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MULTIPLE-SCALE ANALYSIS OF PLANE WAVE REFRACTION AT A DIELECTRIC SLAB WITH KERR-TYPE NONLINEARITY

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Abstract—Multiple-scale analysis is employed for the analysis of plane wave refraction at a nonlinear slab. It will be demonstrated that the perturbation method will lead to a nonuniformly valid approximation to the solution of the nonlinear wave equation. To construct a uniformly valid approximation, we will exploit multiple-scale analysis. Using this method, we will derive the zeroth-order approximation to the solution of the nonlinear wave equation analytically. This approximate solution clearly shows the effects of self-phase modulation (SPM) and cross-phase modulation (XPM) on plane wave refraction at the nonlinear slab. In fact, the obtained zeroth-order approximation is very accurate and there is not any need for derivation of higher-order approximations. As will be shown, the proposed method can be generalized to the rigorous study of nonlinear wave propagation in one-dimensional photonic band-gap structures.

1. INTRODUCTION

In recent years, there have been many studies on nonlinear wave propagation in one-dimensional photonic band-gap structures as a result of which a rich set of phenomena such as bistability, gap and Bragg solitons has been discovered in these structures [1, 4]. There is an increasing interest among researchers for theoretical study and experimental observation of these phenomena as well as their applications to optical signal processing [5]. The coupled mode theory or the nonlinear Shrödinger equation (NLSE) have been conventionally used for investigation of nonlinear wave propagation in these structures near the Bragg wavelength and where the grating is very shallow [5]. However, rigorous study of nonlinear wave propagation in these

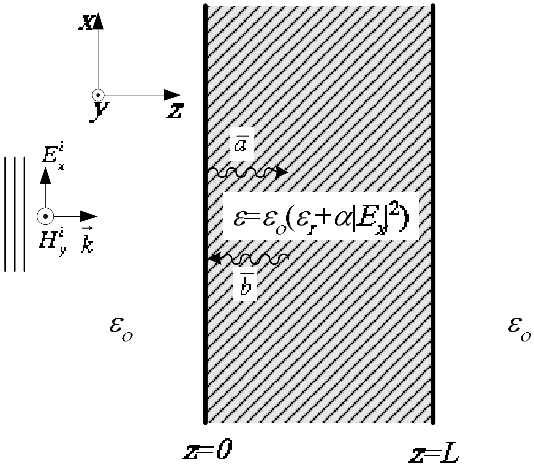

Figure 1. Plane wave refraction at a slab with Kerr-type nonlinearity.

structures opens up a way to interpret the phenomena which cannot be accurately explained using the conventional methods. A rigorous study of wave refraction in a nonlinear slab along with a method applicable to nonlinear photonic band-gap structures will provide a reliable approach for analyzing nonlinear wave propagation in these structures.

To our knowledge, the problem of plane wave refraction at nonlinear slabs has not been addressed in the literature. The majority of authors have concentrated on the analysis of nonlinear slab-guided waves where the nonlinear propagation modes of a nonlinear slab sandwiched between two or more layers of homogeneous linear dielectrics have been studied [6, 8].

It is the aim of this work to study the refraction of normally incident plane wave at a dielectric slab with Kerr-type nonlinearity. As a first step, plane wave refraction in the slab is modeled by wave propagation in a nonlinear transmission line. A multiple-scale analysis will then be exploited for derivation of the solution to the resulting nonlinear wave equation.

2. FORMULATION

Fig. 1 depicts a plane wave incident on a nonlinear slab. It is assumed that the slab is made of Kerr-type nonlinear material for which the refractive index is given by $n = n_o + n_2|E_x|^2$. Therefore, the relative permittivity of the slab is [10]

$$\varepsilon_r = (n_o + n_2|E_x|^2)^2 \approx n_o^2 + 2n_o n_2|E_x|^2 \tag{1}$$

where n_o is the linear refractive index of medium which is dimensionless, and n_2 is the nonlinear refractive index with a unit of m^2/V^2. Since the problem is one-dimensional and the incident plane wave is normal to the slab, only E_x and H_y field components exist. Maxwell's equations in the slab are given as follows

$$\frac{dE_x}{dz} = -j\omega\mu_o H_y \tag{2}$$

$$\frac{dH_y}{dz} = -j\omega\epsilon_o\varepsilon_r E_x. \tag{3}$$

Hence, the nonlinear wave equation in the slab is

$$\frac{d^2E_x}{dz^2} + k_o^2(n_o^2 + \alpha|E_x|^2)E_x = 0 \tag{4}$$

in which $k_o = \omega\sqrt{\mu_o\epsilon_o}$ and $\alpha = 2n_on_2$.

According to (2) and (3), we can model wave propagation in the slab with wave propagation in a nonlinear transmission line with an inductance per unit length of $L = \mu_o$ and a capacitance per unit length of $C = \epsilon_o(n_o^2 + \alpha|E_x|^2)$ where the electric field E_x and magnetic field H_y play the role of voltage and current on this line, respectively. In the present work, we concentrate on the derivation of transmission characteristics of the nonlinear slab. To this end, an ideal voltage source is assumed to be connected to the equivalent nonlinear transmission line at $z = 0$. Since the region $z > L$ is free space, we terminate the equivalent nonlinear transmission line to free-space characteristic impedance $Z_o = \sqrt{\mu_o/\epsilon_o}$. Therefore, the nonlinear equation given by (4) must be solved with a hard boundary condition at $z = 0$ and an impedance boundary condition at $z = L$.

It should be noted that α is typically very small compared with the linear relative permittivity of nonlinear materials. This allows us to use perturbation method for solving the nonlinear wave equation. In this method, the electric field E_x is expanded to a power series of α [9], i.e.,

$$E_x = \sum_{m=0}^{\infty} \alpha^m E_m = E_o + \alpha E_1 + \alpha^2 E_2 + \cdots \tag{5}$$

If this expansion is substituted in the nonlinear wave equation and the terms with equal powers of α are balanced, the differential equations for determining the coefficients of this expansion are derived. For example, the differential equations for the zeroth- and first-order perturbation approximation are given by

$$\frac{d^2E_o}{dz^2} + k_o^2 n_o^2 E_o = 0 \tag{6}$$

$$\frac{d^2E_1}{dz^2} + k_o^2 n_o^2 E_1 = -k_o^2 |E_o|^2 E_o. \tag{7}$$

Note that the boundary conditions for (6) are those of the original problem whereas the boundary conditions for (7) and other higher-order approximations are zero. In other words, we assume $E_m = 0$ and $\frac{dE_m}{dz} = 0$ at $z = 0$ and $z = L$ for every $m \geq 1$. From (6), it is obvious that the zeroth-order approximation E_o is given by

$$E_o = a_o e^{-j\beta z} + b_o e^{j\beta z} \tag{8}$$

in which a_o and b_o are complex numbers representing the amplitude and phase of the forward and backward waves and $\beta = k_o n_o$. It can be easily shown that

$$|E_o|^2 = |a_o|^2 + |b_o|^2 + a_o b_o^* e^{-2j\beta z} + a_o^* b_o e^{2j\beta z}. \tag{9}$$

Now, from (9), it is obvious that if $|E_o|^2$ is multiplied by E_o, some terms proportional to $e^{-j\beta z}$ and $e^{j\beta z}$ will appear in the right-hand side of (7). These terms are in fact the solutions of the homogeneous differential equation. This means that some nonphysical terms (secular terms [9]) of the form $ze^{-j\beta z}$ and $ze^{j\beta z}$ will appear in the solution of E_1. This implies that if the slab thickness approaches infinity the solution will be unbounded. In fact, the solution given by perturbation series (5) does not converge uniformly to the solution of the original nonlinear wave equation, i.e. with increasing z the error of the approximate solution increases rapidly. Note that although the individual terms of the perturbation series are secular, the secularity disappears when the series is summed up [9]. For removing this secularity, use will be made of multiple-scale analysis to be explained in the next section.

2.1. Multiple-Scale Analysis for Removing Secularity

For eliminating the most secular terms to all orders, a new variable $\zeta = \alpha z$ is introduced. Even though the exact solution $E_x(z)$ is a function of z alone, multiple-scale analysis seeks solutions which are functions of both ζ and z treated as *independent* variables. We wish to emphasize that expressing E_x as a function of two variables is merely a mathematical technique to remove secularity; the actual solution has z and ζ related by $\zeta = \alpha z$ so that z and ζ are ultimately not independent.

The formal procedure consists of assuming a perturbation expansion of the form

$$E_x(z) = E_o(z, \zeta) + \alpha E_1(z, \zeta) + \cdots \tag{10}$$

Chain rule for partial differentiation is to be used for computing derivatives of $E_x(z)$. Hence, for the first-order derivative with respect to z, we have

$$\frac{dE_x}{dz} = \left(\frac{\partial E_o}{\partial z} + \frac{\partial E_o}{\partial \zeta}\frac{d\zeta}{dz}\right) + \alpha\left(\frac{\partial E_1}{\partial z} + \frac{\partial E_1}{\partial \zeta}\frac{d\zeta}{dz}\right) + \cdots \tag{11}$$

However, since $\zeta = \alpha z$,

$$\frac{dE_x}{dz} = \frac{\partial E_o}{\partial z} + \alpha\left(\frac{\partial E_o}{\partial \zeta} + \frac{\partial E_1}{\partial z}\right) + O(\alpha^2). \tag{12}$$

Similarly, it can be shown that for the second-order derivative with respect to z, we have

$$\frac{d^2 E_x}{dz^2} = \frac{\partial^2 E_o}{\partial z^2} + \alpha\left(2\frac{\partial^2 E_o}{\partial \zeta \partial z} + \frac{\partial^2 E_1}{\partial z^2}\right) + O(\alpha^2). \tag{13}$$

If this relation is substituted in the nonlinear wave equation and the terms with equal powers of α are balanced, we will obtain the following differential equations for $E_o(z,\zeta)$ and $E_1(z,\zeta)$

$$\frac{\partial^2 E_o}{\partial z^2} + \beta^2 E_o = 0 \tag{14}$$

$$\frac{\partial^2 E_1}{\partial z^2} + \beta^2 E_1 = -k_o^2|E_o|^2 E_o - 2\frac{\partial^2 E_o}{\partial \zeta \partial z}. \tag{15}$$

It is obvious that E_o is given by

$$E_o(z,\zeta) = a(\zeta)e^{-j\beta z} + b(\zeta)e^{j\beta z}. \tag{16}$$

$a(\zeta)$ and $b(\zeta)$ will be determined under the condition that secular terms do not appear in the solution to (15). From (16), the right-hand side of (15) is

$$\left(2j\beta\frac{da}{d\zeta} - k_o^2 a(|a|^2 + 2|b|^2)\right)e^{-j\beta z} - \left(2j\beta\frac{db}{d\zeta} + k_o^2 b(2|a|^2 + |b|^2)\right)e^{j\beta z}$$
$$-k_o^2 a^2 b^* e^{-3j\beta z} - k_o^2 a^* b^2 e^{3j\beta z}. \tag{17}$$

If the coefficients of $e^{-j\beta z}$ and $e^{j\beta z}$ are nonzero, then the solution to E_1 would be secular. To preclude the appearance of secularity, we require that $a(\zeta)$ and $b(\zeta)$ satisfy

$$\begin{cases} \frac{da}{d\zeta} = -\frac{jk_o^2}{2\beta}a(|a|^2 + 2|b|^2) \\ \frac{db}{d\zeta} = \frac{jk_o^2}{2\beta}b(2|a|^2 + |b|^2) \end{cases} \tag{18}$$

For solving these equations, we write $a(\zeta)$ and $b(\zeta)$ in their polar representation, i.e.

$$a(\zeta) = R_1(\zeta)e^{j\theta_1(\zeta)} \tag{19}$$

$$b(\zeta) = R_2(\zeta)e^{j\theta_2(\zeta)}. \tag{20}$$

It can be easily shown that $dR_1/d\zeta = 0$ and $dR_2/d\zeta = 0$ and the differential equations for determining the angles are

$$\begin{cases} \frac{d\theta_1}{d\zeta} = -\frac{k_o}{2n_o}(R_1^2 + 2R_2^2) \\ \frac{d\theta_2}{d\zeta} = \frac{k_o}{2n_o}(2R_1^2 + R_2^2) \end{cases} \tag{21}$$

Hence, $a(\zeta)$ and $b(\zeta)$ are given by

$$a(\zeta) = R_1(0)e^{j\theta_1(0)}e^{-j\frac{k_o}{2n_o}(R_1^2(0)+2R_2^2(0))\zeta} \tag{22}$$

$$b(\zeta) = R_2(0)e^{j\theta_2(0)}e^{j\frac{k_o}{2n_o}(2R_1^2(0)+R_2^2(0))\zeta}. \tag{23}$$

Using complex representation of $a(\zeta)$ and $b(\zeta)$, we will arrive at the following relations

$$a(\zeta) = ae^{-j\frac{k_o}{2n_o}(|a|^2+2|b|^2)\zeta} \tag{24}$$

$$b(\zeta) = be^{j\frac{k_o}{2n_o}(2|a|^2+|b|^2)\zeta} \tag{25}$$

in which $a = R_1(0)e^{j\theta_1(0)}$ and $b = R_2(0)e^{j\theta_2(0)}$ are complex constants which are found by satisfying the boundary conditions. This solution clearly shows the simultaneous effects of SPM and XPM on wave refraction. In another word, the nonlinearity of the medium has imposed a nonlinear phase shift on the forward and backward waves. As can be seen, the forward and backward waves impose nonlinear phase shift on themselves (SPM) and on each other (XPM).

Now, we concentrate our attention to the calculation of a and b from the boundary conditions. It is assumed that the nonlinear transmission line is connected to a voltage source with a phasor E_s at $z = 0$. From (16), the boundary condition at $z = 0$ is

$$E_s = a + b. \tag{26}$$

Also, it is assumed that the nonlinear transmission line is terminated to impedance Z_L at $z = L$. Therefore, the boundary condition at this interface will be

$$\frac{E_o(L, \alpha L)}{H_o(L, \alpha L)} = Z_L. \tag{27}$$

Hence, we need to find an expression for $H_o(L, \alpha L)$. It can be simply shown that the magnetic field at $z = L$ is

$$\begin{aligned} H_o(L, \alpha L) &= \frac{a}{Z_c}\left(1 + \frac{\alpha}{2n_o^2}(|a|^2 + 2|b|^2)\right) e^{-j(\beta+\phi_1)L} \\ &\quad -\frac{b}{Z_c}\left(1 + \frac{\alpha}{2n_o^2}(2|a|^2 + |b|^2)\right) e^{j(\beta+\phi_2)L} \end{aligned} \tag{28}$$

in which $Z_c = Z_o/n_o$, $\phi_1 = \frac{k_o\alpha}{2n_o}(|a|^2 + 2|b|^2)L$, and $\phi_2 = \frac{k_o\alpha}{2n_o}(2|a|^2 + |b|^2)L$. For the sake of simplicity, we assume E_s to be a real number and normalize a and b to this real quantity. Using (16) and (28) for computing $E_o(L, \alpha L)$, we will derive the following nonlinear equations for determination of $\bar{a} = a/E_s$ and $\bar{b} = b/E_s$

$$\bar{a} + \bar{b} = 1 \tag{29}$$

$$\frac{\bar{b}}{\bar{a}} e^{j\frac{3k_o}{2n_o}\alpha L E_s^2(|\bar{a}|^2+|\bar{b}|^2)} = \frac{\bar{Z}\left(1 + \frac{\alpha}{2n_o^2}E_s^2(|\bar{a}|^2 + 2|\bar{b}|^2)\right) - 1}{\bar{Z}\left(1 + \frac{\alpha}{2n_o^2}E_s^2(2|\bar{a}|^2 + |\bar{b}|^2)\right) + 1} e^{-2j\beta L} \tag{30}$$

where $\bar{Z} = Z_L/Z_c$ is the load impedance normalized to the characteristic impedance of the line in the linear regime. Since $\bar{a}$ and $\bar{b}$ are complex numbers, these relations form a system of four nonlinear equations to be numerically solved. These quantities are the only unknowns that characterize the solution. After their evaluation, we would be able to derive the electric and magnetic fields. These equations can be easily solved using the conventional methods for finding the roots of a nonlinear system of equations. The fact that facilitates the derivation of the roots of this system is that there exists a good initial guess for the solution when E_s is relatively weak. In this case, $\bar{a}$ and $\bar{b}$ are nearly equal to their counterparts in the linear regime. Hence, for derivation of the roots of the system for a specific value of the initial field, E_s can initially be assumed to be weak, so that the solution in the linear regime can be regarded as the initial guess for the derivation of $\bar{a}$ and $\bar{b}$. Later this parameter is gradually increased to the desired value so that the solution in the last step can be regarded as the initial guess for the derivation of the solution in the present step until the solution for desired value of E_s is found.

3. NUMERICAL RESULTS

Using the method outlined in the last section, we have studied the plane wave refraction in a nonlinear slab made of Type-RN Corning glass ($n_o = 2.46$ and $n_2 = 1.25 \times 10^{-18}\,\mathrm{m}^2/\mathrm{V}^2$) at $\lambda_o = 1.53\,\mu$m. We have assumed $L = \lambda_o$ and $Z_L = Z_o$. The variation of the magnitude of the reflection coefficient at $z = 0$ with respect to E_s has been depicted in Fig. 2. As it can be seen, in contrast to the linear regime in which the reflection coefficient is independent of the magnitude of the incident electric field, the reflection coefficient increases as the intensity of the electric field E_s increases. The variation of transmission coefficient seen at $z = L$ also has been depicted in Fig. 3. It shows that the transmission coefficient increases slightly as the intensity of the incident electric field increases.

The normalized magnitude and real part of the electric field phasor in the slab for three different values of E_s have been shown in Fig. 4 and Fig. 5, respectively. These figures clearly illustrate the simultaneous effects of SPM and XPM on the wave refraction in the nonlinear slab. As E_s is increased, the distribution of the field in the slab deviates rapidly from the field distribution in the linear regime.

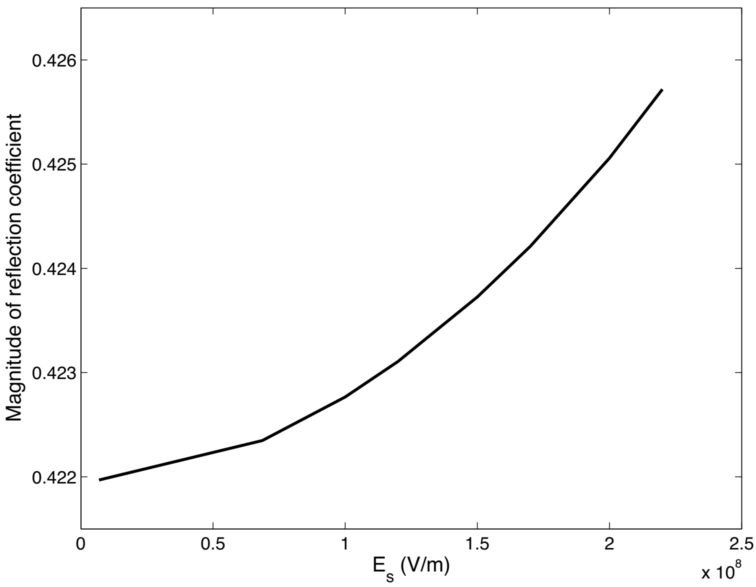

Figure 2. Variation of magnitude of reflection coefficient at $z = 0$ with E_s.

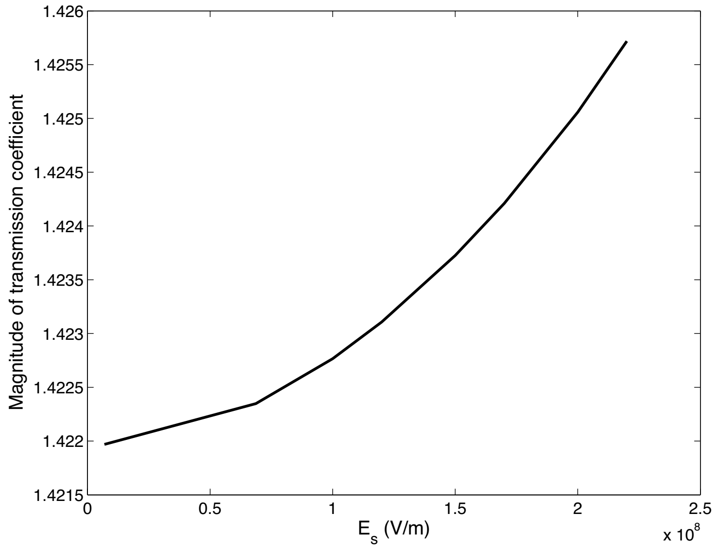

Figure 3. Variation of magnitude of transmission coefficient at $z = L$ with E_s.

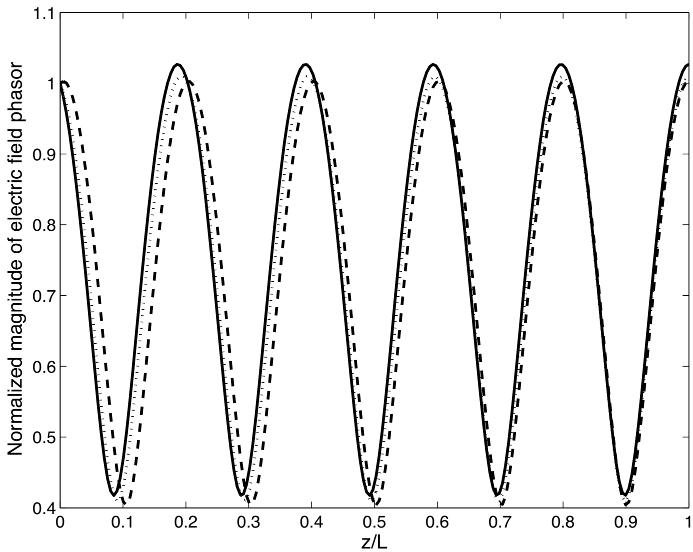

Figure 4. Normalized magnitude of electric field phasor, $|E_x/E_s|$, for three different values of E_s. Solid line: $E_s = 6.87 \times 10^6$ V/m. Dotted line: $E_s = 12 \times 10^7$ V/m. Dashed line: $E_s = 22 \times 10^7$ V/m.

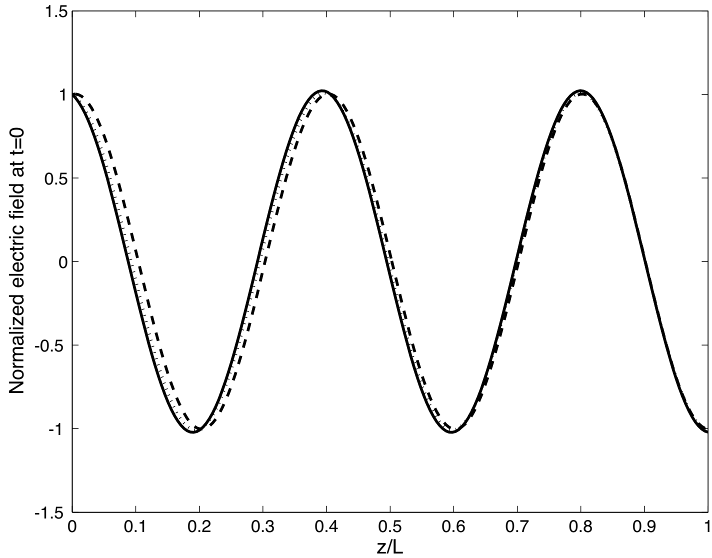

Figure 5. Normalized electric field at $t = 0$ (real part of E_x/E_s) for three different values of E_s. Solid line: $E_s = 6.87 \times 10^6$ V/m. Dotted line: $E_s = 12 \times 10^7$ V/m. Dashed line: $E_s = 22 \times 10^7$ V/m.

4. CONCLUSIONS

A perturbation approach based on a multiple-scale analysis was proposed for the study of plane wave refraction at a slab with Kerr-type nonlinearity. A number of alternative analytical and numerical techniques could have been used for this analysis. For example, a Jacobian elliptic function can be used as the analytical solution of the nonlinear wave equation [11]. The latter suffers from a number of disadvantages. Firstly, since Jacobian elliptic functions do not have explicit representations, satisfaction of complicated boundary conditions such as impedance boundary condition, which was considered in the present article, is very difficult. Secondly, in this approach, the solution cannot be easily subdivided into forward and backward waves and the reflection and transmission coefficients cannot be easily defined. Thirdly, the generalization of this approach for the study of nonlinear wave propagation in one-dimensional photonic band-gap structures is very involved. A number of numerical methods such as FDTD or BPM can also be regarded as candidates for the study of the present problem, but these methods are computationally inefficient and suffer from high memory requirements. In contrast to the aforementioned methods, our proposed method clearly shows the simultaneous effects of SPM and XPM on the wave refraction and it can be simply generalized to the rigorous analysis of nonlinear wave propagation in one-dimensional photonic band-gap structures. As shown, the only numerical step in our method is the solution of a system of nonlinear equations which can be simply accomplished using existing efficient methods such as Newton-Raphson method.

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