^2} + \frac{2}{15}(15\sigma - 16\delta) \frac{d(\Delta \phi)}{dZ} - 8e^{-\Delta T} \sin(\Delta \phi) = 0 \quad (66)
\]

Equations (65) and (66) show that there exists damping in pulse separation and phase difference. However, a damping in phase difference exists for \(\sigma/\delta > 16/15\). This is demonstrated in Figure 1(c) where \(\sigma = \delta = 0.005\).

Figure 1c. \(m = 2, \beta = 3\sigma + 8/5\delta; \sigma = 0.002, \delta = 0.002, \beta = 0.0092\).

4. CONCLUSIONS

In this paper, the SSI of the NLSE in presence of nonlinear gain, saturable amplifiers and bandpass filters are investigated. It was observed that the SSI can be suppressed in presence of these perturbation terms for various values of the degree of nonlinear gain. The QPT, due to these perturbation terms, was developed and the analytical reasoning of the suppression of the SSI was established.

Thus, in the applied soliton community two solitons can be injected into a single channel, close to one another and also suppress their mutual interaction so that performance enhancement can be
achieved. This conclusion is based on numerical and analytical results due to the quasi-particle theory of SSI.

In future, the SSI for non-Kerr law solitons are to be studied and the QPT corresponding to these non-Kerr law nonlinearities are to be developed. Moreover, the soliton-soliton interaction due to other perturbation terms namely higher order dispersion, self-steepening terms and Raman scattering, just to name a few, will be considered in a future publication. Also the results of this paper can be generalized due to 3-soliton interaction and more. Those results are still awaited at this time and will be reported in future publication.

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