Progress In Electromagnetics Research, PIER 54, 109–135, 2005

USE OF SEMI-INVERSION METHOD FOR THE DIRICHLET PROBLEM IN ROUGH SURFACE SCATTERING

V. I. Tatarskii

Zel Technologies and NOAA ETL
325 Broadway, Boulder, CO 80305, USA

Abstract—The scattering problem from the random interface with the Dirichlet boundary condition can be formulated as an integral equation $x = \widehat{K}y$ with respect to surface sources y (here, $\widehat{K}$ is the integral operator). Starting with an approximate operator $\widehat{K}_0$, for which the inverse operator $\widehat{M} = \widehat{K}_0^{-1}$ is known, the series in powers of the operator $\widehat{Z} = \widehat{M}(\widehat{K}_0 - \widehat{K})$ is derived. As an approximate kernel, we consider the kernel depending only on the difference of arguments: $K_0 = K_0(\mathbf{r} - \mathbf{r}')$, for which the kernel of the operator $\widehat{M}$ can be found in terms of generalized functions. The norm of the difference operator $||\widehat{Z}||$ is found; the conditions of convergency $||\widehat{Z}|| \leqslant 1$ were obtained.

1. INTRODUCTION

If we consider the problem of wave scattering by some body or surface, the equation for the sources of the scattered field may be presented in the form of the first or the second kind of integral equation (IE). Usually the Green's integral theorem and the corresponding Helmholtz integral are used in the derivation of such equations. The extinction theorem [20] is a useful tool to derive IE in the scattering problem. Examples of such equations in rough surface scattering theory can be found in many publications, e.g., [8, 13, 18, 1, 3, 10, 11, 19] (we mention here only a few of them). Different algorithms for the solution of these equations were suggested in the above-mentioned papers.

In the present paper, we consider a general method that can be applied to an arbitrary Fredholm IE of the first or second kind. This method allows us to construct the solution of a given IE with the kernel K if an analytical solution of a "close" IE, having an "approximate" kernel K_0, is known (we call this method a semi-inversion method, [SIM]). If we know the solution of an IE with the kernel K_0, it is possible to derive a new IE of the second kind (both for the first and the second kind of the original IE), which provides the solution of the original IE in the form of a series in powers of difference $(K - K_0)$. The success of this method depends on an appropriate choice of K_0, and this step cannot be formalized. It is important that SIM may be used for the acceleration of convergency of an iterative solution of IE. Examples of using methods similar to SIM can be found in [14, 16]. The SIM was extensively used in the different problems of gratings [6, 7, 15]. The review paper [15] contains more than 100 references related to SIM.

There exist many types of kernels for which it is possible to obtain analytical solution; we can mention degenerate kernels, the finite sum of degenerate kernels, the kernels of convolution type, and the kernels defining the known integral transforms [14, 16]. Such types of kernels may be used as K_0. The important problem of convergency of obtained series arises.

In this paper, we illustrate the SIM as applied to a scattering problem from a rough surface with the Dirichlet boundary condition. For the problem of wave scattering by a random (i.e., an arbitrary) surface the simplest possible method is to replace the precise kernel with the kernel of convolution type. But we do not exclude the possibility of using more sophisticated kernels, for instance the kernel, which leads to the small-slope approximation [18]. In Section 2, we consider the first kind IE (8) for the normal derivative of the field at the scattering surface. In Section 3, we describe the SIM procedure using operator notations and introduce the norm of the operator, which determines the convergence of SIM. In Sections 4 and 5, we consider the approximate IE, which was suggested by Meecham and Lysanov (ML) [12, 9] and find the corresponding inverse kernel in a form of generalized function. In Sections 6 and 7, we evaluate the norm of the operator, which determines convergence of SIM, and formulate the condition of convergence.

2. FORMULATION OF THE SCATTERING PROBLEM

We consider the scattering of waves by the irregular rough interface between two media (each of them is infinite in a positive or negative z direction). We assume that the boundary condition at the interface, which has the equation $z = \zeta(\mathbf{r})$, $\mathbf{r} = (x, y)$, comes to the vanishing of the complex amplitude† of a wave field $E(\mathbf{r}, z)$ at the surface (the Dirichlet boundary condition), i.e.,

$$E(\mathbf{r}, \zeta(\mathbf{r})) = 0. \tag{1}$$

If we consider an acoustical problem, this boundary condition corresponds to the scattering of sound, propagated in water, from a water-air interface. In the case of electromagnetic (EM) waves, this boundary condition corresponds to the scattering of horizontally polarized waves from an ideal conductor in a one-dimensional scattering problem. In the case of two-dimensional EM problems, this boundary condition may be used if the cross-polarization is negligible.

Outside the boundary, the field $E(\mathbf{r}, z)$ satisfies the Helmholtz equation,

$$\Delta_2 E(\mathbf{r}, z) + \frac{\partial^2 E(\mathbf{r}, z)}{\partial z^2} + k^2 E(\mathbf{r}, z) = 0 \tag{2}$$

where Δ_2 is the $2D$ Laplasian operator,

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \mathbf{r}^2}, \quad \Delta_2 = \nabla^2, \quad \nabla = \frac{\partial}{\partial \mathbf{r}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \tag{3}$$

† We assume that the time dependence of the wave is given by the factor $\exp(-i\omega t)$, and we may consider the complex amplitude E as time-independent.

and $k = \omega/c$ is the wave number.

For the Dirichlet problem, the scattered field $E_{\text{sc}}(\mathbf{r},z)$ can be expressed in terms of the normal derivative $F(\mathbf{r})$ of the field E at the surface as follows:

$$E_{\text{sc}}(\mathbf{r},z) = \iint G\left(\mathbf{r}, z; \mathbf{r}', \zeta\left(\mathbf{r}'\right)\right) F\left(\mathbf{r}'\right) d\Sigma\left(\mathbf{r}'\right). \tag{4}$$

Here,

$$d\Sigma\left(\mathbf{r}'\right) = \sqrt{1 + \left(\nabla\zeta\left(\mathbf{r}'\right)\right)^2} d^2 r' \tag{5}$$

is the element of the surface area, and

$$G\left(\mathbf{r}, z; \mathbf{r}', z'\right) = -\frac{\exp\left[ik\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2 + \left(z-z'\right)^2}\right]}{4\pi\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2 + \left(z-z'\right)^2}} \tag{6}$$

is the Green's function. Formula (4) contains the Green's function with the first argument $(\mathbf{r}, z)$, which corresponds to the point of observation, and the second argument $(\mathbf{r}', \zeta(\mathbf{r}'))$, which corresponds to the variable point of integration, located at the surface.

The total field E is the sum of the incident field, $E_{\text{inc}}(\mathbf{r}, z)$, and the scattered field, $E_{\text{sc}}(\mathbf{r}, z)$. In terms of E_{inc} and E_{sc}, the boundary condition (1) takes the form

$$E_{\text{sc}}(\mathbf{r},\zeta(\mathbf{r})) + E_{\text{inc}}(\mathbf{r},\zeta(\mathbf{r})) = 0. \tag{7}$$

Substituting (4) into (7), we obtain the first kind IE with respect to an unknown normal derivative of the field at the surface, F:

$$\iint G\left(\mathbf{r}, \zeta(\mathbf{r}); \mathbf{r}', \zeta\left(\mathbf{r}'\right)\right) \sqrt{1 + \left(\nabla\zeta\left(\mathbf{r}'\right)\right)^2} F\left(\mathbf{r}'\right) d^2 r' = -E_{\text{inc}}(\mathbf{r},\zeta(\mathbf{r})). \tag{8}$$

In (8), both points $(\mathbf{r}, \zeta(\mathbf{r}))$ and $(\mathbf{r}', \zeta(\mathbf{r}'))$ belong to the scattering surface.

It is possible to derive the second kind IE for the same problem by applying the normal derivative operator to (4); such approach was used, for example, in [8].

3. SEMI-INVERSION PROCEDURE

In this section, we use operator notations. The IE of the type

$$\iint K\left(\mathbf{r}, \mathbf{r}'\right) X\left(\mathbf{r}'\right) d^2 r' = Y(\mathbf{r}) \tag{9}$$

will be presented as

$$\widehat{K}X = Y. \tag{10}$$

Here, $\widehat{K}$ denotes an integral operator with the kernel $K(\mathbf{r}, \mathbf{r}')$. The product of two operators $\widehat{K}_1\widehat{K}_2 = \widehat{W}$ is the integral operator, having the kernel

$$W(\mathbf{r}, \mathbf{r}') = \iint K_1(\mathbf{r}, \mathbf{r}'') K_2(\mathbf{r}'', \mathbf{r}') d^2r''.$$

The inverse operator, $\widehat{M} = \widehat{K}^{-1}$, is determined by

$$\widehat{K}\widehat{M} = \widehat{M}\widehat{K} = \widehat{1}. \tag{11}$$

Here, $\widehat{1}$ is the identical operator with the kernel $\delta(\mathbf{r} - \mathbf{r}')$. In the explicit form, (11) looks like

$$\iint K(\mathbf{r}, \mathbf{r}'')M(\mathbf{r}'', \mathbf{r}')d^2r'' = \iint M(\mathbf{r}, \mathbf{r}'')K(\mathbf{r}'', \mathbf{r}')d^2r'' = \delta(\mathbf{r} - \mathbf{r}'). \tag{12}$$

We can present (8) in the form (9) if we set

$$\begin{aligned} K(\mathbf{r}, \mathbf{r}') &= G(\mathbf{r}, \zeta(\mathbf{r}); \mathbf{r}', \zeta(\mathbf{r}')), \\ X(\mathbf{r}') &= \sqrt{1 + (\nabla\zeta(\mathbf{r}'))^2} F(\mathbf{r}'), \\ Y(\mathbf{r}) &= -E_{\text{inc}}(\mathbf{r}, \zeta(\mathbf{r})). \end{aligned} \tag{13}$$

Let us assume that if we replace the precise kernel $K(\mathbf{r}, \mathbf{r}')$ by some approximate kernel $K_0(\mathbf{r}, \mathbf{r}')$, we will be able to solve (9) or (10). In other words, we can find, in the explicit form, the operator $\widehat{M}$ or the corresponding kernel $M(\mathbf{r}, \mathbf{r}')$, which is inverse to the operator $\widehat{K}_0$:

$$\widehat{M}\widehat{K}_0 = \widehat{K}_0\widehat{M} = \widehat{1}. \tag{14}$$

Let us rewrite (10) as follows:

$$\left(\widehat{K} - \widehat{K}_0 + \widehat{K}_0\right) X = Y,$$

or

$$\widehat{K}_0 X = Y + \left(\widehat{K}_0 - \widehat{K}\right) X. \tag{15}$$

If we apply the operator $\widehat{M}$ to (15) from the left and take into account (14) , we obtain the equation

$$X = \widehat{M}Y + \widehat{M}\left(\widehat{K}_0 - \widehat{K}\right)X. \tag{16}$$

This equation is the second kind integral equation, which can be iterated. If we denote

$$\widehat{Z} = \widehat{M}\left(\widehat{K}_0 - \widehat{K}\right), \tag{17}$$

we obtain the iterative series for X:

$$\begin{aligned} X &= \widehat{M}Y + \widehat{Z}\widehat{M}Y + \widehat{Z}^2\widehat{M}Y + \widehat{Z}^3\widehat{M}Y + \cdots \\ &= \left(\widehat{1} + \widehat{Z} + \widehat{Z}^2 + \widehat{Z}^3 + \cdots\right)\widehat{M}Y. \end{aligned} \tag{18}$$

If the operator $\widehat{K}_0$ is close to the precise operator $\widehat{K}$, i.e., if the approximate kernel presents the precise kernel with good accuracy, the difference $(\widehat{K}_0 - \widehat{K})$ and the operator $\widehat{Z}$ may be "small," i.e., its norm may be small.

A similar procedure can be used in the case of an IE of the second kind,

$$X - \widehat{Q}X = Y. \tag{19}$$

An approximate equation has the form

$$X - \widehat{Q}_0X = Y \tag{20}$$

and we assume that the solution of (20),

$$X = \widehat{P}Y, \quad \widehat{P} = \left(\widehat{1} - \widehat{Q}_0\right)^{-1} \tag{21}$$

is known, i.e., that we know in an explicit form the kernel of the operator $\widehat{P}$. In this case, if we present (19) in the form

$$X - \widehat{Q}_0X = Y + \left(\widehat{Q} - \widehat{Q}_0\right)X,$$

apply to this equation the operator $\widehat{P}$, and denote

$$\widehat{\mathcal{Z}} = \widehat{P}\left(\widehat{Q} - \widehat{Q}_0\right), \tag{22}$$

we obtain the solution of (19) in the form

$$X = \left(\widehat{1} + \widehat{\mathcal{Z}} + \widehat{\mathcal{Z}}^2 + \widehat{\mathcal{Z}}^3 + \cdots\right)\widehat{P}Y. \tag{23}$$

Formulae (18) and (17) on the one hand, and (23) and (22) on the other hand, are similar. Thus, in both cases of an IE of the first kind and the second kind, it is possible to present the solution in powers of the difference between the initial and approximating operator.

3.1. Norm of the Operator $\widehat{Z}$

There exist many different definitions of norms of functions and operators. We will use the definition of the norm $||u(\mathbf{r})||$ of a function $u(\mathbf{r})$ that corresponds to the space of continuous complex functions:

$$||u(\mathbf{r})|| = \max_{\mathbf{r}} \ [\ |u(\mathbf{r})|\ ]. \tag{24}$$

The norm $||\widehat{A}||$ of the operator $\widehat{A}$ is defined by the formula,

$$\left|\left|\widehat{A}\right|\right| = \max_{u} \frac{\left|\left|\widehat{A}u\right|\right|}{||u||}. \tag{25}$$

The norm $||\widehat{A}||$ shows how much the norm of the vector (function) u may rise after applying the operator $\widehat{A}$. If $||\widehat{A}|| < 1$, the norm of the transformed function will be less than the norm of the original function. The norm of a product of two operators satisfies the inequality,

$$\left|\left|\widehat{A}\widehat{B}\right|\right| \leqslant \left|\left|\widehat{A}\right|\right| \times \left|\left|\widehat{B}\right|\right|. \tag{26}$$

If $||\widehat{Z}|| < 1$, the series in powers of $\widehat{Z}$ in (18) converges. The smaller is $||\widehat{Z}||$, the faster is convergence.

If the norm of the function is defined by formula (24), the corresponding norm of an operator is given by the simple formula,

$$\left|\left|\widehat{A}\right|\right| = \max_{\mathbf{r}} \iint \left|A\left(\mathbf{r},\mathbf{r}'\right)\right| d^2r'. \tag{27}$$

Using (26) and (17) we obtain

$$\left|\left|\widehat{Z}\right|\right| \leqslant \left|\left|\widehat{M}\right|\right| \times \left|\left|\left(\widehat{K}_0 - \widehat{K}\right)\right|\right|. \tag{28}$$

In the next section, we will concretize the approximate kernel $\widehat{K}_0$, find the inverse operator $\widehat{M}$, and its norm $||\widehat{M}||$.

4. APPROXIMATE EQUATION

The main integral in Equation (8) may be presented in the form of (9) if we use the notations in (13). To obtain an approximate equation, which can be solved in the explicit form, we replace the exact kernel $K(\mathbf{r},\mathbf{r}')$ for the approximate kernel $K_0(\mathbf{r},\mathbf{r}')$ as follows:

$$K_0\left(\mathbf{r},\mathbf{r}'\right) = K_0\left(\mathbf{r}-\mathbf{r}'\right) = -\frac{\exp\left\{ik\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2}\right\}}{4\pi\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2}}. \tag{29}$$

This formula corresponds to neglecting the term

$$\delta\varphi = k\frac{\left[\zeta(\mathbf{r})-\zeta(\mathbf{r}')\right]^2}{2\left|\mathbf{r}-\mathbf{r}'\right|}$$

in the expansion of $k\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2+\left[\zeta(\mathbf{r})-\zeta(\mathbf{r}')\right]^2}$ in powers of $u = \zeta(\mathbf{r})-\zeta(\mathbf{r}')$. Because the value $\delta\varphi$ enters in the $\exp(i\delta\varphi)$, the necessary condition for neglecting this term is

$$k\frac{\left[\zeta(\mathbf{r})-\zeta(\mathbf{r}')\right]^2}{2\left|\mathbf{r}-\mathbf{r}'\right|} \ll 1. \tag{30}$$

This approximation was independently suggested in 1956 by Meecham [12] and Lysanov [9]. The physical meaning of the condition in (30) becomes clearer if we present it in the form

$$\left|\zeta(\mathbf{r})-\zeta\left(\mathbf{r}'\right)\right| \ll \sqrt{\frac{\lambda\left|\mathbf{r}-\mathbf{r}'\right|}{\pi}}, \tag{31}$$

where $\lambda = 2\pi/k$. Thus, the difference in elevations at any distance $|\mathbf{r}-\mathbf{r}'|$ must be small in comparison with the radius of the first Fresnel zone for this distance, $\sqrt{\lambda\left|\mathbf{r}-\mathbf{r}'\right|}$. This means that a diffracted field fills in all possible shadowing zones, which may appear according to geometric optics for small grazing angles of an incident wave. Therefore, the Meecham-Lysanov approximation (MLA) by itself may be valid only if all shadowing zones for small grazing angles of an incident wave are filled by the diffracted field. This condition, however, cannot guarantee that MLA will be correct. In this paper, we will use MLA as a starting point for constructing an expansion (18).

4.1. Solution of the Approximate Equation

To determine the inverse operator $\widehat{M}$, we consider the equation,

$$\iint K_0\left(\mathbf{r}-\mathbf{r}'\right) X\left(\mathbf{r}'\right) d^2r' = Y\left(\mathbf{r}\right). \tag{32}$$

Using (29) we can write

$$-\iint \frac{\exp\left\{ik\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2}\right\}}{4\pi\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2}} X\left(\mathbf{r}'\right) d^2r' = Y\left(\mathbf{r}\right). \tag{33}$$

Equation (33) has the form of convolution and can be solved by Fourier transform. Using the Weyl-Sommerfeld (WS) formula for the Green's function G,

$$-\frac{\exp\left[ik\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2+\left(z-z'\right)^2}\right]}{4\pi\sqrt{\left(\mathbf{r}-\mathbf{r}'\right)^2+\left(z-z'\right)^2}} = \frac{1}{8i\pi^2}\iint \exp\left[i\mathbf{q}\left(\mathbf{r}-\mathbf{r}'\right)+i\nu\left(q\right)\left|z-z'\right|\right]\frac{d^2q}{\nu\left(q\right)} \tag{34}$$

where

$$\nu\left(q\right) = \begin{cases} \sqrt{k^2-q^2} > 0 & \text{for } q < k \\ i\sqrt{q^2-k^2}, & \text{for } q > k \end{cases}, \tag{35}$$

it is possible to find the solution of (33). The solution is presented by the formula, in which the order of integration is important and cannot be changed:

$$X(\mathbf{r}) = \lim_{h\downarrow 0}\frac{i}{2\pi^2}\iint \nu(q)\exp[i\mathbf{q}\mathbf{r}-i\nu(q)h]\left[\iint Y(\mathbf{r}')\exp(-i\mathbf{q}\mathbf{r}')d^2r'\right]d^2q. \tag{36}$$

The integral over $\mathbf{r}'$ in (36) determines the Fourier transform $\widetilde{Y}\left(\mathbf{q}\right)$:

$$\widetilde{Y}\left(\mathbf{q}\right) = \frac{1}{4\pi^2}\iint \exp\left(-i\mathbf{q}\mathbf{r}'\right) Y\left(\mathbf{r}'\right) d^2r'.$$

Let us analyze (36). First of all, the factor $\exp\left[-i\nu\left(q\right)h\right]$ for $q > k$ according to (35) has the form $\exp\left[+\sqrt{q^2-k^2}h\right]$ and tends to

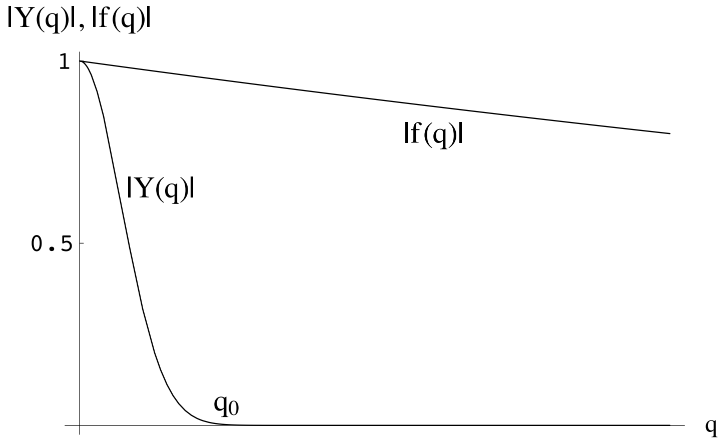

Figure 1. The additional factor $f(\mathbf{q})$ is close to 1 in the region $q < q_0$, where the function $Y(\mathbf{q})$ is concentrated and does not change the value of the integral.

∞ while $q \to \infty$. Thus, the integral over $\mathbf{q}$ in (36) may converge only if the integral over r' as a function of q, i.e., the function $\widetilde{Y}(\mathbf{q})$, tends to zero faster than $\exp\left[-\sqrt{q^2 - k^2}h\right]$ while $q \to \infty$. If the characteristic scale of $\widetilde{Y}(\mathbf{q})$ is some q_0, this means that $|\widetilde{Y}(\mathbf{q})|$ must tend to zero fast enough for $q \gg q_0$ (Figure 1). We assume that this condition is fulfilled. Therefore, if we introduce, under the integral over $\mathbf{q}$ in (36), the additional factor $f(q)$, which is very close to unity for $q < q_0$, the value of the integral will not change. For $q \gg q_0$, the function $f(q)$ tends to zero and, because of this, the behavior of $f(q)$ in this region does not influence the value of the integral over $\mathbf{q}$ in (36). This method is typical for the theory of generalized functions (distributions) see, e.g., [2]. As a result, after introducing such an additional factor in (36), we will be able to change the order of integration and obtain a formula for $M(\mathbf{r} - \mathbf{r}')$ in the form of generalized function.

We may choose the additional factor $f(q)$ in the form

$$f(q) = \exp\left[2i\nu(q)h\right] \tag{37}$$

for the same very small $h \to 0$. Because the function $\widetilde{Y}(\mathbf{q})$ decays for $q \to \infty$, this additional factor does not change the value of integrals over $\mathbf{q}$ if h is small enough. As a result, we can present (36) in the form

$$X(\mathbf{r}) = \frac{i}{2\pi^2}\iint \nu(q)\exp\left[i\mathbf{qr} + i\nu(q)h\right]\left[\iint Y(\mathbf{r}')\exp(-i\mathbf{qr}')d^2r'\right]d^2q. \tag{38}$$

The factor $\exp\left[i\nu\left(q\right)h\right]$ tends to 0 while $q \to \infty$. Because of this, in (38) we may change the order of integration and obtain the formula,

$$X(\mathbf{r}) = \iint Y(\mathbf{r}')d^2r'\left[\lim_{h\downarrow 0}\frac{i}{2\pi^2}\iint \nu(q)\exp\left[i\mathbf{q}(\mathbf{r}-\mathbf{r}') + i\nu(q)h\right]d^2q\right]. \tag{39}$$

Here, the integral over $\mathbf{q}$ converges due to the factor $\exp\left[i\nu\left(q\right)h\right]$.

Formula (39) corresponds to the operator relation $X = \widehat{M}Y$ with the kernel $M\left(\mathbf{r}-\mathbf{r}'\right)$ of the form

$$M(\mathbf{r}-\mathbf{r}') = \lim_{h\downarrow 0}\frac{i}{2\pi^2}\iint \nu(q)\exp\left[i\mathbf{q}(\mathbf{r}-\mathbf{r}') + i\nu(q)h\right]d^2q. \tag{40}$$

5. EVALUATION OF THE FUNCTION $M\left(\mathbf{R}-\mathbf{R}'\right)$

The integral in (40) can be found using the particular form of the Weyl-Sommerfeld (WS) formula,

$$G_0\left(\mathbf{r},h;\mathbf{r}',0\right) = \frac{1}{8i\pi^2}\iint \exp\left\{i\left[\mathbf{q}\left(\mathbf{r}-\mathbf{r}'\right) + \nu\left(\mathbf{q}\right)h\right]\right\}\frac{d^2q}{\nu\left(\mathbf{q}\right)}. \tag{41}$$

Differentiating (41) with respect to h, we obtain

$$-8i\pi^2\frac{\partial^2}{\partial h^2}G_0\left(\mathbf{r},h;\mathbf{r}',0\right) = \iint \exp\left\{i\mathbf{q}\left(\mathbf{r}-\mathbf{r}'\right) + i\nu h\right\}\nu\left(\mathbf{q}\right)d^2q. \tag{42}$$

Let us denote $\rho = \left|\mathbf{r}-\mathbf{r}'\right|$, $R = \sqrt{\rho^2 + h^2}$. Then, for any function $F\left(R\right)$,

$$\frac{dR}{dh} = \frac{h}{R}, \quad \frac{\partial F\left(R\right)}{\partial h} = \frac{h}{R}\frac{dF\left(R\right)}{dR},$$
$$\frac{\partial^2 F\left(R\right)}{\partial h^2} = \frac{1}{R}\frac{dF\left(R\right)}{dR} + \frac{h^2}{R}\frac{d}{dR}\left(\frac{1}{R}\frac{dF\left(R\right)}{dR}\right),$$

and for $F\left(R\right)=-\exp\left(ikR\right)/4\pi R$, after simple calculations, we obtain

$$\begin{aligned} M\left(\mathbf{r},\mathbf{r}'\right) &= \lim_{h\to 0}\frac{i}{2\pi^2}\iint d^2q\sqrt{k^2-\mathbf{q}^2}e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')+ih\nu(\mathbf{q})} \\ &= 4\lim_{h\to 0}\frac{\partial^2 G_0\left(\mathbf{r},h;\mathbf{r}',0\right)}{\partial h^2} \\ &= -\lim_{h\to 0}\frac{1}{\pi R}\frac{d}{dR}\left\{\left[\frac{1}{R}+\frac{ikh^2}{R^2}-\frac{h^2}{R^3}\right]\exp\left(ikR\right)\right\}. \end{aligned} \tag{43}$$

If we perform differentiation, we obtain another formula,

$$\begin{aligned} M(\mathbf{r},\mathbf{r}') &= M\left(R\left(\rho\right)\right) \\ &= \lim_{h\to 0}\frac{k^3}{\pi}\left\{\frac{1-ikR}{k^3R^3}+k^2h^2\frac{k^2R^2+3ikR-3}{k^5R^5}\right\}\exp(ikR). \end{aligned} \tag{44}$$

Assuming that $k^*=k$, we can separate the real and imaginary parts of $M=M_1+iM_2$. We also will use dimensionless values $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

$$M_1=k^3\mathfrak{M}_1;\quad M_2=k^3\mathfrak{M}_2.$$

In terms of $\rho_0=k\rho$ and $\varepsilon=kh$, we can present $\mathfrak{M}_1$ and $\mathfrak{M}_2$ by the formulae,

$$\mathfrak{M}_1 = \lim_{\varepsilon\to 0}\frac{\left[\rho_0^2(1+\varepsilon^2)+\varepsilon^4-2\varepsilon^2\right]\cos\sqrt{\rho_0^2+\varepsilon^2}+\sqrt{\rho_0^2+\varepsilon^2}(\rho_0^2-2\varepsilon^2)\sin\sqrt{\rho_0^2+\varepsilon^2}}{\pi(\rho_0^2+\varepsilon^2)^{5/2}}, \tag{45}$$

and

$$\mathfrak{M}_2 = \lim_{\varepsilon\to 0}\frac{\left[\rho_0^2(1+\varepsilon^2)+\varepsilon^4-2\varepsilon^2\right]\sin\sqrt{\rho_0^2+\varepsilon^2}-\sqrt{\rho_0^2+\varepsilon^2}(\rho_0^2-2\varepsilon^2)\cos\sqrt{\rho_0^2+\varepsilon^2}}{\pi(\rho_0^2+\varepsilon^2)^{5/2}}. \tag{46}$$

We emphasize that $\mathfrak{M}_2\left(\rho_0,\varepsilon\right)$ has the finite limit for $\varepsilon\to 0$, which is the regular function in the point $\rho_0=0$ without singularity:

$$\mathfrak{M}_2\left(\rho_0,0\right)=\frac{\sin\rho_0-\rho_0\cos\rho_0}{\pi\rho_0^3};\quad \mathfrak{M}_2\left(0,0\right)=\frac{1}{3\pi}.$$

The function $\mathfrak{M}_2\left(\rho_0,0\right)$ is shown in Figure 2.

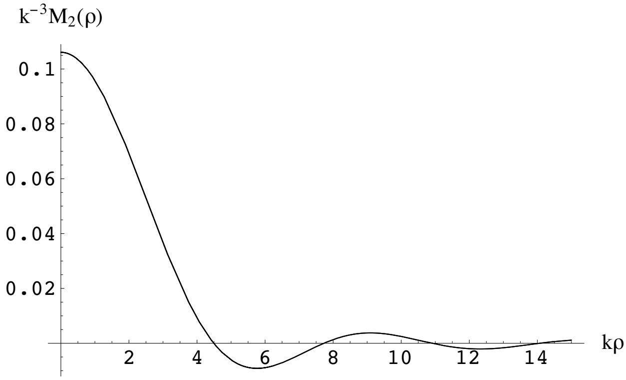

Figure 2. The function $\mathfrak{M}_2(\rho_0, 0)$.

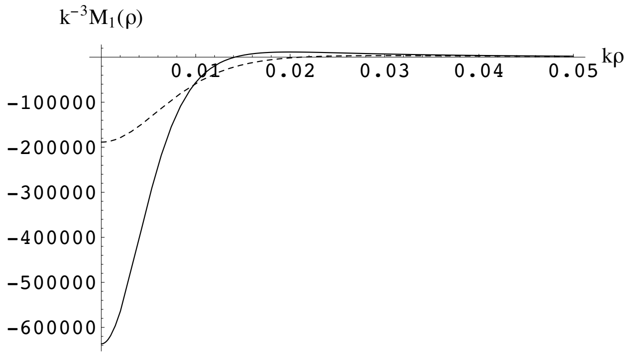

Figure 3. The functions $\mathfrak{M}_1(\rho, 0.01)$ (solid line) and $\mathfrak{M}_1(\rho, 0.015)$ (dashed line).

In contrast to $\mathfrak{M}_2(\rho_0)$, the function $\mathfrak{M}_1(\rho_0)$ significantly depends on $\varepsilon = kh$. The functions $\mathfrak{M}_1(\rho_0, 0.01)$ and $\mathfrak{M}_1(\rho_0, 0.015)$ are shown in Figure 3.

The function $\mathfrak{M}_1(\rho_0, \varepsilon)$ has no uniform limit for $\varepsilon \to 0$, and only integrals of the type

$$\int_0^{\rho} F(\rho)\, \mathfrak{M}_1(\rho_0, \varepsilon)\, \rho d\rho$$

may have the independent of ε limit while $\varepsilon \to 0$. This is a typical situation for generalized functions, and $\mathfrak{M}_1(\rho_0)$ presents an example of a generalized function.

The integrals of the type

$$\int_0^a F(\rho)\,\mathfrak{M}_{1,2}(\rho_0,\varepsilon)\,\rho d\rho$$

converge at $a \to \infty$ very slowly, only due to oscillations of $\mathfrak{M}_{1,2}(\rho_0,\varepsilon)$. Such oscillations always take place in wave problems. However, to analyze an absolute convergence of series (18), we use a norm of the operator $\widehat{Z}$ and must consider $\max\left(\left|\left|\widehat{Z}x\right|\right| / ||x||\right)$ for any functions x, including non-oscillating functions. In such a consideration, the modulus $|\mathfrak{M}(\rho)|$ appears, which is a non-oscillating function:

$$|\mathfrak{M}(\rho_0,\varepsilon)| = \frac{\sqrt{\rho_0^6+\rho_0^4(1-\varepsilon^2+\varepsilon^4)-2\rho_0^2\varepsilon^2(2+\varepsilon^2-\varepsilon^4)+\varepsilon^4(4+\varepsilon^4)}}{\pi(\rho_0^2+\varepsilon^2)^{5/2}}. \tag{47}$$

The function $|\mathfrak{M}(\rho_0,\varepsilon)| \approx 1/\left(\pi\rho_0^2\right)$ while $\rho_0 \to \infty$. Therefore, the integral

$$\int_0^a |\mathfrak{M}(\rho_0,\varepsilon)|\,\rho_0 d\rho_0 \sim \ln a$$

logarithmically diverges while $a \to \infty$. To avoid this divergence, in some integrals we will assume that the wave number k has a very small imaginary part, and we will introduce the additional factor of the form

$$\exp(-\rho/A) = \exp(-\rho_0/A_0), \tag{48}$$

where $A_0 = kA \gg 1$. Because of $A_0 \gg 1$, we will still consider k as a real number everywhere except the factor $\exp(ikR)$ in (44) for M. If we account for this factor in (47), we obtain

$$|\mathfrak{M}(\rho_0,\varepsilon)| = \exp(-\rho_0/A_0)\times$$
$$\frac{\sqrt{\rho_0^6+\rho_0^4(1-\varepsilon^2+\varepsilon^4)-2\rho_0^2\varepsilon^2(2+\varepsilon^2-\varepsilon^4)+\varepsilon^4(4+\varepsilon^4)}}{\pi\left(\rho_0^2+\varepsilon^2\right)^{5/2}}. \tag{49}$$

The role of the scale A is to restrict the effective domain of integration. A similar restriction always appears in real problems because of the finiteness of all real beams in contrast to infinite and non-realizable plane waves. The artificial damping factor of the form (48) is usually included to ensure the absolute convergency of integrals in wave problems (see discussion of this issue in [8]).

5.1. Verifying the Solution

We consider an example in which we are able to solve (32) in two different ways: (a) using the Fourier transform, and (b) using explicit form (45) and (46) of the inverse kernel M. In this example the function $Y(\mathbf{r})$ is Gaussian:

$$Y(\mathbf{r}) = \exp\left(-\frac{r^2}{2a^2}\right). \tag{50}$$

Using (36) we can evaluate all the integrals, and after simple manipulations obtain the formula

$$\begin{aligned} X(\mathbf{r}) &= -2a^2k^3\int_1^\infty \sqrt{x^2-1}\exp\left(-\frac{k^2a^2}{2}x^2\right)J_0(krx)\,xdx \\ &\quad +2ia^2k^3\int_0^1 \sqrt{1-x^2}\exp\left(-\frac{k^2a^2}{2}x^2\right)J_0(krx)\,xdx \\ &= x_1(kr,ka)+ix_2(kr,ka). \end{aligned} \tag{51}$$

If we find the same functions using M_1 and M_2 we obtain

$$X(\mathbf{r}) = \lim_{\varepsilon\to 0}\left[X_{1M}(kr,ka,\varepsilon)+iX_{2M}(kr,ka,\varepsilon)\right]. \tag{52}$$

Here,

$$\begin{aligned} X_{1M}(kr,ka,\varepsilon) &= 2\pi k\exp\left(-\frac{(kr)^2}{2(ka)^2}\right) \\ &\quad \int_0^\infty \mathfrak{M}_1(\rho_0,\varepsilon)\exp\left(-\frac{\rho_0^2}{2k^2a^2}\right)I_0\left(\frac{(kr)\rho_0}{(ka)^2}\right)\rho_0 d\rho_0, \\ X_{2M}(kr,ka,\varepsilon) &= 2\pi k\exp\left(-\frac{(kr)^2}{2(ka)^2}\right) \\ &\quad \int_0^\infty \mathfrak{M}_2(\rho_0,\varepsilon)\exp\left(-\frac{\rho_0^2}{2k^2a^2}\right)I_0\left(\frac{(kr)\rho_0}{(ka)^2}\right)\rho_0 d\rho_0, \end{aligned} \tag{53}$$

$I_0(u)$ is the Bessel function of an imaginary argument, and we marked by the subscript M the functions $X_{1,2}$ obtained using operator $\widehat{M}$.

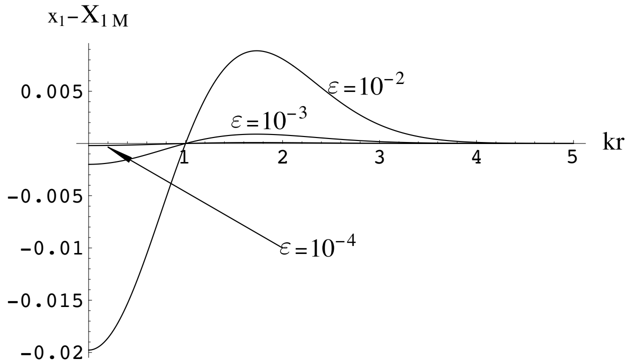

Figure 4. The difference $x_1\left(kr,1\right)-X_{1M}\left(kr,1,10^{-n}\right)$ for $n=2,3$, and 4.

The difference $x_1\left(kr,1\right)-X_{1M}\left(kr,1,\varepsilon=10^{-n}\right)$ for $n=2,3$, and 4 is shown in Figure 4. These plots show the rate of convergence of $X_{1M}\left(\varepsilon\right)\rightarrow x_1$ depending on ε. The relative difference is less than 1.3% for $\varepsilon=0.01$ and less than 0.13% for $\varepsilon=10^{-3}$. It is easy to show that $\left|x_1\left(0,ka\right)-X_{1M}\left(0,ka,\varepsilon\right)\right|\sim\varepsilon$.

Because the function $\mathfrak{M}_2\left(\rho_0,\varepsilon\right)$ has the final limit while $\varepsilon\rightarrow 0$, there is no need to check the rate of convergency for X_2. The example considered illustrates a good convergence of the solution obtained using the operator $\widehat{M}\left(\varepsilon\right)$ to the precise solution X.

6. NORM OF THE OPERATOR $\widehat{Z}$

6.1. Norm of the Operator $\widehat{M}$

According to (27), we have

$$\left|\left|\widehat{M}\right|\right|=\max_{\mathbf{r}}\iint\left|M\left(\mathbf{r},\mathbf{r}'\right)\right|d^2r'. \tag{54}$$

Substituting (49) we obtain

$$\begin{aligned}\left|\left|\widehat{M}\left(\varepsilon\right)\right|\right| &= k^3\max_{\mathbf{r}}\iint d^2r'\exp\left(-\rho_0/A_0\right)\times\\ &\quad\left.\frac{\sqrt{\rho_0^6+\rho_0^4(1-\varepsilon^2+\varepsilon^4)-2\rho_0^2\varepsilon^2(2+\varepsilon^2-\varepsilon^4)+\varepsilon^4(4+\varepsilon^4)}}{\pi\left(\rho_0^2+\varepsilon^2\right)^{5/2}}\right|_{\rho_0=k|\mathbf{r}-\mathbf{r}'|}\\ &= k\mathcal{M}\left(\varepsilon\right)\end{aligned} \tag{55}$$

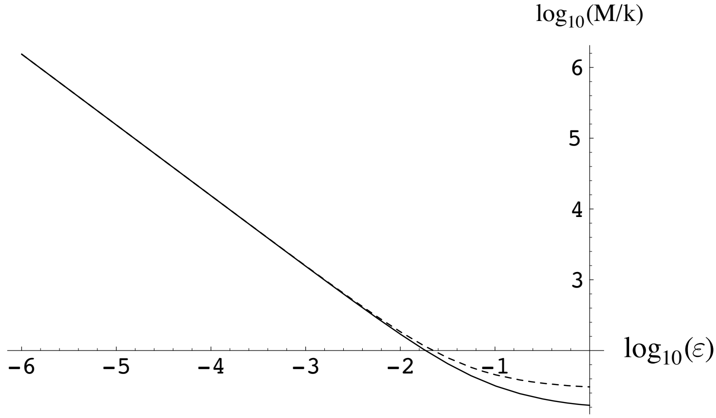

Figure 5. The function $\mathcal{M}(\varepsilon)$ in the double decimal logarithmic scale for $A_0 = 10^4$ (solid) and $A_0 = 10^7$ (dashed).

where $\mathcal{M}(\varepsilon)$ is the dimensionless quantity,

$$\mathcal{M}(\varepsilon) = 2\pi \int_0^\infty \exp(-\rho_0/A_0) \times \frac{\sqrt{\rho_0^6 + \rho_0^4(1-\varepsilon^2+\varepsilon^4) - 2\rho_0^2\varepsilon^2(2+\varepsilon^2-\varepsilon^4) + \varepsilon^4(4+\varepsilon^4)}}{\pi(\rho_0^2+\varepsilon^2)^{5/2}} \rho_0 d\rho_0. \quad (56)$$

By simple analysis it is easy to find that for $\varepsilon < 10^{-2}$ the function $\mathcal{M}(\varepsilon)$ can be approximated by

$$\mathcal{M}(\varepsilon) \approx \frac{C(A_0)}{\varepsilon}. \quad (57)$$

The value of $C(A_0)$ can be determined numerically. The function $C(A_0)$ for $10^4 < A_0 < 10^9$ has a good approximation given by the formula

$$C(A_0) = 8/\left(3\sqrt{3}\right) + 0.050 \log_{10} A_0. \quad (58)$$

The plot of the function $\mathcal{M}(\varepsilon)$ in the double logarithmic scale is shown in Figure 5 for $A_0 = 10^4$ (solid) and $A_0 = 10^7$ (dashed). Note that dependence on A_0 is very slow (logarithmic).

Finally, for the norm $\left|\left|\widehat{M}(\varepsilon)\right|\right|$ we obtain the estimate,

$$\left|\left|\widehat{M}(\varepsilon)\right|\right| \leqslant k\frac{C(A_0)}{\varepsilon} = k\frac{1.54 + 0.050 \lg A_0}{\varepsilon}. \quad (59)$$

6.2. Norm of the Operator $\widehat{V} = \widehat{K}_0 - \widehat{K}$

We have

$$\left|\left|\widehat{V}\right|\right| = \max_{\mathbf{r}''} \iint \left|V\left(\mathbf{r}'', \mathbf{r}'\right)\right| d^2 r' \tag{60}$$

where

$$V\left(\mathbf{r}'', \mathbf{r}'\right) = K_0\left(\mathbf{r}'', \mathbf{r}'\right) - K\left(\mathbf{r}'', \mathbf{r}'\right)$$

$$= \frac{\exp\left\{ik\sqrt{\left(\mathbf{r}''-\mathbf{r}'\right)^2 + \left[\zeta\left(\mathbf{r}''\right) - \zeta\left(\mathbf{r}'\right)\right]^2}\right\}}{4\pi\sqrt{\left(\mathbf{r}''-\mathbf{r}'\right)^2 + \left[\zeta\left(\mathbf{r}''\right) - \zeta\left(\mathbf{r}'\right)\right]^2}} - \frac{\exp\left\{ik\sqrt{\left(\mathbf{r}''-\mathbf{r}'\right)^2}\right\}}{4\pi\sqrt{\left(\mathbf{r}''-\mathbf{r}'\right)^2}}. \tag{61}$$

If we denote

$$u^2 = \left[\zeta\left(\mathbf{r}''\right) - \zeta\left(\mathbf{r}'\right)\right]^2, \quad \rho = \left|\mathbf{r}''-\mathbf{r}'\right|, \tag{62}$$

we obtain

$$\left|V\left(\mathbf{r}'', \mathbf{r}'\right)\right|^2 = \frac{1}{16\pi^2\rho^2} + \frac{1}{16\pi^2\left(\rho^2 + u^2\right)} - \frac{2\cos\left[k\left(\sqrt{\rho^2 + u^2} - \rho\right)\right]}{16\pi^2\rho\sqrt{\rho^2 + u^2}}. \tag{63}$$

The function $\left|V\left(\mathbf{r}'', \mathbf{r}'\right)\right|^2$ is the oscillating function of u. The upper envelope $B^2\left(\rho, u\right)$ of $\left|V\left(\mathbf{r}'', \mathbf{r}'\right)\right|^2$ corresponds to $\cos\left[k\left(\sqrt{\rho^2 + u^2} - \rho\right)\right] = -1$ and is equal to

$$B^2\left(\rho, u\right) = \left[\frac{1}{4\pi\rho} + \frac{1}{4\pi\sqrt{\rho^2 + u^2}}\right]^2;$$

i.e.,

$$B\left(\rho, u\right) = \frac{\sqrt{\rho^2 + u^2} + \rho}{4\pi\rho\sqrt{\rho^2 + u^2}}. \tag{64}$$

In the region $u \ll \rho$ we obtained another asymptotic

$$V(\mathbf{r}'', \mathbf{r}') = \frac{\exp\{ik\rho\}}{4\pi\rho} - \frac{\exp\left\{ik\sqrt{\rho^2+u^2}\right\}}{4\pi\sqrt{\rho^2+u^2}} \approx -\frac{e^{ik\rho}}{4\pi\rho^2}\left(\frac{ik\rho-1}{2\rho}\right)u^2 + O(u^4),$$

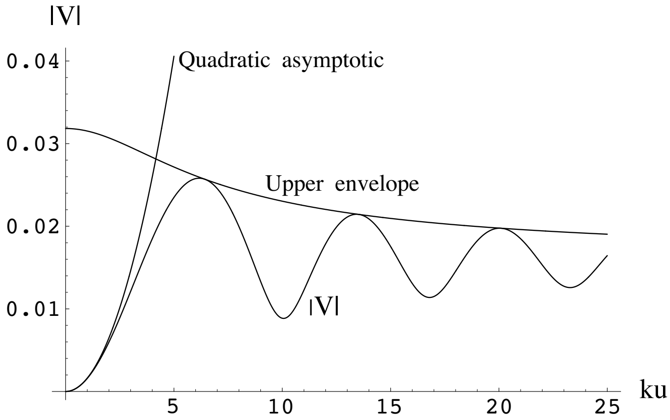

Figure 6. The functions $|V(\rho)|$, $B(\rho,u)$, and $A(\rho,u)$ for $k\rho = 5$ vs ku.

and the modulus of V in the region $u \ll \rho$ is given by the formula

$$\left|V\left(\mathbf{r}'',\mathbf{r}'\right)\right| \approx \frac{\sqrt{1+k^2\rho^2}}{8\pi\rho^3}u^2 \equiv A(\rho,u,k). \tag{65}$$

The dependencies $|V(\rho,u)|$, $B(\rho,u)$, and $A(\rho,u)$ on ku for $k\rho = 5$ are shown in Figure 6.

To obtain the monotonous in u upper estimate of $|V(\mathbf{r}',\mathbf{r}'')|$, we must use such monotonously non-decreasing function $E(\rho,u)$ that

$$E(\rho,u+\delta u) - E(\rho,u) \geqslant 0 \text{ for } \delta u > 0 \text{ and } \left|V\left(\mathbf{r}'-\mathbf{r}''\right)\right| \leqslant E(\rho,u).$$

To construct such a function, we find the point of intersection of the functions $A(\rho,u)$ and $B(\rho,u)$. The point of intersection of the two asymptotic $u = u_0$ is determined by the equation $B(\rho,u_0) = A(\rho,u_0)$. The positive solution of this algebraic equation, $u_0(\rho)$, is given by

$$u_0(\rho,k) = \frac{1}{\sqrt{2}}\sqrt{\rho^2\left(\frac{4}{\sqrt{1+k^2\rho^2}}-1\right)+\frac{\sqrt{\rho^4(1+k^2\rho^2)\left(1+k^2\rho^2+8\sqrt{1+k^2\rho^2}\right)}}{1+k^2\rho^2}}. \tag{66}$$

We determine the monotonously non-decreasing function $E(\rho,u,k)$ which serves as an upper bound for $|V(\rho)|$ for all u:

$$E(\rho,u,k) = \begin{cases} A(\rho,u,k) & \text{if} \quad u < u_0(\rho,k) \\ A(\rho,u_0(\rho,k),k) & \text{if} \quad u > u_0(\rho,k) \end{cases}. \tag{67}$$

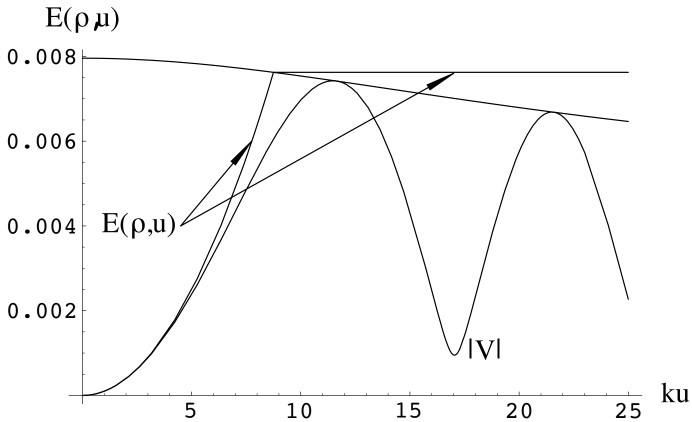

Figure 7. The functions $|V(\rho,u)|$, $B(\rho,u)$, and $E(\rho,u)$ for $k\rho = 20$ vs ku.

The plot of the functions $E(\rho,u,k)$ for value of $k\rho = 20$ is presented in Figure 7.

Now we may perform the following estimations:

$$|V(\rho,u,k)| \leqslant E(\rho,u,k), \tag{68}$$

and because $E(\rho,u,k)$ is the monotonously non-decreasing function of u, we may write

$$|V(\rho,u,k)| \leqslant E(\rho,u,k) \leqslant E(\rho,u_{\max}(\rho),k), \tag{69}$$

where $u_{\max}(\rho)$ is the maximum possible value of $u = \zeta(\mathbf{r}'') - \zeta(\mathbf{r}')$.

We will consider the case when $\zeta(\mathbf{r})$ is a statistically homogeneous, random function. In this case, the difference $u = \zeta(\mathbf{r}'') - \zeta(\mathbf{r}')$ is a random value, having for any fixed distance ρ some probability density function (PDF) $W(u,\rho)$. We assume that

$$W(u,\rho) = 0 \quad \text{if} \quad |u| > u_{\max}(\rho). \tag{70}$$

This assumption is absolutely natural because all realizable physical surfaces must have only finite bursts. For instance, if we approximate the PDF of u by a Gaussian low, this approximation may be accurate only in the region of relatively small u, say $|u| < \sqrt{P\langle u^2\rangle}$, where $\langle\cdots\rangle$ denotes the statistical averaging. (We assume that $\langle\zeta\rangle = 0$.) Here, P is some number about 3-5. The tails of the PDF in the region $|u| > \sqrt{P\langle u^2\rangle}$ must be replaced for 0. Therefore, in this case we have the estimate,

$$u_{\max}^2(\rho) = P\left\langle \left[\zeta\left(\mathbf{r}''\right) - \zeta\left(\mathbf{r}'\right)\right]^2 \right\rangle. \tag{71}$$

The quantity $\left\langle [\zeta(\mathbf{r}'') - \zeta(\mathbf{r}')]^2 \right\rangle$, appearing in (71), is a so-called structure function (see, e.g., [17], §5). This function may be expressed in terms of a correlation function of the surface as follows:

$$\left\langle \left[\zeta\left(\mathbf{r}''\right) - \zeta\left(\mathbf{r}'\right)\right]^2 \right\rangle = \left\langle \left[\zeta\left(\mathbf{r}''\right)\right]^2 \right\rangle + \left\langle \left[\zeta\left(\mathbf{r}'\right)\right]^2 \right\rangle - 2\left\langle \zeta\left(\mathbf{r}'\right)\zeta\left(\mathbf{r}''\right)\right\rangle.$$

For statistically homogeneous surfaces, the value $\left\langle [\zeta(\mathbf{r})]^2 \right\rangle = \sigma^2$ is independent of $\mathbf{r}$ and the correlation coefficient

$$\frac{\langle \zeta(\mathbf{r}')\,\zeta(\mathbf{r}'')\rangle}{\sigma^2} = b\left(\overrightarrow{\rho}\right)$$

depends only on the vector $\overrightarrow{\rho} = \mathbf{r}' - \mathbf{r}''$. For simplicity we will consider the case of statistically isotropic surfaces for which $b(\overrightarrow{\rho}) = b(\rho)$ depends only on the modulus of $|\overrightarrow{\rho}| = \rho$, i.e., on the distance between the points $\mathbf{r}'$ and $\mathbf{r}''$. Thus, we obtain for $u_{\max}^2$ the estimate,

$$u_{\max}^2(\rho) = P\left\langle \left[\zeta\left(\mathbf{r}''\right) - \zeta\left(\mathbf{r}'\right)\right]^2 \right\rangle = 2P\sigma^2\left[1 - b(\rho)\right]. \tag{72}$$

It is important for the convergency of integrals that $u_{\max}^2(\rho) \sim \rho^2$ for small ρ.

After substituting (69) and (72) in (60) we obtain

$$\begin{aligned} \left\|\widehat{V}\right\| &= \max_{\mathbf{r}''} \iint \left|V(\mathbf{r}'', \mathbf{r}')\right| d^2 r' \\ &\leqslant \max_{\mathbf{r}''} \iint E\left(\rho, \sqrt{2P\sigma^2[1 - b(\rho)]}, k\right) d^2\rho \\ &= \iint E\left(\rho, \sqrt{2P\sigma^2\left[1 - b(\rho)\right]}, k\right) d^2\rho \\ &= 2\pi \int_0^\infty E\left(\rho, \sqrt{2P\sigma^2\left[1 - b(\rho)\right]}, k\right) \rho d\rho. \end{aligned} \tag{73}$$

According to (48) we introduce the additional factor $\exp(-\rho/A)$ in the integrand in (73) and use the formula,

$$\left\|\widehat{V}\right\| \leqslant 2\pi \int_0^\infty E\left(\rho, \sqrt{2P\sigma^2\left[1 - b(\rho)\right]}, k\right) \exp(-\rho/A)\, \rho d\rho. \tag{74}$$

In the following numerical calculations we will use the following model for $b(\rho)$:

$$b(\rho) = \exp\left(-\rho^2/l^2\right) \tag{75}$$

where l is the correlation radius for elevations.

7. CONDITIONS OF CONVERGENCE

According to (28), (59), and (74), the final formula for $\left\|\widehat{Z}\right\|$ has the form

$$\left\|\widehat{Z}\right\| \leqslant \left\|\widehat{M}\right\| \times \left\|\widehat{V}\right\| \leqslant \mathcal{Z}(k\sigma, kl, kA, \varepsilon) =$$

$$\frac{1.54 + 0.050\log_{10}(kA)}{\varepsilon} \times 2\pi \int_0^\infty E\left(\rho, \sqrt{2P\sigma^2 D(\rho)}, k\right) \exp(-\rho/A)\rho d\rho. \tag{76}$$

The condition of convergence,

$$\mathcal{Z}(k\sigma, kl, kA, \varepsilon) < 1, \tag{77}$$

can be resolved with respect to $k\sigma$ and takes the form

$$k\sigma < k\sigma_{\rm cr}(\varepsilon, kl, kA). \tag{78}$$

The function $k\sigma_{\rm cr}(\varepsilon, kl, kA)$ can be found numerically from (77). The results of calculations for $\varepsilon = 10^{-2}$ are presented in Figure 8.

In the regions $kl \gg kA$ and $1 \ll kl \ll kA$, the following asymptotic formulae can be derived:

$$k\sigma_{\rm cr} \approx \frac{l\sqrt{2\varepsilon}}{A\sqrt{P(1.54 + 0.05\log_{10}(kA))}} \qquad \text{for } kl \gg kA,$$

$$k\sigma_{\rm cr} \approx \frac{\sqrt{2\varepsilon}}{\sqrt{P[1.54 + 0.05\log_{10}(kA)][0.4343\log_{10}(A/l) - 0.2886]}} \quad \text{for } 1 \ll kl \ll kA. \tag{79}$$

Here, $0.2886 = 0.5\times$Euler constant.

The best conditions for convergency appear if $l_0 > A_0$. For example, for $l_0 = 10^7$ and $A_0 = 10^3$ we obtain $k\sigma \leqslant 632$ for $\varepsilon = 10^{-2}$ and $k\sigma \leqslant 63.2$ for $\varepsilon = 10^{-4}$.

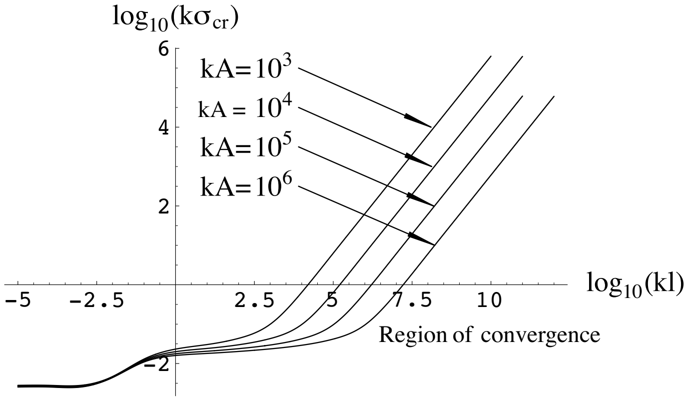

Figure 8. The function $k\sigma_{\rm cr}\,(\varepsilon, kl, kA)$ for $\varepsilon = 10^{-2}$ in double decimal logarithmic scale for different kA.

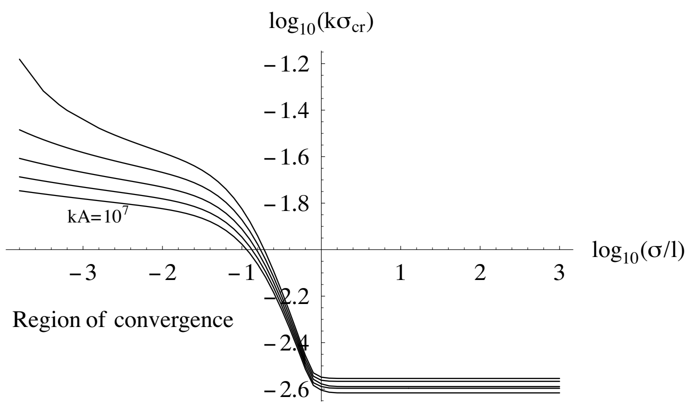

Figure 9. The value $k\sigma_{\rm cr}$ as the function of the slope $\gamma = \sigma/l$ for $kA = 10^2$ (upper curve), $kA = 10^3$, $kA = 10^4$, $kA = 10^5$, $kA = 10^6$, and $kA = 10^7$ (lower curve).

The condition $l_0 \gg A_0$ in a certain way corresponds to such footprint that is small in comparison with l_0, so it may have practical meaning. Thus, we may expect good convergency in the case $D \ll l_0$; i.e., the size D of footprint is smaller than the correlation scale.

We may describe the finite beam by the function $\exp\left(-r^2/A_0^2\right)$. In our calculations, if we replace the factor $\exp\left(-r/A_0\right)$ for $\exp\left(-r^2/A_0^2\right)$, we obtain results that are very close to those presented in Figure 8.

We also can present the condition of convergency in the coordinates $(\sigma/l, k\sigma_{\rm cr})$, if we resolve the equation

$$\mathcal{Z}\left(k\sigma, \frac{k\sigma}{\gamma}, kA, \varepsilon\right) < 1,$$

with respect to $k\sigma_{\rm cr}$ for given slope $\gamma = \sigma/l$. The results are presented in the Figure 9 for different kA.

We note that for very small $k\sigma_{\rm cr} < 2.5 \times 10^{-3}$ the iterations converge even for large slopes.

8. SUMMARY AND DISCUSSION

1. In this paper we suggest a method of constructing a series presenting the solution of an integral equation of the first kind $\widehat{K}X = Y$ in powers of the operator $\widehat{Z} = \widehat{M}(\widehat{K}_0 - \widehat{K})$, where $\widehat{K}_0$ is an approximate integral operator for which the equation $\widehat{K}_0 X = Y$ may be solved in the form $X = \widehat{M}Y$. If the difference $\widehat{K}_0 - \widehat{K}$ is small, this series may converge.

2. We consider the problem of scalar wave scattering from a rough surface with the Dirichlet boundary condition, which may be described by the integral equation of the first kind (8).

3. As an approximate operator $\widehat{K}_0$ we choose the operator having a kernel depending on the difference of arguments (Meecham [12] and [9]). Such equation may be solved by the Fourier transform.

4. We found the explicit form of the kernel of $\widehat{M}$ in terms of generalized functions ([2, 5]), which are defined in terms of functions, depending on the small parameter $\varepsilon \to 0$. We illustrate a good convergence of a solution obtained using $\widehat{M}$ to the solution obtained by the Fourier transform.

5. The norm $||\widehat{Z}||$ of the operator $\widehat{Z}$, corresponding to the space of complex continuous functions [4, 5], was found. Some statistical characteristics of a surface were used for estimations.

6. The condition of convergence of the series, $||\widehat{Z}|| < 1$ was investigated. This condition may be presented in the forms $k\sigma \leqslant k\sigma_{\rm cr}(\varepsilon, kl, kA)$ or $k\sigma \leqslant k\sigma_{\rm cr}(\varepsilon, \gamma, kA)$ where k is the wave number, σ is the rms of surface elevations, l is the correlation length of surface elevations, γ is the rms slope of surface, and A is the extinction length of the scattered waves. The functions $k\sigma_{\rm cr}(\varepsilon, kl, kA)$ and $k\sigma_{\rm cr}(\varepsilon, \gamma, kA)$ were obtained numerically (see Figures 8 and 9).

7. The best conditions of convergence correspond to $l \gg A$. Physically, such a situation may be realized in the case of finite beams if the footprint is small in comparison with the correlation length. It is important that the convergence condition was obtained for any form

of incident wave, and because of this, is independent of the angle of incidence of the wave. Thus, the series obtained must converge even in the case of small grazing angles.

It is useful to compare the results of this paper with the results, obtained in [10, 11], because the method of [10, 11] is rather close to that considered here. In [10], the scattering from 1D surface with the Dirichlet boundary condition was considered. The difference $\delta G = G_2 (\mathbf{r},\zeta(\mathbf{r}); \mathbf{r}', \zeta(\mathbf{r}')) - G_2(\mathbf{r},0;\mathbf{r}',0)$, where $G_2 \sim H_0^{(1)}(kR)$ is the Green's function for a cylindrical source, was used as an expansion parameter. But in contrast to the present paper, the numerical Fourier transform was used to solve the corresponding IE of the convolution type, which doubles the number of integrations and creates the problem of stability. To analyze the convergence of the expansion obtained, in [10] was used another norm: the spectral radius ρ, i.e., the greatest modulus of the eigenvalue of the matrix, which represents the operator δG. The condition $\rho < 1$, which was analyzed in [10] for periodic interface, similarly to the condition $||\widehat{Z}|| < 1$, ensures the absolute convergence of the expansion. It is unclear, however, how the value of ρ was obtained in [10, 11]. In particular, it is unclear why the dependence on the angle of incidence, θ_0, in Figures 2 and 3 in [10] appears, because the spectral radius ρ depends only on the operator δG and may not depend on the angle of incidence, which enters only in the right-hand side of the IE (see equations (4), (6) and (1) in [10]). The parameter A, which describes attenuation of the wave and seems to be an essential parameter for analyzing the condition of absolute convergence (see discussion in [8]), does not appear in [10]. Possibly, the reason for this is that in [10] the convergence was analyzed not by the direct calculation of ρ (in this case dependence on θ_0 would not appear), but by analyzing a few different examples. Because of this, some cases in which expansion diverges, were not found (in Figure 3 in [10] there is one point where $\rho = 1$ in the region $\tan\varepsilon < 1$; possibly, if considering more examples, the points where $\rho > 1$ may appear). In the present paper, the inequality $||\widehat{Z}|| < 1$, which was analyzed analytically, guarantees convergence of expansion for any conditions. It is necessary to emphasize that conditions of convergency presented in Figures 8 and 9 are only sufficient conditions and may be too restrictive.

ACKNOWLEDGMENT

I am grateful to Drs. M. I. Charnotskii, I. M. Fuks and two anonymous reviewers for helpful comments.

REFERENCES

1. Charnotskii, M. I. and V. I. Tatarskii, *WRM*, Vol. 5, 361–380, 1995.
2. Gelfand, I. M. and G. E. Shilov, *Generalized Functions*, Vol. 1. Properties and operations, Translated from the Russian by Eugene Saletan, xviii+423, 46FXX, Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1964 [1977].
3. Kapp, D. A. and G. S. Brown, *IEEE Transactions AP*, Vol. 44, 711–721, 1996.
4. Kreyszig, E., *Advanced Engineering Mathematics*, Seventh edition, Section 19.3, Formula 11, John Wiley & Sons, Inc., New York, 1993.
5. Kreyszig, E., *Introductory Functional Analysis with Applications*, John Wiley & Sons, Inc., New York, 1989.
6. Lerer, A. M. and A. G. Schuchinsky, *IEEE Transactions on Microwave Theory and Techniques*, Vol. MTT-41, 11, 2002–2009, 1993.
7. Lerer, A., I. Donets, and S. Bryzgalo, *J. of Electromagnetic Waves and Application*, Vol. 10, No. 6, 765–790, 1996.
8. Liszka, E. G. and J. J. McCoy, *J. Acoust. Soc. Am.*, Vol. 71, 1093–1100, 1982.
9. Lysanov, Yu. P. and Zh. Akust, Vol. 2, 182–187, 1956 (in Russian).
10. McDaniel, S. T., *WRM*, Vol. 8, 3–14, 1998.
11. McDaniel, S. T., *WRM*, Vol. 9, 401–413, 1999.
12. Meecham W. C., *J. Acoust. Soc. Am.*, Vol. 28, 370–377, 1956.
13. Milder, D. M., *J. Acoust. Soc. Am.*, Vol. 89, 529–541, 1991.
14. Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics*, Part I, Chapt. 8, McGraw-Hill, New York, 1953.
15. Nosich, A. I., *IEEE Antenna and Propagation Magazine*, Vol. 41, No. 3, 34–49, 1999.
16. Polyanin, A. D. and A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press, Boca Raton, Boston, London, New York, Washington DC, 1998.
17. Tatarskii, V. I., "The effects of the turbulent atmosphere on wave propagation," Translated from the Russian by the Israel Program for Scientific Translations, Jerusalem, 1971. Available from the U.S. Dept. of Comm., Nat. Tech. Inf. Serv., Springfield, VA, 22151.
18. Tatarskii, V. I., *WRM*, Vol. 3, 127–146, 1993.
19. Voronovich, A. G., *Wave Scattering from Rough Surfaces*, Second

edition, Springer, Berlin, 1999.

20. Wolf, E., "A generalized extinction theorem and its role in scattering theory," *Coherence and Quantum Optics*, L. Mandel and E. Wolf (eds.), Plenum, New York, 1973.

Valerian I. Tatarskii was born on October 13, 1929 in Kharkov, USSR. He received the M.S. degree in physics and mathematics from Moscow State University in 1952, the Ph.D. degree in physics and mathematics from the Acoustical Institute USSR Academy of Sciences in 1957, and the Doctor of Sciences in physics and mathematics from Gorky State University in 1964. From 1953–1990 he was a Junior Scientist, Senior Scientist, Head of the Laboratory, and Head of the Department with USSR Academy of Sciences. From 1991–2001 he was a Senior Research Scientist at the University of Colorado/CIRES & NOAA/ERL. From 2001 to the present he has been Senior Scientist at Zel Technology & NOAA/ETL. He has published more than 130 titles in wave propagation in random and turbulent media, theory of turbulence, quantum and statistical optics, and mathematics. He published the following books: Wave Propagation in a Turbulent Media (1959 in Russian, 1961 and 1967 in English), The Effect of the Turbulent Atmosphere on Wave Propagation (1965 in Russian, 1971 revised edition in English), and, in coauthorship with S. M. Rytov and Yu. A. Kravtsov, Principles of Statistical Radiophysics (1978 in Russian, 1989 in English).