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ELECTROMAGNETIC WAVE EQUATION IN DIFFERENTIAL-FORM REPRESENTATION

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Abstract—Differential-form formalism has been often applied, in stead of the more commonplace Gibbsian vector calculus, to express the basic electromagnetic equations in simple and elegant form. However, representing higher-order equations has met with unexpected difficulties, in particular, when dealing with general linear electromagnetic media. In the present study, wave equations involving scalar operators of the fourth order are derived for the electromagnetic two-form and the potential one-form, for the general linear bi-anisotropic medium. This generalizes previous coordinate-free approaches valid for certain special classes of media. The analysis is based on some multivector and dyadic identities derived in the Appendix.

1. INTRODUCTION

Differential-form formalism appears to be an ideal tool for electromagnetic analysis because it allows one to express the Maxwell equations in a most compact and elegant form [1–8] as

$$\boldsymbol{d}\wedge\boldsymbol{\Phi}=0,\quad \boldsymbol{d}\wedge\boldsymbol{\Psi}=\boldsymbol{\gamma}. \tag{1}$$

Here, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are two electromagnetic two-forms[†], elements of the space of two-forms, $\mathbb{F}_2$

$$\boldsymbol{\Phi}=\boldsymbol{B}+\boldsymbol{E}\wedge\boldsymbol{d}\tau,\qquad \boldsymbol{\Psi}=\boldsymbol{D}-\boldsymbol{H}\wedge\boldsymbol{d}\tau, \tag{2}$$

and $\boldsymbol{\gamma}\in\mathbb{F}_3$ is the electric source three-form

$$\boldsymbol{\gamma}=\varrho-\boldsymbol{J}\wedge\boldsymbol{d}\tau. \tag{3}$$

All these field and source differential forms are functions of the four-dimensional vector variable $\boldsymbol{x}$ or three-dimensional spatial vector variable $\boldsymbol{r}$ and the normalized temporal variable $\tau=ct$. Connection between the basic electromagnetic field two-forms in a homogeneous and time-invariant linear medium is given by a simple algebraic equation in terms of a medium dyadic $\overline{\overline{\mathsf{M}}}$,

$$\boldsymbol{\Psi}=\overline{\overline{\mathsf{M}}}|\boldsymbol{\Phi}, \tag{4}$$

which involves 36 scalar parameters, in general. Magnetic charges and currents, often used as equivalent sources, are neglected here for simplicity. Here we assume that $\overline{\overline{\mathsf{M}}}$ is a constant dyadic.

To solve the fields from a given source of electric charges and currents in a general linear (bi-anisotropic) medium, one can simplify the problem in terms of a potential one-form $\boldsymbol{\alpha}$ by expressing

$$\boldsymbol{\Phi}=\boldsymbol{d}\wedge\boldsymbol{\alpha}. \tag{5}$$

In fact, the potential then satisfies the equation

$$\boldsymbol{d}\wedge\overline{\overline{\mathsf{M}}}|(\boldsymbol{d}\wedge\boldsymbol{\alpha})=(\boldsymbol{d}\wedge\overline{\overline{\mathsf{M}}}\lfloor\boldsymbol{d})|\boldsymbol{\alpha}=\boldsymbol{\gamma}. \tag{6}$$

This is a three-form equation, i.e., the quantity on both sides is a three-form while the unknown quantity is a one-form. Because three-forms have the same dimension 4 as one-forms, the number of scalar

† Notation of [8] is used throughout in this paper. For a brief introduction to the notation and basic algebraic rules, see the Appendix of [9].

equations in (6) equals the number of scalar unknowns. (6) can be transformed to a vector equation in the form

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{d})|\boldsymbol{\alpha} = \boldsymbol{g}, \tag{7}$$

when operating by a quadrivector $\boldsymbol{e}_N \in \mathbb{E}_4$ and applying the rule [8]

$$\boldsymbol{e}_N\lfloor(\boldsymbol{d}\wedge\overline{\overline{\mathsf{M}}})\lfloor\boldsymbol{d} = \boldsymbol{d}\rfloor(\boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}})\lfloor\boldsymbol{d} = \boldsymbol{d}\rfloor\overline{\overline{\mathsf{M}}}_g\lfloor\boldsymbol{d}. \tag{8}$$

The modified medium dyadic $\overline{\overline{\mathsf{M}}}_g = \boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}}$ is a metric dyadic mapping two-forms to bivectors. In (7) the dyadic differential operator $\overline{\overline{\mathsf{D}}}(\boldsymbol{d})$ is defined by

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{d}) = \overline{\overline{\mathsf{M}}}_g\lfloor\lfloor\boldsymbol{dd} = -\boldsymbol{d}\rfloor\overline{\overline{\mathsf{M}}}_g\lfloor\boldsymbol{d} = \boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_g, \tag{9}$$

and

$$\boldsymbol{g} = \boldsymbol{e}_N\lfloor\boldsymbol{\gamma} \tag{10}$$

is a vector source corresponding to the original source three-form. It was shown in [9] that the same second-order dyadic operator $\overline{\overline{\mathsf{D}}}(\boldsymbol{d})$ also appears in an equation for the electromagnetic two-form $\boldsymbol{\Phi}$ when both electric and magnetic sources are present.

In the three-dimensional Gibbsian vector analysis the vector Helmholtz equation for a time-harmonic vector field $\boldsymbol{f}(\boldsymbol{r})$ can be written as [10]

$$\overline{\overline{\mathsf{L}}}(\nabla)\cdot\boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{s}(\boldsymbol{r}), \tag{11}$$

where $\overline{\overline{\mathsf{L}}}(\nabla)$ is the Helmholtz dyadic operator, second order in ∇, and $\boldsymbol{s}(\boldsymbol{r})$ is some vector source function. The formal solution of (11) can be written as

$$\boldsymbol{f}(\boldsymbol{r}) = \overline{\overline{\mathsf{L}}}^{-1}(\nabla)\cdot\boldsymbol{s}(\boldsymbol{r}). \tag{12}$$

The inverse dyadic operator can be expanded in the general form

$$\overline{\overline{\mathsf{L}}}^{-1}(\nabla) = \frac{\overline{\overline{\mathsf{L}}}_a(\nabla)}{\mathrm{det}\overline{\overline{\mathsf{L}}}(\nabla)}, \tag{13}$$

where the dyadic adjoint and scalar determinant operators can be expressed in terms of Gibbsian dyadic operations [11, 10] as

$$\overline{\overline{\mathsf{L}}}_a(\nabla) = \overline{\overline{\mathsf{L}}}^{(2)T}(\nabla) = \frac{1}{2}\overline{\overline{\mathsf{L}}}^T(\nabla)\,{}^\times_\times\,\overline{\overline{\mathsf{L}}}^T(\nabla), \tag{14}$$

$$\mathrm{det}\overline{\overline{\mathsf{L}}}(\nabla) = \frac{1}{6}\overline{\overline{\mathsf{L}}}(\nabla)\,{}^\times_\times\,\overline{\overline{\mathsf{L}}}(\nabla) : \overline{\overline{\mathsf{L}}}(\nabla). \tag{15}$$

(13) can also be written as the operator rule

$$\overline{\overline{\mathsf{L}}}_a(\nabla)\cdot\overline{\overline{\mathsf{L}}}(\nabla) = L(\nabla)\overline{\overline{\mathsf{I}}}, \qquad L(\nabla) = \det\overline{\overline{\mathsf{L}}}(\nabla). \tag{16}$$

Thus, the expansion (13) can be used to replace the equation (11) involving a dyadic operator by one with a scalar operator

$$L(\nabla)\ \boldsymbol{f}(\boldsymbol{r}) = \overline{\overline{\mathsf{L}}}_a(\nabla)\cdot\boldsymbol{s}(\boldsymbol{r}). \tag{17}$$

The determinant operator $L(\nabla) = \det\overline{\overline{\mathsf{L}}}(\nabla)$ is in general of the fourth order in ∇.

It is the purpose of the present paper to transform the wave equation (7) to a form similar to that in (17), involving a scalar operator, i.e., to find the operators corresponding to $L(\nabla)$ and $\overline{\overline{\mathsf{L}}}_a(\nabla)$. The problem was previously addressed in [9] and [8] where the scalar-operator equation was formulated for some special classes of media. Here we consider the most general linear bi-anisotropic medium. The analysis requires some multivector and dyadic identities discussed in the Appendix.

2. OPERATOR EXPANSIONS

If we wish to express the solution of (7) formally as

$$\boldsymbol{\alpha} = \overline{\overline{\mathsf{D}}}^{-1}(\boldsymbol{d})|\boldsymbol{g}, \tag{18}$$

a problem arises because a dyadic $\overline{\overline{\mathsf{D}}}(\boldsymbol{d})$ of the form (9) does not have a regular algebraic inverse. In fact, because we have [8]

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{d})|\boldsymbol{d} = -(\boldsymbol{d}\rfloor\overline{\overline{\mathsf{M}}}_g\lfloor\boldsymbol{d})|\boldsymbol{d} = -\boldsymbol{d}\rfloor\overline{\overline{\mathsf{M}}}_g|(\boldsymbol{d}\wedge\boldsymbol{d}) = 0, \tag{19}$$

$\overline{\overline{\mathsf{D}}}(\boldsymbol{d})$ maps any multiple of $\boldsymbol{d}$ to the zero vector and, thus, fails to have an inverse. Similarly, we also have $\boldsymbol{d}|\overline{\overline{\mathsf{D}}}(\boldsymbol{d}) = 0$. Differentiating both sides of the vector equation (7) by $\boldsymbol{d}|$ reduces it to the form $0 = 0$ because of

$$\boldsymbol{d}|\boldsymbol{g} = \boldsymbol{g}|\boldsymbol{d} = -(\boldsymbol{e}_N\lfloor\boldsymbol{\gamma})|\boldsymbol{d} = -\boldsymbol{e}_N|(\boldsymbol{\gamma}\wedge\boldsymbol{d}) = \boldsymbol{e}_N|(\boldsymbol{d}\wedge\boldsymbol{\gamma}), \tag{20}$$

and the charge conservation law $\boldsymbol{d}\wedge\boldsymbol{\gamma} = 0$.

In Appendix B we have derived some algebraic rules for expanding double-wedge powers of dyadics of the form $\boldsymbol{\beta}\boldsymbol{\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}}$, where $\boldsymbol{\beta}\in\mathbb{F}_1$ is a one-form and $\overline{\overline{\mathsf{B}}}\in\mathbb{E}_2\mathbb{E}_2$ is a metric dyadic mapping two-forms to bivectors. Let us now apply these rules to form the double-wedge powers of the dyadic operator $\overline{\overline{\mathsf{D}}}(\boldsymbol{d}) = \boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_g$. From (A13) we first have

$$\overline{\overline{\mathsf{D}}}^{(2)}(\boldsymbol{d}) = \frac{1}{2}\overline{\overline{\mathsf{D}}}(\boldsymbol{d})^\wedge_\wedge\overline{\overline{\mathsf{D}}}(\boldsymbol{d}) = \frac{1}{2}\boldsymbol{dd}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}(\boldsymbol{d})) = \frac{1}{2}\boldsymbol{dd}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_g)), \tag{21}$$

while (A20) gives us the expansion

$$\overline{\overline{\mathsf{D}}}^{(3)}(\boldsymbol{d}) = \frac{1}{3}\boldsymbol{dd}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}^{(2)}(\boldsymbol{d})) = \frac{1}{6}\boldsymbol{dd}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_g)))). \tag{22}$$

Finally, we have $\overline{\overline{\mathsf{D}}}^{(4)}(\boldsymbol{d}) = 0$, which is equivalent to the nonexistence of a regular inverse $\overline{\overline{\mathsf{D}}}^{-1}(\boldsymbol{d})$.

Because the dyadic $\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}^{(2)}(\boldsymbol{d})$ belongs to the space $\mathbb{E}_4\mathbb{E}_4$, it must be a multiple of the dyadic $\boldsymbol{e}_N\boldsymbol{e}_N$ and (22) has the form

$$\overline{\overline{\mathsf{D}}}^{(3)}(\boldsymbol{d}) = (\boldsymbol{dd}\rfloor\rfloor\boldsymbol{e}_N\boldsymbol{e}_N)D(\boldsymbol{d}), \tag{23}$$

where $D(\boldsymbol{d})$ is the scalar operator

$$\begin{aligned} D(\boldsymbol{d}) &= \frac{1}{3}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}^{(2)}(\boldsymbol{d})) \\ &= \frac{1}{6}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_g)))). \end{aligned} \tag{24}$$

Obviously, $D(\boldsymbol{d})$ is an operator of the fourth order in $\boldsymbol{d}$.

To proceed, let us invoke the identity (A35) from Appendix B. Replacing again $\boldsymbol{\beta}$ by $\boldsymbol{d}$ and $\overline{\overline{\mathsf{B}}}$ by $\overline{\overline{\mathsf{M}}}_g$ we have

$$\begin{aligned} &\frac{1}{2}\boldsymbol{d}\wedge(\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{M}}}{}_g^T{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}{}_g^T)))|(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_g) \\ &\quad = \frac{1}{3}(\boldsymbol{d}\wedge\overline{\mathsf{I}}^T)\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_g)^{(2)}). \end{aligned} \tag{25}$$

This can be interpreted in terms of the operator $\overline{\overline{\mathsf{D}}}(\boldsymbol{d})$ of (9) as an operator identity of the form

$$\boldsymbol{d}\wedge\overline{\overline{\mathsf{D}}}_a(\boldsymbol{d})|\overline{\overline{\mathsf{D}}}(\boldsymbol{d}) = (\boldsymbol{d}\wedge\overline{\mathsf{I}}^T)D(\boldsymbol{d}), \tag{26}$$

with the adjoint dyadic operator defined by

$$\overline{\overline{\mathsf{D}}}_a(\boldsymbol{d}) = \frac{1}{2}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{M}}}{}_g^T{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}^T(\boldsymbol{d})) = \frac{1}{2}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{M}}}{}_g^T{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{M}}}{}_g^T)). \tag{27}$$

(26) resembles the Gibbsian operator rule (16) and it plays the central role in constructing the wave equation.

3. FIELD AND POTENTIAL EQUATIONS

Applying now the operators on both sides of (26) on the potential one-form $\boldsymbol{\alpha}$ and invoking (7), we obtain the equation

$$D(\boldsymbol{d})(\boldsymbol{d} \wedge \boldsymbol{\alpha}) = \boldsymbol{d} \wedge \overline{\overline{\mathsf{D}}}_a(\boldsymbol{d})|\boldsymbol{g}. \tag{28}$$

This is actually a wave equation for the electromagnetic two-form $\boldsymbol{\Phi}$:

$$D(\boldsymbol{d})\boldsymbol{\Phi} = \boldsymbol{d} \wedge \overline{\overline{\mathsf{D}}}_a(\boldsymbol{d})|\boldsymbol{g}. \tag{29}$$

Canceling the $\boldsymbol{d}\wedge$ operator from both sides of (28) we obtain a corresponding equation for the potential one-form as

$$D(\boldsymbol{d})\boldsymbol{\alpha} = \overline{\overline{\mathsf{D}}}_a(\boldsymbol{d})|\boldsymbol{g}. \tag{30}$$

Canceling operators can be justified by the fact that the potential is not a unique quantity. In fact, if $\boldsymbol{\alpha}$ is a solution for (30), $\boldsymbol{\Phi} = \boldsymbol{d} \wedge \boldsymbol{\alpha}$ is a solution of (29). Somewhere hidden lies a condition for the potential in the bi-anisotropic medium defined by the medium dyadic $\overline{\overline{\mathsf{M}}}_g$. For restricted classes of bi-anisotropic media called the Q-media and generalized Q-media the corresponding condition could be explicitly given and it was labeled as the (generalized) Lorenz condition [8].

Equations (29) and (30) with the operators $D(\boldsymbol{d})$ and $\overline{\overline{\mathsf{D}}}_a(\boldsymbol{d})$ defined by (24) and (27) form the final result of this study. They have the required form similar to that of (17) in expressing the wave equation (7) in terms of a scalar operator $D(\boldsymbol{d})$ of the fourth order.

4. Q-MEDIUM

To check the resulting equations let us consider a special medium whose medium dyadic can be expressed in terms of a dyadic $\overline{\overline{\mathsf{Q}}} \in \mathbb{E}_1\mathbb{E}_1$ as

$$\overline{\overline{\mathsf{M}}}_g = \overline{\overline{\mathsf{Q}}}^{(2)} = \frac{1}{2}\overline{\overline{\mathsf{Q}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{Q}}}. \tag{31}$$

Such a medium was labeled as 'Q-medium' in [9, 8] and the field two-form from electric sources in such a medium was shown to obey the wave equation

$$(\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\boldsymbol{\Phi} = \boldsymbol{d} \wedge \overline{\overline{\mathsf{Q}}}^{-1}|\boldsymbol{g}. \tag{32}$$

This corresponds to the potential equation

$$(\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\boldsymbol{\alpha} = \overline{\overline{\mathsf{Q}}}^{-1}|\boldsymbol{g}. \tag{33}$$

Let us check whether (30) reduces to (33) for the chosen medium (31). Applying the identity

$$\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{Q}}}^{(2)} = (\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\overline{\overline{\mathsf{Q}}} - (\overline{\overline{\mathsf{Q}}}|\boldsymbol{d})(\boldsymbol{d}|\overline{\overline{\mathsf{Q}}}) \tag{34}$$

on (24), we obtain

$$\begin{aligned} D(\boldsymbol{d}) &= \frac{1}{3}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{Q}}}^{(2)}{}^\wedge_\wedge(\boldsymbol{dd}\rfloor\rfloor\overline{\overline{\mathsf{Q}}}^{(2)})^{(2)}) \\ &= \frac{1}{12}(\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{Q}}}{}^\wedge_\wedge\overline{\overline{\mathsf{Q}}}{}^\wedge_\wedge((\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\overline{\overline{\mathsf{Q}}}{}^\wedge_\wedge\overline{\overline{\mathsf{Q}}} - 2\overline{\overline{\mathsf{Q}}}{}^\wedge_\wedge(\overline{\overline{\mathsf{Q}}}|\boldsymbol{dd}|\overline{\overline{\mathsf{Q}}}))) \\ &= (\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(2\overline{\overline{\mathsf{Q}}}^{(4)}(\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd}) - \overline{\overline{\mathsf{Q}}}^{(3)}{}^\wedge_\wedge(\overline{\overline{\mathsf{Q}}}|\boldsymbol{dd}|\overline{\overline{\mathsf{Q}}})). \end{aligned} \tag{35}$$

At this point we apply the inverse formula for a dyadic $\overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{E}_1$ [8]

$$\overline{\overline{\mathsf{A}}}^{-1} = \frac{\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(3)T}}{\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||\overline{\overline{\mathsf{A}}}^{(4)}}, \tag{36}$$

whence the scalar operator is simplified to

$$\begin{aligned} D(\boldsymbol{d}) &= (\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})(\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||\overline{\overline{\mathsf{Q}}}^{(4)})(2\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd} - \overline{\overline{\mathsf{Q}}}^{-1T}||(\overline{\overline{\mathsf{Q}}}|\boldsymbol{dd}|\overline{\overline{\mathsf{Q}}}))) \\ &= (\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})^2(\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||\overline{\overline{\mathsf{Q}}}^{(4)}). \end{aligned} \tag{37}$$

Applying the identity (A39) in the form

$$\overline{\overline{\mathsf{Q}}}^{(4)}\lfloor\lfloor\boldsymbol{dd} = (\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\overline{\overline{\mathsf{Q}}}^{(3)} - \overline{\overline{\mathsf{Q}}}^{(2)}{}^\wedge_\wedge(\overline{\overline{\mathsf{Q}}}|\boldsymbol{dd}|\overline{\overline{\mathsf{Q}}}) = (\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||\overline{\overline{\mathsf{Q}}}^{(4)})\boldsymbol{e}_N\boldsymbol{e}_N\lfloor\lfloor\boldsymbol{dd}, \tag{38}$$

the adjoint dyadic operator (27) transposed can be expanded as

$$\begin{aligned} \overline{\overline{\mathsf{D}}}_a^T(\boldsymbol{d}) &= \frac{1}{2}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{M}}}_g{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}(\boldsymbol{d})) \\ &= \frac{1}{2}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{Q}}}^{(2)}{}^\wedge_\wedge((\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd})\overline{\overline{\mathsf{Q}}} - \overline{\overline{\mathsf{Q}}}|\boldsymbol{dd}|\overline{\overline{\mathsf{Q}}})) \\ &= \frac{1}{2}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(3\overline{\overline{\mathsf{Q}}}^{(3)}(\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd}) - \overline{\overline{\mathsf{Q}}}^{(2)}{}^\wedge_\wedge(\overline{\overline{\mathsf{Q}}}|\boldsymbol{dd}|\overline{\overline{\mathsf{Q}}})) \\ &= \frac{1}{2}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(2\overline{\overline{\mathsf{Q}}}^{(3)}(\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd}) + \overline{\overline{\mathsf{Q}}}^{(4)}\lfloor\lfloor\boldsymbol{dd}) \\ &= (\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||\overline{\overline{\mathsf{Q}}}^{(4)})(\overline{\overline{\mathsf{Q}}}^{-1T}(\overline{\overline{\mathsf{Q}}}||\boldsymbol{dd}) + \frac{1}{2}\boldsymbol{dd}). \end{aligned} \tag{39}$$

Because of the property $\boldsymbol{d}|\boldsymbol{g} = 0$, we can see that, inserting (37) and (39) transposed in (30) reproduces (33) after cancelation of operators.

5. CONCLUSION

In the present study we have derived the wave equation for the electromagnetic two-form or the potential one-form in terms of a scalar operator of the fourth order. The method was based on a set of multivector and dyadic identities. The resulting equation was verified through a special medium case for which the wave equation has been previously derived.

APPENDIX A. DYADIC IDENTITIES

In this Appendix, some multivector and dyadic identities needed in the preceding analysis are briefly given. Some of them are reproduced from [8] while others have been derived here.

A.1. Bac cab rules

The well-known 'bac cab rule' in Gibbsian vector algebra

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b}) \tag{A1}$$

takes various forms in multivector and dyadic algebra. The basic rule between a dual vector $\boldsymbol{\alpha}$ and two vectors $\boldsymbol{b},\boldsymbol{c}$ is [8]

$$\boldsymbol{\alpha}\rfloor(\boldsymbol{b}\wedge\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{\alpha}|\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{\alpha}|\boldsymbol{b}) \tag{A2}$$

and its dual has the form

$$\boldsymbol{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = \boldsymbol{\beta}(\boldsymbol{a}|\boldsymbol{\gamma}) - \boldsymbol{\gamma}(\boldsymbol{a}|\boldsymbol{\beta}). \tag{A3}$$

Denoting the grade of a multivector or dual multivector by a superscript, some generalizations of the bac can rule are

$$\boldsymbol{\alpha}\rfloor(\boldsymbol{b}\wedge\boldsymbol{c}^2) \;=\; \boldsymbol{b}\wedge(\boldsymbol{\alpha}\rfloor\boldsymbol{c}^2) + \boldsymbol{c}^2(\boldsymbol{\alpha}|\boldsymbol{b}), \tag{A4}$$

$$\boldsymbol{\alpha}\rfloor(\boldsymbol{b}\wedge\boldsymbol{c}^3) \;=\; \boldsymbol{b}\wedge(\boldsymbol{\alpha}\rfloor\boldsymbol{c}^3) - \boldsymbol{c}^3(\boldsymbol{\alpha}|\boldsymbol{b}), \tag{A5}$$

$$\boldsymbol{\alpha}\rfloor(\boldsymbol{b}^2\wedge\boldsymbol{c}^2) \;=\; \boldsymbol{b}^2\wedge(\boldsymbol{\alpha}\rfloor\boldsymbol{c}^2) + \boldsymbol{c}^2\wedge(\boldsymbol{\alpha}\rfloor\boldsymbol{b}^2), \tag{A6}$$

$$\boldsymbol{\alpha}^3\rfloor(\boldsymbol{b}\wedge\boldsymbol{c}^3) \;=\; \boldsymbol{b}(\boldsymbol{\alpha}^3|\boldsymbol{c}^3) - \boldsymbol{c}^3\lfloor(\boldsymbol{\alpha}^3\lfloor\boldsymbol{b}) \tag{A7}$$

All these rules are special cases of the 'mother of bac cab rules' [8]

$$\boldsymbol{\alpha}\rfloor(\boldsymbol{b}^p\wedge\boldsymbol{c}^q) = \boldsymbol{b}^p\wedge(\boldsymbol{\alpha}\rfloor\boldsymbol{c}^q) + (-1)^{pq}\boldsymbol{c}^q\wedge(\boldsymbol{\alpha}\rfloor\boldsymbol{b}^p), \tag{A8}$$

and they can be proven through basis expansions of all quantities.

A.2. First Identity

In order to avoid possible conflicts with the potential one-form, let us replace $\boldsymbol{\alpha}$ by $\boldsymbol{\beta}$ and apply the bac cab rule (A4) written as

$$\boldsymbol{\beta}\rfloor(\boldsymbol{b}\wedge\boldsymbol{C}) = \boldsymbol{b}\wedge(\boldsymbol{\beta}\rfloor\boldsymbol{C}) + \boldsymbol{C}(\boldsymbol{\beta}|\boldsymbol{b}), \tag{A9}$$

where $\boldsymbol{\beta}$ is a dual vector, $\boldsymbol{b}$ a vector and $\boldsymbol{C}$ a bivector. Assuming $\boldsymbol{b} = \boldsymbol{\beta}\rfloor\boldsymbol{B}$ where $\boldsymbol{B}$ is another bivector and noting that $\boldsymbol{\beta}|\boldsymbol{b} = 0$, we can write (A9) as

$$\boldsymbol{\beta}\rfloor((\boldsymbol{\beta}\rfloor\boldsymbol{B})\wedge\boldsymbol{C}) = (\boldsymbol{\beta}\rfloor\boldsymbol{B})\wedge(\boldsymbol{\beta}\rfloor\boldsymbol{C}). \tag{A10}$$

A dyadic product of two such identities gives

$$\boldsymbol{\beta\beta}\rfloor\rfloor((\boldsymbol{\beta\beta}\rfloor\rfloor\boldsymbol{B}_1\boldsymbol{B}_2)^{\wedge}_{\wedge}\boldsymbol{C}_1\boldsymbol{C}_2) = (\boldsymbol{\beta\beta}\rfloor\rfloor\boldsymbol{B}_1\boldsymbol{B}_2)^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\boldsymbol{C}_1\boldsymbol{C}_2), \tag{A11}$$

where the bivectors $\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C}_1, \boldsymbol{C}_2$ are arbitrary. Because this identity is linear in the dyads $\boldsymbol{B}_1\boldsymbol{B}_2$ and $\boldsymbol{C}_1\boldsymbol{C}_2$ we can replace these by the respective arbitrary dyadics $\overline{\overline{\mathsf{B}}}$ and $\overline{\overline{\mathsf{C}}}$, elements of the space $\mathbb{E}_2\mathbb{E}_2$, whence the identity takes the more general form

$$\boldsymbol{\beta\beta}\rfloor\rfloor((\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{\wedge}_{\wedge}\overline{\overline{\mathsf{C}}}) = (\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{C}}}). \tag{A12}$$

As a special case $\overline{\overline{\mathsf{C}}} = \overline{\overline{\mathsf{B}}}$ we arrive at the expansion identity needed in the analysis

$$(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)} = \frac{1}{2}\boldsymbol{\beta\beta}\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})). \tag{A13}$$

This shows us that the following orthogonality is valid:

$$\boldsymbol{\beta}\rfloor(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)} = (\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)}\lfloor\boldsymbol{\beta} = 0. \tag{A14}$$

A.3. Second Identity

Another useful identity can be derived from the bac cab rule (A6)

$$\boldsymbol{\beta}\rfloor(\boldsymbol{B}\wedge\boldsymbol{C}) = \boldsymbol{B}\wedge(\boldsymbol{\beta}\rfloor\boldsymbol{C}) + \boldsymbol{C}\wedge(\boldsymbol{\beta}\rfloor\boldsymbol{B}), \tag{A15}$$

valid for any dual vector $\boldsymbol{\beta}$ and two bivectors $\boldsymbol{B}, \boldsymbol{C}$. Assuming that $\boldsymbol{C} = \boldsymbol{\beta}\rfloor\boldsymbol{k}$, where $\boldsymbol{k} \in \mathbb{E}_3$ is a trivector, we have $\boldsymbol{\beta}\rfloor\boldsymbol{C} = 0$ and the identity reduces to

$$\boldsymbol{\beta}\rfloor(\boldsymbol{B}\wedge(\boldsymbol{\beta}\rfloor\boldsymbol{k})) = (\boldsymbol{\beta}\rfloor\boldsymbol{k})\wedge(\boldsymbol{\beta}\rfloor\boldsymbol{B}). \tag{A16}$$

Again, this gives rise to the dyadic identity

$$\boldsymbol{\beta\beta}\rfloor\rfloor(\boldsymbol{B}_1\boldsymbol{B}_2{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\boldsymbol{k}_1\boldsymbol{k}_2)) = (\boldsymbol{\beta\beta}\rfloor\rfloor\boldsymbol{k}_1\boldsymbol{k}_2)^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\boldsymbol{B}_1\boldsymbol{B}_2), \tag{A17}$$

and, more generally, to

$$\boldsymbol{\beta\beta}\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{K}}})) = (\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{K}}}){}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}}), \tag{A18}$$

where $\overline{\overline{\mathsf{K}}} \in \mathbb{E}_3\mathbb{E}_3$ is a dyadic mapping three-forms to trivectors and $\overline{\overline{\mathsf{B}}} \in \mathbb{E}_2\mathbb{E}_2$ is as in (A13). Assuming that, in particular, $\overline{\overline{\mathsf{K}}} = \overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})$, whence from (A13) we have $\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{K}}} = 2(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)}$, (A18) takes on the form

$$\boldsymbol{\beta\beta}\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)}) = (\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}}) = 3(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(3)}. \tag{A19}$$

This gives rise to the expansion rule

$$\begin{aligned}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(3)} &= \frac{1}{3}\boldsymbol{\beta\beta}\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)})\\ &= \frac{1}{6}\boldsymbol{\beta\beta}\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})))).\end{aligned} \tag{A20}$$

This shows us that the following orthogonality is valid:

$$\boldsymbol{\beta}\rfloor(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(3)} = (\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(3)}\lfloor\boldsymbol{\beta} = 0. \tag{A21}$$

As the next step, one can show that the fourth double-wedge power vanishes: $(\boldsymbol{\beta\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(4)} = 0$.

A.4. Third Identity

Let us start from a known identity valid for dyadics $\overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{E}_1$ in three-dimensional vector space spanned by the vector basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ with the corresponding reciprocal dual basis $\{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_2\}$ [8],

$$(\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(2)T})|\overline{\overline{\mathsf{A}}} = (\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\overline{\overline{\mathsf{A}}}^{(3)})(\overline{\overline{\mathsf{I}}} - \boldsymbol{e}_4\boldsymbol{\varepsilon}_4)^T. \tag{A22}$$

Let us now extend the space to four dimensions through a basis vector $\boldsymbol{e}_4$ and its reciprocal dual vector $\boldsymbol{\varepsilon}_4$. We can choose the dyadic in (A22) as $\overline{\overline{\mathsf{A}}} = \boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}}$, where $\overline{\overline{\mathsf{B}}} \in \mathbb{E}_2\mathbb{E}_2$ is not restricted to three dimensions. Thus, (A22) can be written as

$$\begin{aligned}&(\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}\lfloor\lfloor(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)T})|(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})\\ &= (\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(3)})(\overline{\overline{\mathsf{I}}} - \boldsymbol{e}_4\boldsymbol{\varepsilon}_4)^T.\end{aligned} \tag{A23}$$

Let us now apply the identities (A13) and (A21) with $\boldsymbol{\beta}$ replaced by $\boldsymbol{\varepsilon}_4$ to expand

$$\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}\lfloor\lfloor(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)} = \frac{1}{2}\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}\lfloor\lfloor(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}}))), \tag{A24}$$

$$\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(3)} = \frac{1}{3}\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)})). \quad \text{(A25)}$$

Because every element of the dyadic space $\mathbb{E}_4\mathbb{E}_4$ is a multiple of $\boldsymbol{e}_N\boldsymbol{e}_N$, we can write

$$\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)} = B\boldsymbol{e}_N\boldsymbol{e}_N, \quad \text{(A26)}$$

where the scalar B is defined by

$$B = \boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)}). \quad \text{(A27)}$$

Inserting these and

$$\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\boldsymbol{e}_N\boldsymbol{e}_N) = \boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\boldsymbol{e}_{123}\boldsymbol{e}_{123} = 1, \quad \text{(A28)}$$

$$(\overline{\overline{\mathsf{I}}} - \boldsymbol{e}_4\boldsymbol{\varepsilon}_4)^T = -\boldsymbol{e}_4\rfloor(\boldsymbol{\varepsilon}_4\wedge\overline{\overline{\mathsf{I}}}^T) \quad \text{(A29)}$$

in (A23) gives us

$$\frac{1}{2}(\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}\lfloor\lfloor(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}}))^T))|(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}}) = -\frac{B}{3}\boldsymbol{e}_4\rfloor(\boldsymbol{\varepsilon}_4\wedge\overline{\overline{\mathsf{I}}}^T). \quad \text{(A30)}$$

Now we need the following expansion for the general three-vector $\boldsymbol{k}$:

$$\boldsymbol{\varepsilon}_N\lfloor\boldsymbol{k} = \boldsymbol{\varepsilon}_4(\boldsymbol{\varepsilon}_{123}|\boldsymbol{k}) - \boldsymbol{\varepsilon}_{123}\lfloor(\boldsymbol{\varepsilon}_4\rfloor\boldsymbol{k}), \quad \text{(A31)}$$

which can be derived through basis expansions. Expressing this as

$$\boldsymbol{\varepsilon}_{123}\lfloor(\boldsymbol{\varepsilon}_4\rfloor\boldsymbol{k}) = (\overline{\overline{\mathsf{I}}} - \boldsymbol{e}_4\boldsymbol{\varepsilon}_4)^T|(\boldsymbol{\varepsilon}_N\lfloor\boldsymbol{k}), \quad \text{(A32)}$$

we can write (A30) in the form

$$\frac{1}{2}(\overline{\overline{\mathsf{I}}} - \boldsymbol{e}_4\boldsymbol{\varepsilon}_4)^T|(\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}}))^T))|(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}}) = -\frac{B}{3}\boldsymbol{e}_4\rfloor(\boldsymbol{\varepsilon}_4\wedge\overline{\overline{\mathsf{I}}}^T), \quad \text{(A33)}$$

or, multiplying by $\boldsymbol{\varepsilon}_4\wedge$ in the equivalent form

$$\begin{aligned}&\boldsymbol{\varepsilon}_4\wedge(\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}}))^T)|(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})\\ &\quad = \frac{2}{3}(\boldsymbol{\varepsilon}_4\wedge\overline{\overline{\mathsf{I}}}^T)\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)}). \end{aligned} \quad \text{(A34)}$$

Since $\boldsymbol{\varepsilon}_4$ was not specified in any way and since (A34) is homogeneous in $\boldsymbol{\varepsilon}_4$ (of the fifth degree), $\boldsymbol{\varepsilon}_4$ can be replaced by any dual vector $\boldsymbol{\beta}$. Also, $\boldsymbol{\varepsilon}_N$ could be replaced by any dual quadrivector κ. However, we retain $\boldsymbol{\varepsilon}_N$ and the final form of the third identity now becomes

$$\begin{aligned}&\boldsymbol{\beta}\wedge(\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N\lfloor\lfloor(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\beta}\boldsymbol{\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}}))^T)|(\boldsymbol{\beta}\boldsymbol{\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})\\ &\quad = \frac{2}{3}(\boldsymbol{\beta}\wedge\overline{\overline{\mathsf{I}}}^T)\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge(\boldsymbol{\beta}\boldsymbol{\beta}\rfloor\rfloor\overline{\overline{\mathsf{B}}})^{(2)}). \end{aligned} \quad \text{(A35)}$$

A.5. Fourth Identity

As a last algebraic rule we wish to expand the dyadic $\boldsymbol{\beta\gamma}\rfloor\rfloor\overline{\overline{\mathsf{A}}}^{(4)}$ where the dyadic $\overline{\overline{\mathsf{A}}}\in\mathbb{E}_1\mathbb{E}_1$ maps one-forms to vectors and $\boldsymbol{\beta},\boldsymbol{\gamma}$ are one-forms. Starting from the vector expansion [8]

$$\begin{aligned}\boldsymbol{\beta}\rfloor(\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\boldsymbol{a}_3\wedge\boldsymbol{a}_4) \;=\; & -\boldsymbol{\beta}|\boldsymbol{a}_1(\boldsymbol{a}_2\wedge\boldsymbol{a}_3\wedge\boldsymbol{a}_4)+\boldsymbol{\beta}|\boldsymbol{a}_2(\boldsymbol{a}_3\wedge\boldsymbol{a}_4\wedge\boldsymbol{a}_1)\\ & -\boldsymbol{\beta}|\boldsymbol{a}_3(\boldsymbol{a}_4\wedge\boldsymbol{a}_1\wedge\boldsymbol{a}_2)+\boldsymbol{\beta}|\boldsymbol{a}_4(\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\boldsymbol{a}_3),\end{aligned}\tag{A36}$$

substituting $\overline{\overline{\mathsf{A}}}=\sum\boldsymbol{a}_i\boldsymbol{b}_i$ and defining for brevity

$$\overline{\overline{\mathsf{A}}}_i=\boldsymbol{a}_i\boldsymbol{b}_i,\quad \overline{\overline{\mathsf{A}}}_{ij}=(\boldsymbol{a}_i\wedge\boldsymbol{a}_j)(\boldsymbol{b}_i\wedge\boldsymbol{b}_j),\quad \overline{\overline{\mathsf{A}}}_{ijk}=(\boldsymbol{a}_i\wedge\boldsymbol{a}_j\wedge\boldsymbol{a}_k)(\boldsymbol{b}_i\wedge\boldsymbol{b}_j\wedge\boldsymbol{b}_k),\tag{A37}$$

we can expand

$$\begin{aligned}\boldsymbol{\beta\gamma}\rfloor\rfloor\overline{\overline{\mathsf{A}}}^{(4)} \;=\; & \boldsymbol{\beta\gamma}\rfloor\rfloor(\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\boldsymbol{a}_3\wedge\boldsymbol{a}_4)(\boldsymbol{b}_1\wedge\boldsymbol{b}_2\wedge\boldsymbol{b}_3\wedge\boldsymbol{b}_4)\\ =\; & -(\overline{\overline{\mathsf{A}}}_1|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_2+\overline{\overline{\mathsf{A}}}_2|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_1)^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_{34}\\ & -(\overline{\overline{\mathsf{A}}}_1|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_3+\overline{\overline{\mathsf{A}}}_3|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_1)^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_{24}\\ & -(\overline{\overline{\mathsf{A}}}_1|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_4+\overline{\overline{\mathsf{A}}}_4|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_1)^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_{23}\\ & -(\overline{\overline{\mathsf{A}}}_2|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_3+\overline{\overline{\mathsf{A}}}_3|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_2)^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_{14}\\ & -(\overline{\overline{\mathsf{A}}}_2|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_4+\overline{\overline{\mathsf{A}}}_4|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_2)^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_{13}\\ & -(\overline{\overline{\mathsf{A}}}_3|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_4+\overline{\overline{\mathsf{A}}}_4|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}_3)^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}}_{12}\\ =\; & \boldsymbol{\beta\gamma}||(\overline{\overline{\mathsf{A}}}_1\overline{\overline{\mathsf{A}}}_{234}+\overline{\overline{\mathsf{A}}}_2\overline{\overline{\mathsf{A}}}_{134}+\overline{\overline{\mathsf{A}}}_3\overline{\overline{\mathsf{A}}}_{124}+\overline{\overline{\mathsf{A}}}_4\overline{\overline{\mathsf{A}}}_{123}).\end{aligned}\tag{A38}$$

Noting that we can replace every $\overline{\overline{\mathsf{A}}}_{ij}$ by $\overline{\overline{\mathsf{A}}}^{(2)}$, it is easy to finally arrive at the following identity:

$$\boldsymbol{\beta\gamma}\rfloor\rfloor\overline{\overline{\mathsf{A}}}^{(4)}=(\overline{\overline{\mathsf{A}}}||\boldsymbol{\beta\gamma})\overline{\overline{\mathsf{A}}}^{(3)}-\overline{\overline{\mathsf{A}}}^{(2)}{}^{\wedge}_{\wedge}(\overline{\overline{\mathsf{A}}}|\boldsymbol{\gamma\beta}|\overline{\overline{\mathsf{A}}}).\tag{A39}$$

REFERENCES

1. Flanders, H., *Differential Forms*, Academic Press, New York, 1963.
2. Thirring, W., *Classical Field Theory*, Springer, New York, 1979.
3. Schutz, B., *Geometrical Methods of Mathematical Physics*, University Press, Cambridge, 1980.
4. Deschamps, G. A., "Electromagnetics and differential forms," *Proc. IEEE*, Vol. 69, No. 6, 676–696, 1981.

5. Baldomir, D. and P. Hammond, *Geometry of Electromagnetic Systems*, Clarendon Press, Oxford, 1996.
6. Warnick, K. F. and D. V. Arnold, "Electromagnetic Green functions using differential forms," *J. Electro. Waves Appl.*, Vol. 10, No. 3, 427–438, 1996.
7. Hehl, F. W. and Yu. W. Obukhov, *Foundations of Classical Electrodynamics*, Birkhäuser, Boston, 2003.
8. Lindell, I. V., *Differential Forms in Electromagnetics*, Wiley and IEEE Press, New York, 2004.
9. Lindell, I. V. and K. H. Wallén, "Wave equations for bi-anisotropic media in differential forms," *J. Electromag. Waves Appl.*, Vol. 16, No. 11, 1615–1635, 2002.
10. Lindell, I. V., *Methods for Electromagnetic Field Analysis*, 2nd ed., University Press, Oxford, 1995.
11. Gibbs, J. W. and E. B. Wilson, *Vector Analysis*, Dover, New York, 1960. Reprint of the 1909 edition.

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