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EXACT FORMULAS FOR THE LATERAL ELECTROMAGNETIC PULSES GENERATED BY A HORIZONTAL ELECTRIC DIPOLE IN THE INTERFACE OF TWO DIELECTRICS

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Abstract—The Sommerfeld integrals for the electromagnetic field due to a delta-function current in a horizontal electric dipole located on the planar boundary between air and a homogeneous dielectric are examined in detail. Similar to the case of the vertical dipole, the tangential electric fields consist of a delta-function pulse travelling in the air with the velocity c, the oppositely directed delta-function pulse travelling in the dielectric with the velocity $c/\epsilon^{1/2}$ for the component E_ρ and the velocity $c\epsilon^{1/2}$ for the component E_ϕ, and the final static electric fields due to the charge left on the dipole. The appearance of the vertical magnetic field is similar to that of the tangential electric field. It is pointed out that the amplitude of the pulsed field along the boundary is $1/\rho^2$, which is characteristic of the surface-wave or lateral pulse.

1. INTRODUCTION

The electromagnetic fields from vertical and horizontal electric dipoles located on or near the planar interface between two different media like earth and air or sea water and rock have many useful applications in subsurface and closed-to-the-surface communication, radar, and geophysical prospecting and diagnostics [1–9]. A historical account and extensive list of references can be found in the monograph by King, Owens and Wu [9]. In addition, the problem of the transient field due to a dipole source near or on the boundary between two dielectrics have been visited by many investigators, especially Van der Pol [10], Wait [11, 12], Frankena [13], De Hoop and Frankena [14], Ezzeddine, Kong, and Tsang [15], and Wu and King [16].

In [10], the transient calculation evaluates the Hertz potential of a delta-function current in a vertical electric dipole on the boundary between two half-spaces. Unfortunately, as pointed out in the chapter

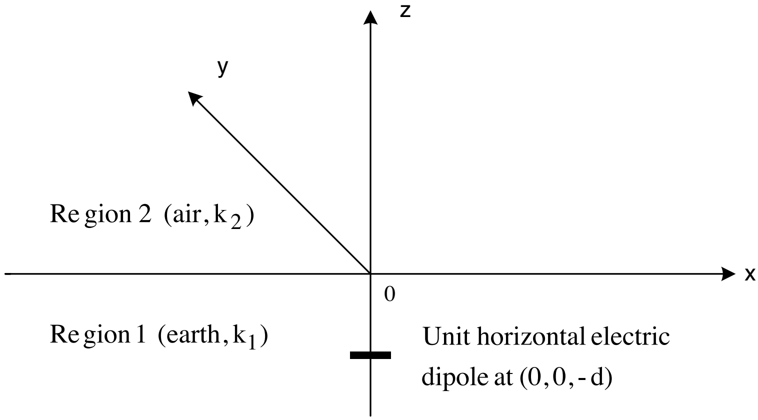

Figure 1. Geometry of a $\hat{x}$-directed horizontal electric dipole on the boundary between two dielectrics like as air and earth.

13 of [9], the electric field components E_ρ and E_z can not be obtained by evaluating the Hertz potential in the time domain. On other hand, the magnetic field B_ϕ can be evaluated readily. In [17, 18], the approximate formulas are derived for lateral electromagnetic pulses due to vertical and horizontal dipole source with delta excitation and Gaussian pulse excitation. Recently, the approximate formulas are derived for lateral electromagnetic pulses from a horizontal electric dipole on the surface of one-dimensionally anisotropic medium [19]. The important introduction to the exact formulas for the components E_z and B_ϕ of the transient electromagnetic field generated by a vertical electric dipole with delta-function current on the boundary between two dielectrics was addressed by Wu and King [16]. The developments in [16] rekindled the interest in the study on the transient electromagnetic field due to a horizontal dipole on the boundary between two dielectrics. The derivation for the case of the horizontal dipole will be more complex because the six components of the transient electromagnetic field are involved.

In the present study, with extension of [16], the exact formulas in terms of elementary function are obtained for three time-dependent components $E_{2z}(\rho, 0; t)$, $B_{2\rho}(\rho, \pi/2; t)$, and $B_{2\phi}(\rho, \pi/2; t)$ from a horizontal electric dipole located on the planar boundary $z = 0$ between two dielectrics.

2. FORMAL REPRESENTATIONS OF TIME-INDEPENDENT FIELD DUE TO UNIT HORIZONTAL ELECTRIC DIPOLE ON THE BOUNDARY BETWEEN TWO DIELECTRICS

The geometry under consideration is shown in Fig. 1, where a unit horizontal electric dipole in the $\hat{x}$ direction is located at $(0, 0, -d)$. When the dipole source and the observation point approach the boundary from below ($d \to 0^+$) and from above ($z \to 0^+$), respectively, with the time dependence of $e^{-i\omega t}$, the frequency-domain formulas for the electromagnetic field in the cylindrical coordinates (ρ, ϕ, z) with $x = \rho \cos\phi$ and $y = \rho \sin\phi$ $(0 \le \phi < 2\pi)$ have been derived in [9, 21]. They are

$$\begin{aligned} \widetilde{E}_{2\rho}(\rho,\phi;\omega) &= \widetilde{E}_{1\rho}(\rho,\phi;\omega) \\ &= -\frac{\omega\mu_0}{4\pi}\int_0^\infty d\lambda\ \lambda\Bigg\{\frac{\sqrt{k_1^2-\lambda^2}\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}} \\ &\quad\times[J_0(\lambda\rho)-J_2(\lambda\rho)] \\ &\quad+\frac{1}{\sqrt{k_2^2-\lambda^2}+\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)+J_2(\lambda\rho)]\Bigg\}\cos\phi, \quad (1) \end{aligned}$$

$$\begin{aligned} \widetilde{E}_{2\phi}(\rho,\phi;\omega) &= \widetilde{E}_{1\phi}(\rho,\phi;\omega) \\ &= \frac{\omega\mu_0}{4\pi}\int_0^\infty d\lambda\ \lambda\Bigg\{\frac{\sqrt{k_1^2-\lambda^2}\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}} \\ &\quad\times[J_0(\lambda\rho)+J_2(\lambda\rho)] \\ &\quad+\frac{1}{\sqrt{k_2^2-\lambda^2}+\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)-J_2(\lambda\rho)]\Bigg\}\sin\phi, \quad (2) \end{aligned}$$

$$\begin{aligned} \widetilde{E}_{2z}(z,\phi;\omega) &= \frac{k_1^2}{k_2^2}\widetilde{E}_{1z}(\rho,\phi;\omega) \\ &= \frac{i\omega\mu_0}{4\pi k_2^2}\int_0^\infty d\lambda\lambda^2\frac{k_2^2\sqrt{k_1^2-\lambda^2}-k_1^2\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}}J_1(\lambda\rho)\cos\phi, \end{aligned} \quad (3)$$

$$\widetilde{B}_{2\rho}(\rho,\phi;\omega) = \widetilde{B}_{1\rho}(\rho,\phi;\omega)$$

$$= -\frac{\mu_0}{8\pi}\int_0^\infty d\lambda\ \lambda\left\{\frac{k_2^2\sqrt{k_1^2-\lambda^2}-k_1^2\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}}\right.$$

$$\times[J_0(\lambda\rho)+J_2(\lambda\rho)]$$

$$\left.+\frac{\sqrt{k_2^2-\lambda^2}-\sqrt{k_1^2-\lambda^2}}{\sqrt{k_2^2-\lambda^2}+\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)-J_2(\lambda\rho)]\right\}\sin\phi, \quad (4)$$

$$\widetilde{B}_{2\phi}(\rho,\phi;\omega) = \widetilde{B}_{1\phi}(\rho,\phi;\omega)$$

$$= -\frac{\mu_0}{8\pi}\int_0^\infty d\lambda\ \lambda\left\{\frac{k_2^2\sqrt{k_1^2-\lambda^2}-k_1^2\sqrt{k_2^2-\lambda^2}}{k_2^2\sqrt{k_1^2-\lambda^2}+k_1^2\sqrt{k_2^2-\lambda^2}}\right.$$

$$\times[J_0(\lambda\rho)-J_2(\lambda\rho)]$$

$$\left.+\frac{\sqrt{k_2^2-\lambda^2}-\sqrt{k_1^2-\lambda^2}}{\sqrt{k_2^2-\lambda^2}+\sqrt{k_1^2-\lambda^2}}[J_0(\lambda\rho)+J_2(\lambda\rho)]\right\}\cos\phi, \quad (5)$$

$$\widetilde{B}_{2z}(z,\phi;\omega) = \widetilde{B}_{1z}(\rho,\phi;\omega)$$

$$= \frac{i\mu_0}{2\pi}\int_0^\infty d\lambda\ \lambda^2\frac{1}{\sqrt{k_2^2-\lambda^2}+\sqrt{k_1^2-\lambda^2}}J_1(\lambda\rho)\sin\phi, \quad (6)$$

where k_j $(j = 1, 2)$ is the wave number in medium j, J_0 and J_2 are the Bessel functions of orders 0 and 2, respectively.

For mathematical conveniences, it is necessary to express the components in terms of ω instead of the wave number. With $\phi = 0$, $k_1 = \omega\epsilon_1^{1/2}/c$, $k_2 = \omega/c$, $\lambda' = c\lambda$, and $\rho' = \rho/c$, where $c = (\mu_0\epsilon_0)^{-1/2}$ is the velocity of light and $\epsilon = \epsilon_1/\epsilon_0$, and taking into account the following relations,

$$J_0(\lambda\rho)+J_2(\lambda\rho) = \frac{2}{\lambda\rho}J_1(\lambda\rho), \quad (7)$$

$$J_0(\lambda\rho)-J_2(\lambda\rho) = 2J_0(\lambda\rho)-\frac{2}{\lambda\rho}J_1(\lambda\rho), \quad (8)$$

where J_1 is the Bessel function of order 1. The formulas for the six field components can be expressed in explicit forms.

$$\widetilde{E}_{2\rho}(\rho',0;\omega) = \widetilde{E}_{1\rho}(\rho',0;\omega)$$

$$= -\frac{\omega\mu_0}{2\pi c}\int_0^\infty d\lambda'\ \lambda'\left\{\frac{\sqrt{\omega^2\epsilon-\lambda'^2}\sqrt{\omega^2-\lambda'^2}}{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}+\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}\right.$$

$$\times\left[J_0(\lambda'\rho') - \frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right] + \frac{1}{\sqrt{\omega^2-\lambda'^2}+\sqrt{\omega^2\epsilon-\lambda'^2}}\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\Bigg\}, \tag{9}$$

$$\begin{aligned}\widetilde{E}_{2\phi}(\rho',\pi/2;\omega) &= \widetilde{E}_{1\phi}(\rho',\pi/2;\omega)\\ &= \frac{\omega\mu_0}{2\pi c}\int_0^\infty d\lambda'\lambda'\Bigg\{\frac{\sqrt{\omega^2\epsilon-\lambda'^2}\sqrt{\omega^2-\lambda'^2}}{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}+\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\\ &\quad + \frac{1}{\sqrt{\omega^2-\lambda'^2}+\sqrt{\omega^2\epsilon-\lambda'^2}}\left[J_0(\lambda'\rho') - \frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right]\Bigg\},\end{aligned} \tag{10}$$

$$\begin{aligned}\widetilde{E}_{2z}(\rho',0;\omega) &= \epsilon\widetilde{E}_{1z}(\rho',0;\omega)\\ &= \frac{i\mu_0}{4\pi\omega c}\int_0^\infty d\lambda'\lambda'^2\frac{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}-\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}+\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}J_1(\lambda'\rho'),\end{aligned} \tag{11}$$

$$\begin{aligned}\widetilde{B}_{2\rho}(\rho',\pi/2;\omega) &= \widetilde{B}_{1\rho}(\rho',\pi/2;\omega)\\ &= -\frac{\mu_0}{4\pi c^2}\int_0^\infty d\lambda'\lambda'\Bigg\{\frac{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}-\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}+\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\\ &\quad + \frac{\sqrt{\omega^2-\lambda'^2}-\sqrt{\omega^2\epsilon-\lambda'^2}}{\sqrt{\omega^2-\lambda'^2}+\sqrt{\omega^2\epsilon-\lambda'^2}}\left[J_0(\lambda'\rho') - \frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right]\Bigg\},\end{aligned} \tag{12}$$

$$\begin{aligned}\widetilde{B}_{2\phi}(\rho',0;\omega) &= \widetilde{B}_{1\phi}(\rho',0;\omega)\\ &= -\frac{\mu_0}{4\pi c^2}\int_0^\infty d\lambda'\ \lambda'\Bigg\{\frac{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}-\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}+\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}\\ &\quad \times\left[J_0(\lambda'\rho') - \frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right]\\ &\quad + \frac{\sqrt{\omega^2-\lambda'^2}-\sqrt{\omega^2\epsilon-\lambda'^2}}{\sqrt{\omega^2-\lambda'^2}+\sqrt{\omega^2\epsilon-\lambda'^2}}\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\Bigg\},\end{aligned} \tag{13}$$

$$\begin{aligned}\widetilde{B}_{2z}(\rho',\pi/2;\omega) &= \widetilde{B}_{1z}(\rho',\pi/2;\omega)\\ &= \frac{i\mu_0}{2\pi c^2}\int_0^\infty d\lambda'\lambda'^2\frac{1}{\sqrt{\omega^2-\lambda'^2}+\sqrt{\omega^2\epsilon-\lambda'^2}}J_1(\lambda'\rho').\end{aligned} \tag{14}$$

3. TIME-DEPENDENT COMPONENT $E_{2\rho}$ DUE TO A HORIZONTAL DIPOLE WITH A DELTA-FUNCTION EXCITATION

3.1. The Integrated Formula for Time-dependent Component $E_{2\rho}$

If the exciting current in a horizontal dipole is a delta-function current with a unit amplitude, the time-dependent component $E_{2\rho}$ can be obtained by the Fourier transform

$$E_{2\rho}(\rho',0;t) \;=\; \frac{1}{\pi}\mathrm{Re}\int_0^\infty d\omega\; e^{-i\omega t}\widetilde{E}_{2\rho}(\rho',0;\omega). \tag{15}$$

Substituting (9) into (15), we get

$$\begin{aligned}E_{2\rho}(\rho',0;t) \;=\; & -\frac{\mu_0}{2\pi^2 c}\int_0^\infty d\omega e^{-i\omega t}\omega \\ & \times\mathrm{Re}\int_0^\infty d\lambda'\;\lambda'\left\{\frac{\sqrt{\omega^2\epsilon-\lambda'^2}\sqrt{\omega^2-\lambda'^2}}{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}+\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}\right. \\ & \times\left[J_0(\lambda'\rho')-\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right] \\ & \left.+\frac{1}{\sqrt{\omega^2\epsilon-\lambda'^2}+\sqrt{\omega^2-\lambda'^2}}\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right\}. \end{aligned}\tag{16}$$

The evaluation of the integral in (16) is a very difficult task. With the definition $\lambda'=\omega\xi$, $d\lambda'=\omega d\xi$, (16) reads as

$$\begin{aligned}E_{2\rho}(\rho',0;t)=&\frac{\mu_0}{2\pi^2 c}\mathrm{Re}\left\{\int_0^\infty \xi d\xi\frac{\sqrt{\epsilon-\xi^2}\sqrt{1-\xi^2}}{\sqrt{\epsilon-\xi^2}+\epsilon\sqrt{1-\xi^2}}\right. \\ &\times\left[\frac{\partial^2}{\partial t^2}\int_0^\infty d\omega e^{-i\omega t}J_0(\omega\xi\rho')+\frac{i}{\xi\rho'}\frac{\partial}{\partial t}\int_0^\infty d\omega e^{-i\omega t}J_1(\omega\xi\rho')\right] \\ &\left.-\frac{1}{\sqrt{\epsilon-\xi^2}+\sqrt{1-\xi^2}}\frac{i}{\xi\rho'}\frac{\partial}{\partial t}\int_0^\infty d\omega e^{-i\omega t}J_1(\omega\xi\rho')\right\}.\end{aligned}\tag{17}$$

The integrals in (17) with respect to ω can be obtained readily by using the infinite integral formula 6.611-1 of [30]. When $t>\xi\rho'$,

$$\int_0^\infty d\omega e^{-i\omega t}J_0(\omega\xi\rho') \;=\; -\frac{i}{\sqrt{t^2-\xi^2\rho'^2}}, \tag{18}$$

$$\int_0^\infty d\omega e^{-i\omega t}J_1(\omega\xi\rho') \;=\; \frac{1}{\xi\rho'}\left[1-\frac{t}{\sqrt{t^2-\xi^2\rho'^2}}\right]. \tag{19}$$

Thus, (17) can be rewritten as

$$E_{2\rho}(\rho', 0; t) = \frac{\mu_0}{2\pi^2 \rho' c} \left[I_1 + I_2 + I_3 \right]. \tag{20}$$

where

$$I_1 = \frac{\partial^2}{\partial t^2} \,\mathrm{Im} \int_0^\infty \xi d\xi \frac{\sqrt{\epsilon - \xi^2}\sqrt{1-\xi^2}}{\sqrt{\epsilon - \xi^2} + \epsilon\sqrt{1-\xi^2}} \frac{1}{\sqrt{t^2/\rho'^2 - \xi^2}}, \tag{21}$$

$$I_2 = -\frac{\partial}{\partial t} \,\mathrm{Im} \int_0^\infty \xi d\xi \frac{\sqrt{\epsilon - \xi^2}\sqrt{1-\xi^2}}{\sqrt{\epsilon - \xi^2} + \epsilon\sqrt{1-\xi^2}} \frac{1}{\xi^2 \rho'} \left[1 - \frac{t}{\sqrt{t^2 - \xi^2 \rho'^2}} \right], \tag{22}$$

$$I_3 = \frac{\partial}{\partial t} \,\mathrm{Im} \int_0^\infty \xi d\xi \frac{1}{\sqrt{\epsilon - \xi^2} + \sqrt{1-\xi^2}} \frac{1}{\xi^2 \rho'} \left[1 - \frac{t}{\sqrt{t^2 - \xi^2 \rho'^2}} \right]. \tag{23}$$

Next, we should evaluate the above three integrals.

3.2. Evaluation of I_1

Following the similar manner used for the evaluation of $E_{2z}(\rho, t)$ due to the vertical dipole in [16], the evaluation of I_1 can be carried out readily.

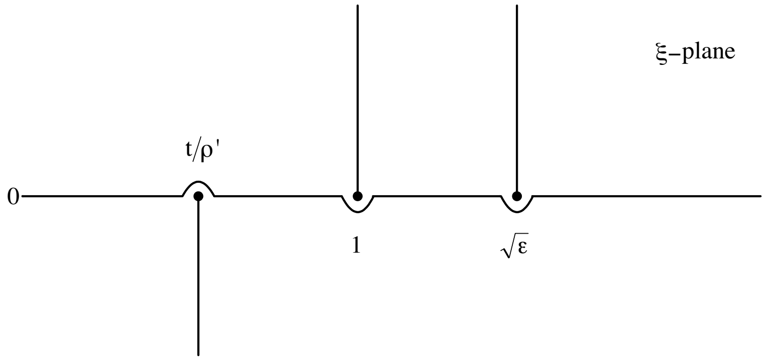

Figure 2. Branch-cut structure for the integrals in (21)–(23).

With the branch-cut structure in Fig. 2, it follows that

$$I_1 = 0, \qquad t/\rho' < 1, \tag{24}$$

and

$$I_1 = \frac{\partial^2}{\partial t^2} \,\mathrm{Im} \Bigg\{ \int_0^1 \xi d\xi \Bigg[\frac{\sqrt{\epsilon - \xi^2}\sqrt{1-\xi^2}}{(\sqrt{\epsilon - \xi^2} + \epsilon\sqrt{1-\xi^2})\sqrt{t^2/\rho'^2 - \xi^2}}$$

$$
\begin{aligned}
&-\frac{\sqrt{\epsilon-\xi^2}\sqrt{1-\xi^2}}{(\sqrt{\epsilon-\xi^2}+\epsilon\sqrt{1-\xi^2})(-\sqrt{t^2/\rho'^2-\xi^2})}\Bigg]\\
&+\int_1^{t/\rho'}\xi d\xi\Bigg[\frac{i\sqrt{\epsilon-\xi^2}\sqrt{\xi^2-1}}{(\sqrt{\epsilon-\xi^2}+i\epsilon\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}}\\
&-\frac{i\sqrt{\epsilon-\xi^2}\sqrt{\xi^2-1}}{(\sqrt{\epsilon-\xi^2}+i\epsilon\sqrt{\xi^2-1})(-\sqrt{t^2/\rho'^2-\xi^2})}\Bigg]\Bigg\},\ 1<t/\rho'<\sqrt{\epsilon}.
\end{aligned}
\tag{25}
$$

Because the integrand of the first integral is *real*, there is no contribution to the imaginary part.

$$
I_1 = \frac{\partial^2}{\partial t^2}\,\mathrm{Im}\int_1^{t/\rho'}\xi d\xi\frac{i2\sqrt{\epsilon-\xi^2}\sqrt{\xi^2-1}}{(\sqrt{\epsilon-\xi^2}+i\epsilon\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}}. \tag{26}
$$

The real and imaginary parts can be separated readily and there is no contribution to the integral for the real part. Then, the integral is simplified as

$$
\begin{aligned}
I_1 &= \frac{\partial^2}{\partial t^2}\int_1^{t/\rho'}\xi d\xi\frac{2(\epsilon-\xi^2)\sqrt{\xi^2-1}}{(\epsilon-1)[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\\
&= \frac{2}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_1^{t/\rho'}\xi^3 d\xi\frac{-\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\\
&\quad+\frac{2\epsilon}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_1^{t/\rho'}\xi d\xi\frac{\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}.
\end{aligned}
\tag{27}
$$

With the change of the variable $\zeta=\xi^2$, $d\zeta=2\xi d\xi$, it follows

$$
\begin{aligned}
I_1 &= \frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_1^{t^2/\rho'^2}\zeta d\zeta\frac{-\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\\
&\quad+\frac{\epsilon}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_1^{t^2/\rho'^2}d\zeta\frac{\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}.
\end{aligned}
\tag{28}
$$

The following two integrals need to be treated.

$$
\vartheta_0^{(1)} = \int_1^{t^2/\rho'^2}\zeta d\zeta\frac{-\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}, \tag{29}
$$

$$
\vartheta_0^{(2)} = \int_1^{t^2/\rho'^2}d\zeta\frac{\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}. \tag{30}
$$

The above two integrals had been solved by Wu and King [16]. The results are

$$\vartheta_0^{(1)} = -\frac{\pi}{2(\epsilon+1)}\left[\frac{t^2}{\rho'^2}-1+\frac{2\epsilon}{\epsilon+1}-\frac{2\epsilon}{\epsilon+1}\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right], \quad (31)$$

$$\vartheta_0^{(2)} = \frac{\pi}{\epsilon+1}\left[1-\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right]. \quad (32)$$

Thus,

$$I_1 = -\frac{\pi}{2(\epsilon^2-1)}\frac{\partial^2}{\partial t^2}\left[\frac{t^2}{\rho'^2}-1-\frac{2\epsilon^2}{\epsilon+1}+\frac{2\epsilon^2}{\epsilon+1}\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right], \quad 1<\frac{t}{\rho'}<\sqrt{\epsilon}. \quad (33)$$

When $t/\rho' > \sqrt{\epsilon}$,

$$\begin{aligned} I_1 = \frac{\partial^2}{\partial t^2}\,\mathrm{Im}\Bigg\{ &\int_1^{\sqrt{\epsilon}} \xi d\xi \frac{2\sqrt{\epsilon-\xi^2}\,i\sqrt{\xi^2-1}}{(\sqrt{\epsilon-\xi^2}+i\epsilon\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}} \\ &+\int_{\sqrt{\epsilon}}^{t/\rho'} \xi d\xi \frac{i2\sqrt{\xi^2-\epsilon}\sqrt{\xi^2-1}}{(\sqrt{\xi^2-\epsilon}+\epsilon\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}}\Bigg\}. \end{aligned} \quad (34)$$

The imaginary part is as follows:

$$\begin{aligned} I_1 &= \frac{2}{\epsilon-1}\frac{\partial^2}{\partial t^2}\Bigg\{\int_1^{\sqrt{\epsilon}} \xi d\xi \frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}} \\ &\quad +\int_{\sqrt{\epsilon}}^{t/\rho'} \xi d\xi \frac{\epsilon(\xi^2-1)\sqrt{\xi^2-\epsilon}-(\xi^2-\epsilon)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\Bigg\} \\ &= \frac{2}{\epsilon-1}\frac{\partial^2}{\partial t^2}\Bigg\{\int_1^{t/\rho'} \xi d\xi \frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}} \\ &\quad +\int_{\sqrt{\epsilon}}^{t/\rho'} \xi d\xi \frac{\epsilon(\xi^2-1)\sqrt{\xi^2-\epsilon}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\Bigg\}. \end{aligned} \quad (35)$$

With the change in variable $\zeta = \xi^2$, it becomes

$$\begin{aligned} I_1 = \frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\Bigg\{ &\int_1^{t^2/\rho'^2} d\zeta \frac{(\epsilon-\zeta)\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}} \\ &+\int_{\epsilon}^{t^2/\rho'^2} d\zeta \frac{\epsilon(\zeta-1)\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\Bigg\} \end{aligned}$$

$$= \frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_1^{t^2/\rho'^2}\zeta d\zeta\frac{-\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}$$
$$+\frac{\epsilon}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_1^{t^2/\rho'^2}d\zeta\frac{\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}$$
$$+\frac{\epsilon}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_\epsilon^{t^2/\rho'^2}\zeta d\zeta\frac{\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}$$
$$+\frac{\epsilon}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_\epsilon^{t^2/\rho'^2}d\zeta\frac{-\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}. \tag{36}$$

The first and second integrals in (36) are shown in (31) and (32). Next, the third and fourth integrals need to be evaluated. They are

$$\vartheta_\epsilon^{(1)} = \int_\epsilon^{t^2/\rho'^2}\zeta d\zeta\frac{\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}, \tag{37}$$
$$\vartheta_\epsilon^{(2)} = \int_\epsilon^{t^2/\rho'^2}d\zeta\frac{-\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}. \tag{38}$$

Also the above two integrals were solved by Wu and King [16]. They are

$$\vartheta_\epsilon^{(1)} = \frac{\pi}{2(\epsilon+1)}\left[\frac{t^2}{\rho'^2}-\epsilon+\frac{2\epsilon}{\epsilon+1}-\frac{2\epsilon^2}{\epsilon+1}\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right], \tag{39}$$
$$\vartheta_\epsilon^{(2)} = -\frac{\pi}{\epsilon+1}\left[1-\epsilon\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right]. \tag{40}$$

With substitutions (31), (32), (39), and (40) into (36), we get

$$I_1 = \frac{\pi}{2(\epsilon+1)}\frac{\partial^2}{\partial t^2}\left(\frac{t^2}{\rho'^2}-\frac{\epsilon^2+1}{\epsilon+1}\right), \qquad \frac{t}{\rho'}>\sqrt{\epsilon}. \tag{41}$$

Then, I_1 can be rewritten as follows:

$$I_1 = \frac{\pi}{2}\frac{\partial^2}{\partial t^2}\, f_1\left(\frac{t}{\rho'}\right). \tag{42}$$

where

$$f_1\left(\frac{t}{\rho'}\right)=\begin{cases} 0, & \dfrac{t}{\rho'}<1 \\ -\dfrac{1}{\epsilon^2-1}\left[\dfrac{t^2}{\rho'^2}-1-\dfrac{2\epsilon^2}{\epsilon+1}+\dfrac{2\epsilon^2}{\epsilon+1}\left((\epsilon+1)\dfrac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right], & 1<\dfrac{t}{\rho'}<\sqrt{\epsilon} \\ \dfrac{1}{\epsilon+1}\left(\dfrac{t^2}{\rho'^2}-\dfrac{\epsilon^2+1}{\epsilon+1}\right), & \dfrac{t}{\rho'}>\sqrt{\epsilon} \end{cases}. \quad (43)$$

It follows that $f_1(1-) = f_1(1+) = 0$ and $f_1(\sqrt{\epsilon}-) = f_1(\sqrt{\epsilon}+) = \frac{\epsilon-1}{(\epsilon+1)^2}$. Thus, $f_1(t/\rho')$ is everywhere continuous.

$$f_1'\left(\frac{t}{\rho'}\right) = \begin{cases} 0, & \dfrac{t}{\rho'}<1 \\ -\dfrac{1}{\epsilon^2-1}\dfrac{2t}{\rho'^2}\left[1-\epsilon^2\left((\epsilon+1)\dfrac{t^2}{\rho'^2}-\epsilon\right)^{-3/2}\right], & 1<\frac{t}{\rho'}<\sqrt{\epsilon} \\ \dfrac{1}{\epsilon+1}\dfrac{2t}{\rho'^2}, & \dfrac{t}{\rho'}>\sqrt{\epsilon} \end{cases}. \quad (44)$$

Since $f_1'(1-) = 0$, $f_1'(1+) = 2/\rho'$, there is a step discontinuity of $2/\rho'$ in $f_1'(t/\rho')$ at $t/\rho' = 1$. Similarly, $f_1'(\sqrt{\epsilon}-) = -2/[\sqrt{\epsilon}(\epsilon+1)\rho']$, $f_1'(\sqrt{\epsilon}+) = 2\sqrt{\epsilon}/[(\epsilon+1)\rho']$, $f_1'(t/\rho')$ has a step discontinuity of $2/(\sqrt{\epsilon}\rho')$ at $t/\rho' = \sqrt{\epsilon}$. Thus,

$$f_1''\left(\frac{t}{\rho'}\right) = \begin{cases} 0, & \dfrac{t}{\rho'}<1 \\ -\dfrac{2}{(\epsilon^2-1)\rho'^2}\left[1+\dfrac{\epsilon^2}{(\epsilon+1)^{3/2}}\left(\dfrac{2t^2}{\rho'^2}+\dfrac{\epsilon}{\epsilon+1}\right)\left(\dfrac{t^2}{\rho'^2}-\dfrac{\epsilon}{\epsilon+1}\right)^{-5/2}\right], & 1<\dfrac{t}{\rho'}<\sqrt{\epsilon} \\ \dfrac{2}{(\epsilon+1)\rho'^2}, & \dfrac{t}{\rho'}>\sqrt{\epsilon} \end{cases}. \quad (45)$$

Finally, the exact expressions for I_1 is obtained readily.

$$I_1 = \frac{c\pi}{\rho}\left[\delta\left(t-\frac{\rho}{c}\right)+\frac{1}{\sqrt{\epsilon}}\delta\left(t-\frac{\sqrt{\epsilon}\rho}{c}\right)\right]$$

$$+\frac{c^2\pi}{(\epsilon^2-1)\rho^2}\begin{cases}0, & \frac{ct}{\rho}<1\\ -\left[1+\frac{\epsilon^2}{(\epsilon+1)^{3/2}}\left(\frac{2c^2t^2}{\rho^2}+\frac{\epsilon}{\epsilon+1}\right)\left(\frac{c^2t^2}{\rho^2}-\frac{\epsilon}{\epsilon+1}\right)^{-5/2}\right], & \\ & 1<\frac{ct}{\rho}<\sqrt{\epsilon}\\ \epsilon-1, & \frac{ct}{\rho}>\sqrt{\epsilon}\end{cases}. \tag{46}$$

3.3. Evaluation of I_2

Following the similar manners used for the evaluation of $B_{2\phi}(\rho,t)$ due the vertical dipole with delta-function excitation in [16], the evaluation of I_2 can be also carried out readily.

$$I_2=0, \qquad t/\rho'<1. \tag{47}$$

When $1<t/\rho'<\sqrt{\epsilon}$,

$$\begin{aligned}I_2 \;=\; & -\frac{\partial}{\partial t}\,\mathrm{Im}\left\{\int_0^\infty \frac{\xi d\xi}{\xi^2\rho'}\left[\frac{i\sqrt{\epsilon-\xi^2}\sqrt{\xi^2-1}}{\sqrt{\epsilon-\xi^2}+i\epsilon\sqrt{\xi^2-1}}\right.\right.\\ & \left.\left.-\frac{t}{\rho'}\frac{i\sqrt{\epsilon-\xi^2}\sqrt{\xi^2-1}}{(\sqrt{\epsilon-\xi^2}+i\epsilon\sqrt{\xi^2-1})(\sqrt{t^2/\rho'^2-\xi^2})}\right]\right\}.\end{aligned} \tag{48}$$

The real and imaginary parts is separated readily and there is no contribution to the integral for the real part. The result reduces to

$$\begin{aligned}I_2 \;=\; & -\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\int_0^\infty \frac{\xi d\xi}{\xi^2}\frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{(\epsilon+1)\xi^2-\epsilon}\right.\\ & \left.-\frac{t}{\rho'}\frac{\partial}{\partial t}\int_0^\infty \frac{\xi d\xi}{\xi^2}\frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\right\}.\end{aligned} \tag{49}$$

With the contour in Fig. 3, this becomes

$$\begin{aligned}I_2 \;=\; & -\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\int_1^{t/\rho'} \frac{\xi d\xi}{\xi^2}\frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{(\epsilon+1)\xi^2-\epsilon}\right.\\ & -\frac{t}{\rho'}\int_1^{t/\rho'} \frac{\xi d\xi}{\xi^2}\frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\\ & -\int_1^{t/\rho'} \frac{\xi d\xi}{\xi^2}\frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{(\epsilon+1)\xi^2-\epsilon}\end{aligned}$$

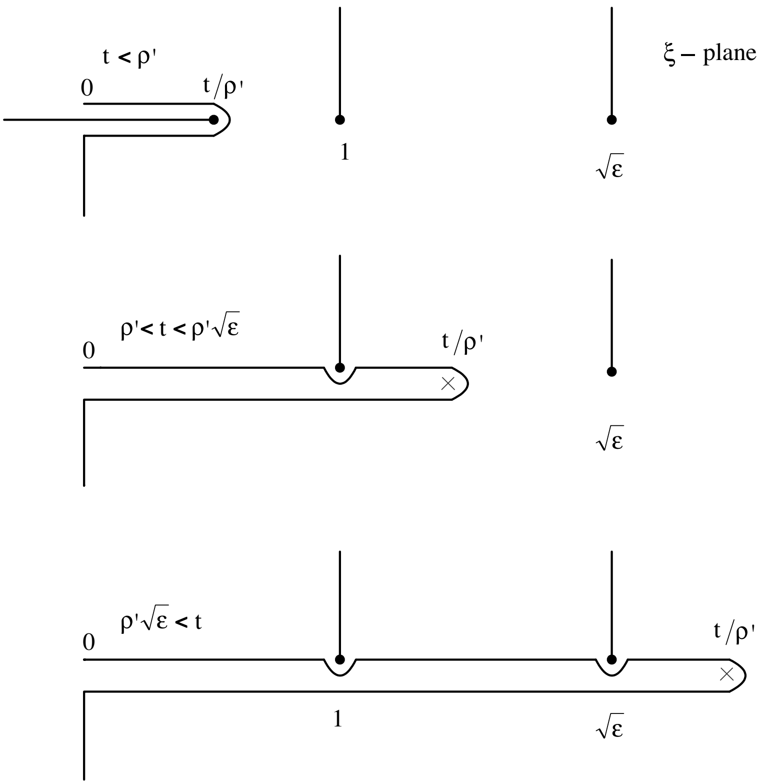

Figure 3. Contours of integration for the integrals in (21)–(23).

$$
\begin{aligned}
&+\frac{t}{\rho'}\int_1^{t/\rho'}\frac{\xi d\xi}{\xi^2}\frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon](-\sqrt{t^2/\rho'^2-\xi^2})}\Bigg\} \\
&= \frac{2}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t/\rho'}\frac{\xi d\xi}{\xi^2}\frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\right\}. \quad (50)
\end{aligned}
$$

With $\zeta=\xi^2$,

$$
\begin{aligned}
I_2 &= \frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{(\epsilon-\zeta)\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\right\} \\
&= \frac{\epsilon}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\right\} \\
&\quad+\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t^2/\rho'^2}d\zeta\frac{-\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\right\}. \quad (51)
\end{aligned}
$$

Then,

$$\vartheta_0^{(3)} = \int_1^{t^2/\rho'^2} \frac{d\zeta}{\zeta} \frac{\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}, \tag{52}$$

$$\vartheta_0^{(4)} = -\vartheta_0^{(2)} = \int_1^{t^2/\rho'^2} d\zeta \frac{-\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}. \tag{53}$$

The first integral need to be solved and the second one has been solved in (32). Let $x' = \zeta - 1$, $T_0 = t^2/\rho'^2 - 1$, and $E_0 = 1/(\epsilon+1)$, then

$$\begin{aligned}
\vartheta_0^{(3)} &= \int_0^{T_0} \frac{x'dx'}{(x'+1)[(\epsilon+1)x'+1]\sqrt{(T_0-x')x'}} \\
&= \frac{1}{\epsilon+1}\int_0^{T_0} \frac{x'dx'}{(x'+1)(x'+E_0)\sqrt{(T_0-x')x'}} \\
&= \frac{1}{\epsilon+1}\frac{1}{1-E_0}\int_0^{T_0}\left(\frac{1}{x'+E_0}-\frac{1}{x'+1}\right)\frac{x'dx'}{\sqrt{(T_0-x')x'}} \\
&= \frac{1}{\epsilon}\left\{\int_0^{T_0}\frac{x'dx'}{(x'+E_0)\sqrt{(T_0-x')x'}} - \int_0^{T_0}\frac{x'dx'}{(x'+1)\sqrt{(T_0-x')x'}}\right\}.
\end{aligned} \tag{54}$$

Let $x = x' + E_0$ and $X_1 = (T_0 + E_0 - x)(x - E_0)$, then

$$\begin{aligned}
\int_0^{T_0}\frac{x'dx'}{(x'+E_0)\sqrt{(T_0-x')x'}} &= \int_{E_0}^{T_0+E_0}\frac{(x-E_0)dx}{x\sqrt{(T_0+E_0-x)(x-E_0)}} \\
&= \int_{E_0}^{T_0+E_0}\frac{dx}{X_1^{1/2}} - E_0\int_{E_0}^{T_0+E_0}\frac{dx}{xX_1^{1/2}} \\
&= \pi\left[1-\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right].
\end{aligned} \tag{55}$$

Similarly, let $x = x' + 1$ and $X_2 = (T_0 + 1 - x)(x - 1)$, then

$$\begin{aligned}
\int_0^{T_0}\frac{x'dx'}{(x'+1)\sqrt{(T_0-x')x'}} &= \int_1^{T_0+1}\frac{(x-1)dx}{x\sqrt{(T_0+1-x)(x-1)}} \\
&= \int_1^{T_0+1}\frac{dx}{X_2^{1/2}} - \int_1^{T_0+1}\frac{dx}{xX_2^{1/2}} \\
&= \pi\left[1-\left(\frac{t^2}{\rho'^2}\right)^{-1/2}\right].
\end{aligned} \tag{56}$$

With substitutions (55) and (56) into (54), we get

$$\vartheta_0^{(3)} = \frac{\pi}{\epsilon}\left[\left(\frac{t^2}{\rho'^2}\right)^{-1/2} - \left((\epsilon+1)\frac{t^2}{\rho'^2} - \epsilon\right)^{-1/2}\right]. \tag{57}$$

Thus,

$$I_2 = \frac{\pi}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\left[-\frac{1}{\epsilon+1} + \left(\frac{t^2}{\rho'^2}\right)^{-1/2} - \frac{\epsilon}{\epsilon+1}\left((\epsilon+1)\frac{t^2}{\rho'^2} - \epsilon\right)^{-1/2}\right]\right\}, \quad 1 < \frac{t}{\rho'} < \sqrt{\epsilon}. \tag{58}$$

When $t/\rho' > \sqrt{\epsilon}$,

$$\begin{aligned} I_2 = & -\frac{\partial}{\partial t}\,\mathrm{Im}\left\{\int_0^{\sqrt{\epsilon}} \frac{\xi d\xi}{\xi^2\rho'} \frac{i\sqrt{\epsilon-\xi^2}\sqrt{\xi^2-1}}{\sqrt{\epsilon-\xi^2} + i\epsilon\sqrt{\xi^2-1}}\right. \\ & -\frac{t}{\rho'}\int_0^{\sqrt{\epsilon}} \frac{\xi d\xi}{\xi^2\rho'} \frac{i\sqrt{\epsilon-\xi^2}\sqrt{\xi^2-1}}{(\sqrt{\epsilon-\xi^2} + i\epsilon\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}} \\ & +\int_{\sqrt{\epsilon}}^{\infty} \frac{\xi d\xi}{\xi^2\rho'} \frac{i\sqrt{\xi^2-\epsilon}\sqrt{\xi^2-1}}{\sqrt{\xi^2-\epsilon} + \epsilon\sqrt{\xi^2-1}} \\ & \left. -\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^{\infty} \frac{\xi d\xi}{\xi^2\rho'} \frac{i\sqrt{\xi^2-\epsilon}\sqrt{\xi^2-1}}{(\sqrt{\xi^2-\epsilon} + \epsilon\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}}\right\}. \end{aligned} \tag{59}$$

The imaginary part is as follows:

$$\begin{aligned} I_2 = & -\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\int_0^{\sqrt{\epsilon}} \frac{\xi d\xi}{\xi^2} \frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{(\epsilon+1)\xi^2-\epsilon}\right. \\ & -\frac{t}{\rho'}\int_0^{\sqrt{\epsilon}} \frac{\xi d\xi}{\xi^2} \frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}} \\ & +\int_{\sqrt{\epsilon}}^{\infty} \frac{\xi d\xi}{\xi^2} \frac{\epsilon(\xi^2-1)\sqrt{\xi^2-\epsilon} - (\xi^2-\epsilon)\sqrt{\xi^2-1}}{(\epsilon+1)\xi^2-\epsilon} \\ & \left. -\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^{\infty} \frac{\xi d\xi}{\xi^2} \frac{\epsilon(\xi^2-1)\sqrt{\xi^2-\epsilon} - (\xi^2-\epsilon)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\right\} \\ = & -\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\int_0^{\infty} \frac{\xi d\xi}{\xi^2} \frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{(\epsilon+1)\xi^2-\epsilon}\right. \\ & -\frac{t}{\rho'}\int_0^{\infty} \frac{\xi d\xi}{\xi^2} \frac{(\epsilon-\xi^2)\sqrt{\xi^2-1}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}} \end{aligned}$$

$$+\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2}\frac{\epsilon(\xi^2-1)\sqrt{\xi^2-\epsilon}}{(\epsilon+1)\xi^2-\epsilon}$$
$$-\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2}\frac{\epsilon(\xi^2-1)\sqrt{\xi^2-\epsilon}}{[(\epsilon+1)\xi^2-\epsilon]\sqrt{t^2/\rho'^2-\xi^2}}\Bigg\}. \tag{60}$$

In terms of the variable $\zeta = \xi^2$, $d\zeta = 2\xi d\xi$, this becomes

$$\begin{aligned} I_2 \;=\; & \frac{\epsilon}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\right\} \\ & +\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t^2/\rho'^2}d\zeta\frac{-\sqrt{\zeta-1}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\right\} \\ & +\frac{\epsilon}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_\epsilon^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{-\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\right\} \\ & +\frac{\epsilon}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_\epsilon^{t^2/\rho'^2}d\zeta\frac{\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}\right\}. \end{aligned} \tag{61}$$

The first and second integrals in (61) have been evaluated in (57) and (53). Next, the third and fourth integrals need to be evaluated. They are

$$\vartheta_\epsilon^{(3)} = \int_\epsilon^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{-\sqrt{\zeta-\epsilon}}{[(\epsilon+1)\zeta-\epsilon]\sqrt{t^2/\rho'^2-\zeta}}, \tag{62}$$

$$\vartheta_\epsilon^{(4)} = -\vartheta_\epsilon^{(2)}. \tag{63}$$

Let $x' = \zeta-\epsilon$, $T_\epsilon = t^2/\rho'^2-\epsilon$, $E_\epsilon = \epsilon^2/(\epsilon+1)$, then

$$\begin{aligned} \vartheta_\epsilon^{(3)} &= -\int_0^{T_\epsilon}\frac{x'dx'}{(x'+\epsilon)[(\epsilon+1)x'+\epsilon^2]\sqrt{(T_\epsilon-x')x'}} \\ &= -\frac{1}{\epsilon+1}\int_0^{T_\epsilon}\frac{x'dx'}{(x'+\epsilon)(x'+E_\epsilon)\sqrt{(T_\epsilon-x')x'}} \\ &= -\frac{1}{\epsilon+1}\frac{1}{\epsilon-E_\epsilon}\int_0^{T_\epsilon}\left(\frac{1}{x'+E_\epsilon}-\frac{1}{x'+\epsilon}\right)\frac{x'dx'}{\sqrt{(T_\epsilon-x')x'}} \\ &= -\frac{1}{\epsilon}\left\{\int_0^{T_\epsilon}\frac{x'dx'}{(x'+E_\epsilon)\sqrt{(T_\epsilon-x')x'}}-\int_0^{T_\epsilon}\frac{x'dx'}{(x'+\epsilon)\sqrt{(T_\epsilon-x')x'}}\right\}. \end{aligned} \tag{64}$$

Let $x = x'+E_\epsilon$ and $Y_1 = (T_\epsilon+E_\epsilon-x)(x-E_\epsilon)$,

$$\int_0^{T_\epsilon}\frac{x'dx'}{(x'+E_\epsilon)\sqrt{(T_\epsilon-x')x'}} = \int_{E_\epsilon}^{T_\epsilon+E_\epsilon}\frac{(x-E_\epsilon)dx}{x\sqrt{(T_\epsilon+E_\epsilon-x)(x-E_\epsilon)}}$$

$$= \int_{E_\epsilon}^{T_\epsilon+E_\epsilon} \frac{dx}{Y_1^{1/2}} - E_\epsilon \int_{E_\epsilon}^{T_\epsilon+E_\epsilon} \frac{dx}{xY_1^{1/2}}$$

$$= \pi\left[1-\epsilon\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right]. \quad (65)$$

Similarly, let $x = x' + \epsilon$ and $Y_2 = (T_\epsilon + \epsilon - x)(x - \epsilon)$

$$\int_0^{T_\epsilon} \frac{x'dx'}{(x'+\epsilon)\sqrt{(T_\epsilon - x')x'}} = \int_\epsilon^{T_\epsilon+\epsilon} \frac{(x-\epsilon)dx}{x\sqrt{(T_\epsilon+\epsilon-x)(x-\epsilon)}}$$

$$= \int_\epsilon^{T_\epsilon+\epsilon} \frac{dx}{Y_2^{1/2}} - \epsilon \int_\epsilon^{T_\epsilon+\epsilon} \frac{dx}{xY_2^{1/2}}$$

$$= \pi\left[1-\epsilon\left(\frac{\epsilon t^2}{\rho'^2}\right)^{-1/2}\right]. \quad (66)$$

With substitutions (65), (66) into (64), we get

$$\vartheta_\epsilon^{(3)} = \pi\left[-\left(\frac{\epsilon t^2}{\rho'^2}\right)^{-1/2} + \left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right]. \quad (67)$$

Thus,

$$I_2 = \frac{\pi}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\left[\frac{\epsilon-1}{\epsilon+1} + (1-\sqrt{\epsilon})\left(\frac{t^2}{\rho'^2}\right)^{-1/2}\right]\right\}, \quad \frac{t}{\rho'} > \sqrt{\epsilon}. \quad (68)$$

Combined with (47), (58), and (68), the result is

$$I_2 = \frac{\pi}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\, f_2\left(\frac{t}{\rho'}\right), \quad (69)$$

where

$$f_2\left(\frac{t}{\rho'}\right) = \begin{cases} 0, & \frac{t}{\rho'} < 1 \\ \frac{t}{\rho'}\left[-\frac{1}{\epsilon+1}+\left(\frac{t^2}{\rho'^2}\right)^{-1/2} - \frac{\epsilon}{\epsilon+1}\left((\epsilon+1)\frac{t^2}{\rho'^2}-\epsilon\right)^{-1/2}\right], & 1 < \frac{t}{\rho'} < \sqrt{\epsilon} \\ \frac{t}{\rho'}\left[\frac{\epsilon-1}{\epsilon+1} + (1-\sqrt{\epsilon})\left(\frac{t^2}{\rho'^2}\right)^{-1/2}\right], & \frac{t}{\rho'} > \sqrt{\epsilon} \end{cases}. \quad (70)$$

Since $f_2(1-) = f_2(1+) = 0$, $f_2(\sqrt{\epsilon}-) = f_2(\sqrt{\epsilon}+) = \sqrt{\epsilon}\left(-\frac{2}{\epsilon+1} + \frac{1}{\sqrt{\epsilon}}\right)$, it is continuous at $t/\rho' = 1$ and $t/\rho' = \sqrt{\epsilon}$.

$$f_2'\left(\frac{t}{\rho'}\right) = \begin{cases} 0, & \dfrac{t}{\rho'} < 1 \\ \dfrac{1}{\rho'}\left[-\dfrac{1}{\epsilon+1} + \dfrac{\epsilon^2}{\epsilon+1}\left((\epsilon+1)\dfrac{t^2}{\rho'^2} - \epsilon\right)^{-3/2}\right], & 1 < \dfrac{t}{\rho'} < \sqrt{\epsilon} \\ \dfrac{1}{\rho'}\dfrac{\epsilon-1}{\epsilon+1}, & \dfrac{t}{\rho'} > \sqrt{\epsilon} \end{cases}. \tag{71}$$

The complete formula for I_2 can be expressed as follows:

$$I_2 = \frac{c^2\pi}{(\epsilon^2-1)\rho^2}\begin{cases} 0, & \dfrac{ct}{\rho} < 1 \\ -1 + \dfrac{\epsilon^2}{(\epsilon+1)^{3/2}}\left(\dfrac{c^2t^2}{\rho^2} - \dfrac{\epsilon}{\epsilon+1}\right)^{-3/2}, & 1 < \dfrac{ct}{\rho} < \sqrt{\epsilon} \\ \epsilon - 1, & \dfrac{ct}{\rho} > \sqrt{\epsilon} \end{cases}. \tag{72}$$

3.4. Evaluation of I_3

Following the same procedures in evaluations of I_2, I_3 can also be evaluated readily. When $t/\rho' < 1$,

$$I_3 = 0. \tag{73}$$

When $1 < t/\rho' < \sqrt{\epsilon}$,

$$\begin{aligned} I_3 \;=\; & \frac{\partial}{\partial t}\,\mathrm{Im}\left\{\int_0^\infty \frac{\xi d\xi}{\xi^2\rho'}\left[\frac{1}{\sqrt{\epsilon-\xi^2} + i\sqrt{\xi^2-1}}\right.\right. \\ & \left.\left. -\frac{t}{\rho'}\frac{1}{(\sqrt{\epsilon-\xi^2} + i\sqrt{\xi^2-1})(\sqrt{t^2/\rho'^2-\xi^2})}\right]\right\}. \end{aligned} \tag{74}$$

The real and imaginary parts is readily separated and only the imaginary part is retained. I_3 reduces to

$$I_3 = -\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\int_0^\infty \frac{\xi d\xi}{\xi^2}\sqrt{\xi^2-1} - \frac{t}{\rho'}\int_0^\infty \frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}}\right\}. \tag{75}$$

With the contour in Fig.3, this becomes

$$\begin{aligned} I_3 &= -\frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\int_1^{t/\rho'}\frac{\xi d\xi}{\xi^2}\sqrt{\xi^2-1}-\frac{t}{\rho'}\int_1^{t/\rho'}\frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}}\right. \\ &\quad \left.-\int_1^{t/\rho'}\frac{\xi d\xi}{\xi^2}\sqrt{\xi^2-1}+\frac{t}{\rho'}\int_1^{t/\rho'}\frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-1}}{-\sqrt{t^2/\rho'^2-\xi^2}}\right\} \\ &= \frac{2}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t/\rho'}\frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}}\right\}. \end{aligned} \tag{76}$$

With the change of the variable $\zeta=\xi^2$, $d\zeta=2\xi d\xi$, it follows

$$I_3 = \frac{1}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{\sqrt{\zeta-1}}{\sqrt{t^2/\rho'^2-\zeta}}\right\}. \tag{77}$$

The integral in (77) can be evaluated readily.

$$\begin{aligned} \vartheta_0^{(5)} &= \int_1^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{\sqrt{\zeta-1}}{\sqrt{t^2/\rho'^2-\zeta}} \\ &= \int_1^{t^2/\rho'^2}\frac{d\zeta}{\sqrt{(t^2/\rho'^2-\zeta)(\zeta-1)}}-\int_1^{t^2/\rho'^2}\frac{d\zeta}{\zeta\sqrt{(t^2/\rho'^2-\zeta)(\zeta-1)}} \\ &= \pi\left[1-\left(\frac{t^2}{\rho'^2}\right)^{-1/2}\right]. \end{aligned} \tag{78}$$

When $t/\rho' > \sqrt{\epsilon}$,

$$\begin{aligned} I_3 &= \frac{\partial}{\partial t}\,\mathrm{Im}\left\{\int_0^{\sqrt{\epsilon}}\frac{\xi d\xi}{\xi^2\rho'^2}\frac{1}{\sqrt{\epsilon-\xi^2}+i\sqrt{\xi^2-1}}\right. \\ &\quad -\frac{t}{\rho'}\int_0^{\sqrt{\epsilon}}\frac{\xi d\xi}{\xi^2\rho'^2}\frac{1}{(\sqrt{\epsilon-\xi^2}+i\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}} \\ &\quad +\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2\rho'^2}\frac{1}{i\sqrt{\xi^2-\epsilon}+i\sqrt{\xi^2-1}} \\ &\quad \left.-\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2\rho'^2}\frac{1}{(i\sqrt{\xi^2-\epsilon}+i\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}}\right\}. \end{aligned} \tag{79}$$

The contributing imaginary part is

$$I_3 = -\frac{1}{(\epsilon-1)\rho'^2}\frac{\partial}{\partial t}\left\{\int_0^{\sqrt{\epsilon}}\frac{\xi d\xi}{\xi^2}\sqrt{\xi^2-1}-\frac{t}{\rho'}\int_0^{\sqrt{\epsilon}}\frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}}\right.$$

$$
\begin{aligned}
&+\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2}\left(\sqrt{\xi^2-1}-\sqrt{\xi^2-\epsilon}\right)\\
&\left.-\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-1}-\sqrt{\xi^2-\epsilon}}{\sqrt{t^2/\rho'^2-\xi^2}}\right\}\\
=\ &-\frac{1}{(\epsilon-1)\rho'^2}\frac{\partial}{\partial t}\left\{\int_0^{\infty}\frac{\xi d\xi}{\xi^2}\sqrt{\xi^2-1}-\frac{t}{\rho'}\int_0^{\infty}\frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}}\right.\\
&\left.-\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2}\sqrt{\xi^2-\epsilon}+\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^{\infty}\frac{\xi d\xi}{\xi^2}\frac{\sqrt{\xi^2-\epsilon}}{\sqrt{t^2/\rho'^2-\xi^2}}\right\}. \qquad (80)
\end{aligned}
$$

In terms of the variable $\zeta=\xi^2$, this becomes

$$
\begin{aligned}
I_3\ =\ &\frac{1}{(\epsilon-1)\rho'^2}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_1^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{\sqrt{\zeta-1}}{\sqrt{t^2/\rho'^2-\zeta}}\right\}\\
&+\frac{1}{(\epsilon-1)\rho'^2}\frac{\partial}{\partial t}\left\{\frac{t}{\rho'}\int_{\epsilon}^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{-\sqrt{\zeta-1}}{\sqrt{t^2/\rho'^2-\zeta}}\right\}. \qquad (81)
\end{aligned}
$$

The first integral in (81) is shown in (78) and the second one need to be evaluated.

$$
\begin{aligned}
\vartheta_{\epsilon}^{(5)}\ =\ &\int_{\epsilon}^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\frac{-\sqrt{\zeta-\epsilon}}{\sqrt{t^2/\rho'^2-\zeta}}\\
=\ &-\int_{\epsilon}^{t^2/\rho'^2}\frac{d\zeta}{\zeta}\cdot\frac{\zeta-\epsilon}{\sqrt{(t^2/\rho'^2-\zeta)(\zeta-\epsilon)}}\\
=\ &-\int_{\epsilon}^{t^2/\rho'^2}\frac{d\zeta}{\sqrt{(t^2/\rho'^2-\zeta)(\zeta-\epsilon)}}+\epsilon\int_{\epsilon}^{t^2/\rho'^2}\frac{d\zeta}{\zeta\sqrt{(t^2/\rho'^2-\zeta)(\zeta-\epsilon)}}\\
=\ &-\pi\left[1-\sqrt{\epsilon}\left(\frac{t^2}{\rho'^2}\right)^{-1/2}\right]. \qquad (82)
\end{aligned}
$$

With substitution (78) and (82) into (81),

$$
I_3\ =\ \frac{\pi}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\left[\frac{t}{\rho'}(\sqrt{\epsilon}-1)\left(\frac{t^2}{\rho'^2}\right)^{-1/2}\right],\qquad \frac{ct}{\rho}>\sqrt{\epsilon}. \qquad (83)
$$

Thus,

$$
I_3\ =\ \frac{\pi}{(\epsilon-1)\rho'}\frac{\partial}{\partial t}\ f_3\left(\frac{t}{\rho'}\right), \qquad (84)
$$

where

$$f_3\left(\frac{t}{\rho'}\right) = \frac{t}{\rho'}\begin{cases} 0 & , \quad \frac{ct}{\rho} < 1 \\ \left[1-\left(\frac{t^2}{\rho'^2}\right)^{-1/2}\right] & , \quad 1 < \frac{ct}{\rho} < \sqrt{\epsilon} \\ (\sqrt{\epsilon}-1)\left(\frac{t^2}{\rho'^2}\right)^{-1/2} & , \quad \frac{ct}{\rho} > \sqrt{\epsilon} \end{cases}. \tag{85}$$

Since $f_3(1-) = f_3(1+) = 0$, $f_3(\sqrt{\epsilon}-) = f_3(\sqrt{\epsilon}+) = \sqrt{\epsilon}-1$. It is continuous at $t/\rho' = 1$ and $t/\rho' = \sqrt{\epsilon}$.

$$f_3'\left(\frac{t}{\rho'}\right) = \begin{cases} 0 & , \quad \frac{ct}{\rho} < 1 \\ \frac{1}{\rho'} & , \quad 1 < \frac{ct}{\rho} < \sqrt{\epsilon} \\ 0 & , \quad \frac{ct}{\rho} > \sqrt{\epsilon} \end{cases}. \tag{86}$$

Finally, the complete expression for I_3 is obtained as follows,

$$I_3 = \frac{c^2\pi}{(\epsilon-1)\rho^2}\begin{cases} 0 & , \quad \frac{ct}{\rho} < 1 \\ 1 & , \quad 1 < \frac{ct}{\rho} < \sqrt{\epsilon} \\ 0 & , \quad \frac{ct}{\rho} > \sqrt{\epsilon} \end{cases}. \tag{87}$$

3.5. Evaluation of $E_{2\rho}(\rho, 0; t)$

Combined with (20), (46), (72), and (87), the electric field component $E_{2\rho}(\rho, 0; t)$ can be expressed as follows:

$$E_{2\rho}(\rho,0;t) = \frac{1}{2\pi\epsilon_0 c\rho^2}\left[\delta\left(t-\frac{\rho}{c}\right)+\frac{1}{\sqrt{\epsilon}}\delta\left(t-\frac{\sqrt{\epsilon}\rho}{c}\right)\right]+\frac{1}{2\pi\epsilon_0(\epsilon+1)\rho^3}$$
$$\times\begin{cases} 0, & \frac{ct}{\rho} < 1 \\ 1-\frac{1}{\epsilon-1}\frac{\epsilon^2}{(\epsilon+1)^{3/2}}\left(\frac{c^2t^2}{\rho^2}+\frac{2\epsilon}{\epsilon+1}\right)\left(\frac{c^2t^2}{\rho^2}-\frac{\epsilon}{\epsilon+v1}\right)^{-5/2}, & 1 < \frac{ct}{\rho} < \sqrt{\epsilon} \\ 2, & \frac{ct}{\rho} > \sqrt{\epsilon} \end{cases} \tag{88}$$

From (88), it is seen that the amplitude of the delta-function in (88) includes the factors $1/\rho^2$. It is concluded that when the horizontal dipole is on the boundary, the far pulsed field along the boundary decreases with $1/\rho^2$, which is characteristic of the surface-wave or lateral pulse.

Assuming that the dipole source is located on the boundary in Region 1(the dielectric), at any distance ρ, the electric field $E_{2\rho}(\rho, 0; t)$ is always 0 until the instant $t = \rho/c$. The magnitude of the electric field component $E_{2\rho}(\rho, 0; t)$ increases momentarily to infinite and decreases to the value

$$E_{2\rho}(\rho, 0; t) = -\frac{1}{2\pi\epsilon_0\rho^3}\frac{3\epsilon^3 + \epsilon^2 - \epsilon + 1}{\epsilon^2 - 1}. \tag{89}$$

The first pulse arrives at $t = \rho/c$ has travelled along the boundary in Region 2 (air) with the velocity c. The magnitude of the field component varies with time according to

$$\begin{aligned} E_{2\rho}(\rho, 0; t) = {} & \frac{1}{2\pi\epsilon_0(\epsilon+1)\rho^3}\left[1 - \frac{1}{\epsilon-1}\frac{\epsilon^2}{(\epsilon+1)^{3/2}}\left(\frac{c^2t^2}{\rho^2} + \frac{2\epsilon}{\epsilon+1}\right)\right. \\ & \left.\cdot\left(\frac{c^2t^2}{\rho^2} - \frac{\epsilon}{\epsilon+1}\right)^{-5/2}\right]. \end{aligned} \tag{90}$$

until $t = \sqrt{\epsilon}\rho/c$ when it approaches the value

$$E_{2\rho}(\rho, 0; t) = \frac{1}{2\pi\epsilon_0(\epsilon+1)\rho^3}\left[1 - \frac{\epsilon+3}{\epsilon^2(\epsilon-1)}\right]. \tag{91}$$

At this instant, the magnitude of the field component $E_{2\rho}(\rho, 0; t)$ increases momentarily to infinite and decreases to the value

$$E_{2\rho}(\rho, 0; t) = \frac{2}{2\pi\epsilon_0(\epsilon^2-1)\rho^3}. \tag{92}$$

The second pulse arrives at $t = \sqrt{\epsilon}\rho/c$ has travelled along the boundary in Region 1 (dielectric) with the velocity $c/\sqrt{\epsilon}$.

4. TIME-DEPENDENT COMPONENT $E_{2\phi}$ DUE TO A HORIZONTAL DIPOLE WITH A DELTA-FUNCTION EXCITATION

4.1. Finite Integration for Time-dependent Component $E_{2\phi}$

Similarly, time-dependent component $E_{2\phi}$ due to a horizontal dipole with a delta-function excitation can be written as follows:

$$E_{2\phi}(\rho', \pi/2; t) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{-i\omega t}\widetilde{E}_{2\phi}(\rho', \pi/2; \omega)d\omega. \tag{93}$$

With substitution (10) into (93), we get

$$\begin{aligned} E_{2\phi}(\rho', \pi/2; t) = & \frac{\mu_0}{2\pi^2 c}\int_0^\infty d\omega e^{-i\omega t}\omega \\ & \times\mathrm{Re}\int_0^\infty d\lambda'\lambda'\left\{\frac{\sqrt{\omega^2\epsilon-\lambda'^2}\sqrt{\omega^2-\lambda'^2}}{\omega^2\sqrt{\omega^2\epsilon-\lambda'^2}+\omega^2\epsilon\sqrt{\omega^2-\lambda'^2}}\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right. \\ & \left.+\frac{1}{\sqrt{\omega^2\epsilon-\lambda'^2}+\sqrt{\omega^2-\lambda'^2}}\left[J_0(\lambda'\rho')-\frac{1}{\lambda'\rho'}J_1(\lambda'\rho')\right]\right\}. \end{aligned} \tag{94}$$

With the definition $\lambda' = \omega\xi$, $d\lambda' = \omega d\xi$, (94) reads as

$$\begin{aligned} E_{2\phi}(\rho', \pi/2; t) = & \frac{\mu_0}{2\pi^2 c}\mathrm{Re}\int_0^\infty \xi d\xi\left\{\frac{\sqrt{\epsilon-\xi^2}\sqrt{1-\xi^2}}{\sqrt{\epsilon-\xi^2}+\epsilon\sqrt{1-\xi^2}}\frac{i}{\xi\rho'}\frac{\partial}{\partial t}\right. \\ & \cdot\int_0^\infty d\omega e^{-i\omega t}J_1(\omega\xi\rho')-\frac{1}{\sqrt{\epsilon-\xi^2}+\sqrt{1-\xi^2}} \\ & \times\left[\frac{\partial^2}{\partial t^2}\int_0^\infty d\omega e^{-i\omega t}J_0(\omega\xi\rho')+\frac{i}{\xi\rho'}\frac{\partial}{\partial t}\right. \\ & \left.\left.\cdot\int_0^\infty d\omega e^{-i\omega t}J_1(\omega\xi\rho')\right]\right\}. \end{aligned} \tag{95}$$

Then, (95) can be rewritten as

$$E_{2\phi}(\rho', \pi/2; t) = \frac{\mu_0}{2\pi^2\rho' c}\left[I_2+I_3+I_4\right], \tag{96}$$

where I_2 and I_3 had been solved and shown in (72) and (87). The next task is to evaluate I_4.

$$I_4 = -\frac{\partial^2}{\partial t^2}\,\mathrm{Im}\int_0^\infty \xi d\xi\frac{1}{\sqrt{\epsilon-\xi^2}+\sqrt{1-\xi^2}}\frac{1}{\sqrt{t^2/\rho'^2-\xi^2}}. \tag{97}$$

4.2. Evaluation of I_4

Following the similar procedures in the evaluation of I_2 and I_3, the evaluation of the integral I_4 can be carried out readily. When $t/\rho' < 1$,

$$I_4 = 0. \tag{98}$$

When $1 < t/\rho' < \sqrt{\epsilon}$,

$$\begin{aligned} I_4 = & -\frac{\partial^2}{\partial t^2} \operatorname{Im}\Bigg\{ \int_0^1 \xi d\xi \Bigg[\frac{1}{(\sqrt{\epsilon - \xi^2} + \sqrt{1-\xi^2})\sqrt{t^2/\rho'^2 - \xi^2}} \\ & - \frac{1}{(\sqrt{\epsilon - \xi^2} + \sqrt{1-\xi^2})(-\sqrt{t^2/\rho'^2 - \xi^2})} \Bigg] \\ & + \int_1^{t/\rho'} \xi d\xi \Bigg[\frac{1}{(\sqrt{\epsilon - \xi^2} + i\sqrt{\xi^2 - 1})\sqrt{t^2/\rho'^2 - \xi^2}} \\ & - \frac{1}{(\sqrt{\epsilon - \xi^2} + i\sqrt{\xi^2-1})(-\sqrt{t^2/\rho'^2 - \xi^2})} \Bigg] \Bigg\}. \end{aligned} \tag{99}$$

Because the integrand of the first integral is *real*, the contributing imaginary part is

$$I_4 = -\frac{\partial^2}{\partial t^2} \operatorname{Im} \int_1^{t/\rho'} \xi d\xi \frac{2}{(\sqrt{\epsilon - \xi^2} + i\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2 - \xi^2}}. \tag{100}$$

Then,

$$I_4 = \frac{2}{\epsilon - 1} \frac{\partial^2}{\partial t^2} \int_1^{t/\rho'} \xi d\xi \frac{\sqrt{\xi^2 - 1}}{\sqrt{t^2/\rho'^2 - \xi^2}}. \tag{101}$$

With $\zeta = \xi^2$, it follows

$$I_4 = \frac{1}{\epsilon - 1} \frac{\partial^2}{\partial t^2} \int_1^{t^2/\rho'^2} d\zeta \frac{\sqrt{\zeta - 1}}{\sqrt{t^2/\rho'^2 - \zeta}}. \tag{102}$$

Let $T_0 = t^2/\rho'^2 - 1$, and $x' = \zeta - 1$, the integral in (102) can be rewritten as follows:

$$\vartheta_0^{(6)} = \int_1^{t^2/\rho'^2} d\zeta \frac{\sqrt{\zeta - 1}}{\sqrt{t^2/\rho'^2 - \zeta}} = \int_0^{T_0} \frac{x' dx'}{\sqrt{(T_0 - x')x'}}. \tag{103}$$

With $x = x' + E_0$, $E_0 = 1/(\epsilon + 1)$, and $X_3 = (T_0 + E_0 - x)(x - E_0)$, thus

$$\begin{aligned} \vartheta_0^{(6)} &= \int_{E_0}^{T_0+E_0} \frac{x dx}{X_3^{1/2}} - E_0 \int_{E_0}^{T_0+E_0} \frac{dx}{X_3^{1/2}} \\ &= \frac{\pi}{2}(T_0 + 2E_0) - \pi E_0 = \frac{\pi}{2}\left(\frac{t^2}{\rho'^2} - 1\right). \end{aligned} \tag{104}$$

Thus,

$$I_4 = \frac{\pi}{2(\epsilon-1)}\frac{\partial^2}{\partial t^2}\left(\frac{t^2}{\rho'^2}-1\right), \quad 1 < \frac{t}{\rho'} < \sqrt{\epsilon}. \tag{105}$$

When $t/\rho' > \sqrt{\epsilon}$,

$$\begin{aligned} I_4 \;=\; & -\frac{\partial^2}{\partial t^2}\,\mathrm{Im}\Bigg\{\int_1^{\sqrt{\epsilon}} \xi d\xi \frac{2}{(\sqrt{\epsilon-\xi^2}+i\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}} \\ & + \int_{\sqrt{\epsilon}}^{\frac{t}{\rho'}} \xi d\xi \frac{2}{(i\sqrt{\xi^2-\epsilon}+i\sqrt{\xi^2-1})\sqrt{t^2/\rho'^2-\xi^2}}\Bigg\}. \end{aligned} \tag{106}$$

The imaginary part is as follows:

$$\begin{aligned} I_4 \;=\; & \frac{2}{\epsilon-1}\frac{\partial^2}{\partial t^2}\left\{\int_1^{\sqrt{\epsilon}} \xi d\xi \frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}} + \int_{\sqrt{\epsilon}}^{t/\rho'} \xi d\xi \frac{\sqrt{\xi^2-1}-\sqrt{\epsilon-\xi^2}}{\sqrt{t^2/\rho'^2-\xi^2}}\right\} \\ =\; & \frac{2}{\epsilon-1}\frac{\partial^2}{\partial t^2}\left\{\int_1^{t/\rho'} \xi d\xi \frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}} - \int_{\sqrt{\epsilon}}^{t/\rho'} \xi d\xi \frac{\sqrt{\epsilon-\xi^2}}{\sqrt{t^2/\rho'^2-\xi^2}}\right\}. \end{aligned} \tag{107}$$

In terms of the variable $\zeta = \xi^2$, this becomes

$$I_4 = \frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\left\{\int_1^{t^2/\rho'^2} d\zeta \frac{\sqrt{\zeta-1}}{\sqrt{t^2/\rho'^2-\zeta}} - \int_{\epsilon}^{t^2/\rho'^2} d\zeta \frac{\sqrt{\epsilon-\zeta}}{\sqrt{t^2/\rho'^2-\zeta}}\right\}. \tag{108}$$

The first integral has been addressed in (104). In the next step, the second integral need to be solved. Let $T_\epsilon = t^2/\rho'^2 - \epsilon$, and $y' = \zeta - \epsilon$, the second integral in (108) can be written as

$$\vartheta_\epsilon^{(6)} = \int_\epsilon^{t^2/\rho'^2} d\zeta \frac{\sqrt{\zeta-\epsilon}}{\sqrt{t^2/\rho'^2-\zeta}} = \int_0^{T_\epsilon} \frac{y'dy'}{\sqrt{(T_\epsilon-y')y'}}. \tag{109}$$

With $y = y' + E_0$, $E_0 = 1/(\epsilon+1)$, and $Y_3 = (T_\epsilon + E_0 - y)(y - E_0)$, this becomes

$$\begin{aligned} \vartheta_\epsilon^{(6)} \;=\; & \int_{E_0}^{T_\epsilon+E_0} \frac{ydy}{Y_3^{1/2}} - E_0 \int_{E_0}^{T_0+E_0} \frac{dy}{Y_3^{1/2}} \\ =\; & \frac{\pi}{2}(T_\epsilon + 2E_0) - \pi E_0 \;=\; \frac{\pi}{2}\left(\frac{t^2}{\rho'^2} - \epsilon\right). \end{aligned} \tag{110}$$

So that

$$I_4 = \frac{\pi}{2}, \quad 1 < \frac{t}{\rho'} < \sqrt{\epsilon}. \tag{111}$$

From (98), (105), and (111), it is obtained readily.

$$I_4 = \frac{\pi}{2}\frac{\partial^2}{\partial t^2}\, f_4\left(\frac{t}{\rho'}\right), \tag{112}$$

where

$$f_4\left(\frac{t}{\rho'}\right) = \begin{cases} 0 & , \quad \frac{t}{\rho'} < 1 \\ \dfrac{1}{\epsilon - 1}\left(\dfrac{t^2}{\rho'^2} - 1\right) & , \quad 1 < \dfrac{t}{\rho'} < \sqrt{\epsilon} \\ 1 & , \quad \dfrac{t}{\rho'} > \sqrt{\epsilon} \end{cases}. \tag{113}$$

Obviously, $f_4(1-) = f_4(1+) = 0$ and $f_4(\sqrt{\epsilon}-) = f_4(\sqrt{\epsilon}+) = 1$, it follows that $f_4(t/\rho')$ is everywhere continuous.

$$f_4'\left(\frac{t}{\rho'}\right) = \begin{cases} 0 & , \quad \dfrac{t}{\rho'} < 1 \\ \dfrac{1}{\epsilon - 1}\dfrac{2t}{\rho'^2} & , \quad 1 < \frac{t}{\rho'} < \sqrt{\epsilon} \\ 0 & , \quad \dfrac{t}{\rho'} > \sqrt{\epsilon} \end{cases}. \tag{114}$$

Since $f_4'(1-) = 0$, $f_4'(1+) = 2/[(\epsilon - 1)\rho']$, $f_1'(t/\rho')$ has a step discontinuity of $2/\rho'$ at $t/[(\epsilon - 1)\rho'] = 1$. Similarly, $f_4'(\sqrt{\epsilon}-) = 2\sqrt{\epsilon}/[(\epsilon - 1)\rho']$, $f_4'(\sqrt{\epsilon}+) = 0$, $f_4'(t/\rho')$ has a step discontinuity of $-2\sqrt{\epsilon}/[(\epsilon - 1)\rho']$ at $t/\rho' = \sqrt{\epsilon}$. Thus,

$$f_4''\left(\frac{t}{\rho'}\right) = \begin{cases} 0 & , \quad \dfrac{t}{\rho'} < 1 \\ \dfrac{2}{(\epsilon - 1)\rho'^2} & , \quad 1 < \dfrac{t}{\rho'} < \sqrt{\epsilon} \\ 0 & , \quad \dfrac{t}{\rho'} > \sqrt{\epsilon} \end{cases}. \tag{115}$$

The final expression for I_4 can be written as follows:

$$\begin{aligned} I_4 \;=\; & \frac{c\pi}{(\epsilon - 1)\rho}\left[\delta\left(t - \frac{\rho}{c}\right) - \sqrt{\epsilon}\delta\left(t - \frac{\sqrt{\epsilon}\rho}{c}\right)\right] \\ & + \frac{c^2\pi}{(\epsilon - 1)\rho^2}\begin{cases} 0 & , \quad \dfrac{ct}{\rho} < 1 \\ 1 & , \quad 1 < \dfrac{ct}{\rho} < \sqrt{\epsilon} \\ 0 & , \quad \dfrac{ct}{\rho} > \sqrt{\epsilon} \end{cases}. \end{aligned} \tag{116}$$

4.3. Evaluation of $E_{2\phi}(\rho, \pi/2; t)$

Combined with (72), (87), (96), and (116), it is obtained readily.

$$E_{2\phi}(\rho,\pi/2;t) = \frac{1}{2\pi\epsilon_0 c(\epsilon-1)\rho^2}\left[\delta\left(t-\frac{\rho}{c}\right) - \sqrt{\epsilon}\delta\left(t-\frac{\sqrt{\epsilon}\rho}{c}\right)\right] + \frac{1}{2\pi\epsilon_0(\epsilon-1)\rho^3} \times \begin{cases} 0, & \dfrac{ct}{\rho}<1 \\ 2-\dfrac{1}{\epsilon+1}+\dfrac{\epsilon^2}{(\epsilon+1)^{5/2}}\left(\dfrac{c^2t^2}{\rho^2}-\dfrac{\epsilon}{\epsilon+1}\right)^{-3/2}, & 1<\dfrac{ct}{\rho}<\sqrt{\epsilon} \\ \dfrac{\epsilon-1}{\epsilon+1}, & \dfrac{ct}{\rho}>\sqrt{\epsilon} \end{cases}. \tag{117}$$

Similarly, at the instant $t = \rho/c$, the magnitude of the electric field $E_{2\phi}(\rho, 0; t)$ increases momentarily to infinite and decreases to the value

$$E_{2\phi}(\rho,0;t) = \frac{1}{2\pi\epsilon_0\rho^3}\frac{\epsilon+1}{\epsilon-1}. \tag{118}$$

The magnitude of the electric field varies with time according to

$$E_{2\phi}(\rho,0;t) = \frac{1}{2\pi\epsilon_0(\epsilon-1)\rho^3}\left[2-\frac{1}{\epsilon+1}+\frac{\epsilon^2}{(\epsilon+1)^{5/2}}\left(\frac{c^2t^2}{\rho^2}-\frac{\epsilon}{\epsilon+1}\right)^{-3/2}\right]. \tag{119}$$

until $t = \sqrt{\epsilon}\rho/c$ when it approaches the value

$$E_{2\rho}(\rho,0;t) = \frac{1}{2\pi\epsilon_0\rho^3}\frac{2\epsilon^2+\epsilon+1}{\epsilon(\epsilon^2-1)}. \tag{120}$$

At this instant, the magnitude of the electric field $E_{2\rho}(\rho, 0; t)$ increases momentarily to infinite and decreases to the value

$$E_{2\rho}(\rho,0;t) = \frac{1}{2\pi\epsilon_0(\epsilon+1)\rho^3}. \tag{121}$$

The second pulse has travelled at $t = \sqrt{\epsilon}\rho/c$ along the boundary in Region 1 (the dielectric) with the velocity $c\sqrt{\epsilon}$.

5. TIME-DEPENDENT COMPONENT B_{2Z} DUE TO A HORIZONTAL DIPOLE WITH A DELTA-FUNCTION EXCITATION

When the horizontal electric dipole is excited by a unit moment that is a δ-function pulse in time, the vertical magnetic field, which is *real*, can also be given by the following Fourier transform.

$$B_{2z}(\rho', \pi/2; t) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty d\omega\ e^{-i\omega t}\widetilde{B}_{2z}(\rho', \pi/2; \omega). \qquad (122)$$

With (14), it follows that

$$\begin{aligned} B_{2z}(\rho', \pi/2; t) &= \mathrm{Re}\frac{i\mu}{2\pi^2 c^2}\int_0^\infty d\omega\ e^{-i\omega t} \\ &\quad \cdot \int_0^\infty d\lambda'\ \lambda'^2 \frac{J_1(\lambda'\rho')}{\sqrt{\omega^2-\lambda'^2}+\sqrt{\omega^2\epsilon-\lambda'^2}}. \end{aligned} \qquad (123)$$

With the definition $\lambda' = \omega\xi$, $d\lambda' = \omega d\xi$, (123) reads as

$$\begin{aligned} B_{2z}(\rho', \pi/2; t) &= \mathrm{Re}\frac{i\mu_0}{2\pi^2 c^2}\frac{\partial^2}{\partial t^2}\ \int_0^\infty \frac{\xi^2 d\xi}{\sqrt{\epsilon-\xi^2}+\sqrt{1-\xi^2}} \\ &\quad \cdot \int_0^\infty d\omega\ e^{-i\omega t}J_1(\omega\xi\rho'). \end{aligned} \qquad (124)$$

Taking into account the relationship in (19), it is obtained readily.

$$\begin{aligned} B_{2z}(\rho', \pi/2; t) &= \mathrm{Re}\frac{i\mu_0}{2\pi^2 c^2}\frac{\partial^2}{\partial t^2}\ \int_0^\infty \frac{\xi^2 d\xi}{\sqrt{\epsilon-\xi^2}+\sqrt{1-\xi^2}}\frac{1}{\xi\rho'} \\ &\quad \cdot\left(1-\frac{t}{\sqrt{t^2-\xi^2\rho'^2}}\right) \\ &= \frac{\mu_0}{2\pi^2 c^2\rho'}\frac{\partial^2}{\partial t^2}\ \ \mathrm{Im}\left[\int_0^\infty \frac{\xi d\xi}{\sqrt{\epsilon-\xi^2}+\sqrt{1-\xi^2}}\right. \\ &\quad \left. -\frac{t}{\rho'}\int_0^\infty \frac{\xi d\xi}{(\sqrt{\epsilon-\xi^2}+\sqrt{1-\xi^2})\cdot\sqrt{t^2/\rho'^2-\xi^2}}\right]. \end{aligned} \qquad (125)$$

Following the same procedures in evaluation of I_2, the evaluations of the integrals (125) can be carried out readily. When $t/\rho' < 1$,

$$B_{2z}(\rho', \pi/2; t) = 0. \qquad (126)$$

When $1 < t/\rho' < \sqrt{\epsilon}$,

$$B_{2z}(\rho', \pi/2; t) = -\frac{t}{\rho'}\int_0^\infty \frac{\xi d\xi}{(\sqrt{\epsilon-\xi^2}+i\sqrt{\xi^2-1})\cdot\sqrt{t^2/\rho'^2-\xi^2}}\Bigg]$$
$$= -\frac{\mu_0}{2\pi^2c^2\rho'}\frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\int_0^\infty \xi d\xi\left[\sqrt{\xi^2-1}-\frac{t}{\rho'}\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}}\right]. \tag{127}$$

With $\zeta = \xi^2$,

$$B_{2z}(\rho', \pi/2; t) = \frac{\mu_0}{2\pi^2c^2\rho'}\frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\left[\frac{t}{\rho'}\int_1^{t^2/\rho'^2} d\zeta\frac{\sqrt{\zeta-1}}{\sqrt{t^2/\rho'^2-\zeta}}\right]. \tag{128}$$

The integral in (128) had been evaluated in (104). Thus

$$B_{2z}(\rho', \pi/2; t) = \frac{\mu_0}{4\pi c^2\rho'}\frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\left[\frac{t}{\rho'}\left(\frac{t^2}{\rho'^2}-1\right)\right], \quad 1 < \frac{t}{\rho'} < \sqrt{\epsilon}. \tag{129}$$

When $t/\rho' > \sqrt{\epsilon}$,

$$\begin{aligned} B_{2z}(\rho', \pi/2; t) = &\frac{\mu_0}{2\pi^2c^2\rho'}\frac{\partial^2}{\partial t^2}\,\mathrm{Im}\Bigg[\int_0^{\sqrt{\epsilon}}\frac{\xi d\xi}{\sqrt{\epsilon-\xi^2}+i\sqrt{\xi^2-1}} \\ &-\frac{t}{\rho'}\int_0^{\sqrt{\epsilon}}\frac{\xi d\xi}{(\sqrt{\epsilon-\xi^2}+i\sqrt{\xi^2-1})\cdot\sqrt{t^2/\rho'^2-\xi^2}} \\ &+\int_{\sqrt{\epsilon}}^\infty\frac{\xi d\xi}{i\sqrt{\xi^2-\epsilon}+i\sqrt{\xi^2-1}} \\ &-\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^\infty\frac{\xi d\xi}{(i\sqrt{\xi^2-\epsilon}+i\sqrt{\xi^2-1})\cdot\sqrt{t^2/\rho'^2-\xi^2}}\Bigg] \\ = &-\frac{\mu_0}{2\pi^2c^2\rho'}\cdot\frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\Bigg[\int_0^{\sqrt{\epsilon}}\xi d\xi\sqrt{\xi^2-1} \\ &-\frac{t}{\rho'}\int_0^{\sqrt{\epsilon}}\xi d\xi\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}} \\ &+\int_{\sqrt{\epsilon}}^\infty\xi d\xi\left(\sqrt{\xi^2-1}-\sqrt{\xi^2-\epsilon}\right) \\ &-\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^\infty\xi d\xi\frac{(\sqrt{\xi^2-1}-\sqrt{\xi^2-\epsilon})}{\sqrt{t^2/\rho'^2-\xi^2}}\Bigg]. \end{aligned} \tag{130}$$

Then

$$\begin{aligned}B_{2z}(\rho',\pi/2;t) = &-\frac{\mu_0}{2\pi^2c^2\rho'}\cdot\frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\\&\times\left[\int_0^\infty \xi d\xi\sqrt{\xi^2-1}-\frac{t}{\rho'}\int_0^\infty \xi d\xi\frac{\sqrt{\xi^2-1}}{\sqrt{t^2/\rho'^2-\xi^2}}\right.\\&\left.-\int_{\sqrt{\epsilon}}^\infty \xi d\xi\sqrt{\xi^2-\epsilon}+\frac{t}{\rho'}\int_{\sqrt{\epsilon}}^\infty \xi d\xi\frac{\sqrt{\xi^2-\epsilon}}{\sqrt{t^2/\rho'^2-\xi^2}}\right]. \quad (131)\end{aligned}$$

In terms of the variable $\zeta=\xi^2$, $d\zeta=2\xi d\xi$, it follows

$$\begin{aligned}B_{2z}(\rho',\pi/2;t) \;=\; &\frac{\mu_0}{2\pi^2c^2\rho'}\frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\left[\frac{t}{\rho'}\left(\int_1^{t^2/\rho'^2}d\zeta\frac{\sqrt{\zeta-1}}{\sqrt{t^2/\rho'^2-\zeta}}\right.\right.\\&\left.\left.-\int_\epsilon^{t^2/\rho'^2}d\zeta\frac{\sqrt{\zeta-\epsilon}}{\sqrt{t^2/\rho'^2-\zeta}}\right)\right]. \quad (132)\end{aligned}$$

The above two integrals have been evaluated in (104) and (110). Then, we get

$$B_{2z}(\rho',\pi/2;t) \;=\; \frac{\mu_0}{4\pi c^2\rho'}\frac{1}{\epsilon-1}\frac{\partial^2}{\partial t^2}\left[\frac{t}{\rho'}(\epsilon-1)\right], \quad \frac{t}{\rho'}>\sqrt{\epsilon}. \quad (133)$$

With substitutions (126), (129), and (133) into (125), we get

$$B_{2z}(\rho',\pi/2;t) \;=\; \frac{\mu_0}{4\pi c^2\rho'}\frac{\partial^2}{\partial t^2}f_7\left(\frac{t}{\rho'}\right), \quad (134)$$

where

$$f_7\left(\frac{t}{\rho'}\right) \;=\; \frac{1}{\epsilon-1}\frac{t}{\rho'}\begin{cases}0 & , \quad \dfrac{t}{\rho'}<1\\ \dfrac{t^2}{\rho'^2}-1 & , \quad 1<\dfrac{t}{\rho'}<\sqrt{\epsilon}\\ \epsilon-1 & , \quad \dfrac{t}{\rho'}>\sqrt{\epsilon}\end{cases}. \quad (135)$$

Since $f(1-)=f(1+)=0$ and $f(\sqrt{\epsilon}-)=f(\sqrt{\epsilon}+)=\sqrt{\epsilon}$, $f_7(t/\rho')$ is everywhere continuous.

$$f_7'\left(\frac{t}{\rho'}\right) \;=\; \frac{1}{(\epsilon-1)\rho'}\begin{cases}0 & , \quad \dfrac{t}{\rho'}<1\\ \dfrac{3t^2}{\rho'^2}-1 & , \quad 1<\dfrac{t}{\rho'}<\sqrt{\epsilon}\\ \epsilon-1 & , \quad \dfrac{t}{\rho'}>\sqrt{\epsilon}\end{cases}. \quad (136)$$

Since $f_7'(1-) = 0$, $f_7'(1+) = 2/[(\epsilon-1)\rho']$, $f'(t/\rho')$ has a step discontinuity of $2/[(\epsilon-1)\rho']$ at $\frac{t}{\rho'} = 1$. Similarly, $f'(\sqrt{\epsilon}-) = (3\epsilon-1)/[(\epsilon-1)\rho']$, $f'(\sqrt{\epsilon}+) = 1/\rho'$, $f'(t/\rho')$ has a step discontinuity of $-2\epsilon/[(\epsilon-1)\rho']$ at $t/\rho' = \sqrt{\epsilon}$. Thus

$$f_7''\left(\frac{t}{\rho'}\right) = \frac{1}{(\epsilon-1)\rho'^3}\begin{cases} 0 & , \quad \dfrac{t}{\rho'} < 1 \\ 6t & , \quad 1 < \dfrac{t}{\rho'} < \sqrt{\epsilon} \\ 0 & , \quad \dfrac{t}{\rho'} > \sqrt{\epsilon} \end{cases}. \tag{137}$$

With substitution (137) into (134) and $\rho' = \rho/c$, the final formula for the vertical magnetic field is expressed as follows:

$$\begin{aligned} B_{2z}(\rho, \pi/2; t) &= \frac{\mu_0}{2\pi\rho^2}\frac{1}{\epsilon-1}\left[\delta\left(t-\frac{\rho}{c}\right) - \epsilon\delta\left(t-\frac{\sqrt{\epsilon}\rho}{c}\right)\right] \\ &\quad -\frac{\mu_0 c}{2\pi\rho^3}\frac{1}{\epsilon-1}\begin{cases} 0 & , \quad \dfrac{ct}{\rho} < 1 \\ \dfrac{3ct}{\rho} & , \quad 1 < \dfrac{ct}{\rho} < \sqrt{\epsilon} \\ 0 & , \quad \dfrac{ct}{\rho} > \sqrt{\epsilon} \end{cases}. \end{aligned} \tag{138}$$

6. DISCUSSIONS AND CONCLUSIONS

We start this paper with the Fourier-Bessel integral representations for the electromagnetic field due to a delta-function current in a horizontal electric dipole located on the planar boundary between air and a homogeneous dielectric. Similar to the case of the vertical dipole, the tangential electric field components due to a horizontal dipole consist of a delta-function pulse travelling in the air with the velocity c, the oppositely directed delta-function pulse travelling in the dielectric with the velocity $c/\epsilon^{1/2}$ for the component E_ρ and the velocity $c\epsilon^{1/2}$ for the component E_ϕ, and the final static electric fields due to the charge left on the dipole. The structures of the three field components $E_{2\rho}(\rho, 0; t)$, $E_{2\phi}(\rho, \pi/2; t)$, and $E_{2z}(\rho, \pi/2; t)$ between the two delta-function are different each other. Also the structures of the field components are, of course, different from those of the field components for the case of the vertical dipole.

As addressed in [16], the time-dependent component $E_{2\rho}(\rho, t)$ due to a vertical dipole can not be expressed in terms of elementary functions. Similarly, also the three time-dependent components

$E_{2z}(\rho, 0; t)$, $B_{2\rho}(\rho, \pi/2; t)$, and $B_{2\phi}(\rho, \pi/2; t)$ due to a horizontal electric dipole with a delta-function excitation are not expressible in terms of elementary functions.

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