

ELECTROMAGNETIC SCATTERING BY A SET OF OBJECTS: AN INTEGRAL METHOD BASED ON SCATTERING PROPERTIES

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Abstract—The paper presents the Scattering Operator Method, which is devoted to the problem of scattering from a set of N cylindrical objects. By contrast with the Scattering Matrix Method, which has been used by many groups in the last twenty years, it applies to any kind of cylinder shape, regardless of the relative location of the cylinders. The theory is based on a mathematical result: it is possible to define in the vicinity of the surface of each cylinder two complementary parts of the field: the total incident field and the field scattered by this cylinder. These two parts are the Calderon projectors of the values of the total fields on the surface of the cylinder. The validity of the method is checked on two examples. It is shown that the theory avoids some problems encountered in integral method like evaluations of singular or hypersingular integrals, or instabilities due to internal resonance of objects.

1. INTRODUCTION

The problem of electromagnetic scattering by a set of N objects has been widely studied in the last fifteen years, due to the increasing interest devoted to photonic crystals, metamaterials or left-handed materials. Some authors have used classical and general methods of Electromagnetics like FDTD or finite-element method [1, 2] while other ones have developed the Scattering Matrix Method (SMM) [3–7]. The SMM, which is quite well adapted to the problem of scattering from a finite-size photonic crystal, allows one to compute the scattering properties of a large set of objects (of the order of 1000 for 2D cylindrical objects) with remarkable precision stability and rapidity

using simple PC. However, this method suffers from some limitations. For example, it cannot handle structures like Split Ring Resonators (SRRs) proposed by J. B. Pendry to make metamaterials with negative permeability [8–10]. The subject of metamaterials [11–14] becomes increasingly important and there exists a need for accurate numerical modeling tools in this domain. Thus, we sought for a generalization of SMM, based on the same basic principles in order to hold the stability of the method, but able to deal with arbitrary kinds of objects, regardless of their relative location.

This generalization, called Scattering Operator Method (SOM) relies on the following mathematical result: it is possible to separate in the vicinity of each object a total incident field (the sum of the imposed incident field and of the fields scattered by the other objects towards this object) and the field scattered by this object. In the present paper, we define as “vicinity of the surface” a thin strip placed in vacuum and including the surface. This result can be expressed in a very simple mathematical form: the total incident field on the object and the field scattered by the object are the Calderon projectors of the total field (and its normal derivative) on the surface of the object. The SOM relies on the use of the same basic concepts as the SMM and the final equation takes the same form, matrices being replaced by operators. The validity of the theory is verified for two examples.

2. PRESENTATION OF THE PROBLEM AND NOTATION

For the sake of simplicity, the problem of scattering by N objects will be presented in the two-dimensional (2D) case with a s -polarized incident plane wave (electric field parallel to the cylinder axes). The extension of the theory to the other polarization or to the 3D case does not in essence change the problem, even though for the 3D case, it leads to a significant increase of the algebraic complexity of the formalism. The problem is schematized in Figure 1. A set of N cylindrical objects ($N = 4$ in the figure), parallel to the z axis, of cross sections C_j with boundaries Σ_j , is illuminated by a s -polarized incident wave with wavelength $\lambda_0 = 2\pi/k_0$ in vacuum. For simplicity, it is assumed that the incident field is imposed by sources placed at infinity, but the final equations hold true for incident fields generated by sources placed at finite distance from the objects (for example a point source). In the special case where this incident wave is a plane wave propagating under the incidence θ^{inc} (angle between the x axis and the wavevector of the incident wave, measured anticlockwise), and using a time dependence

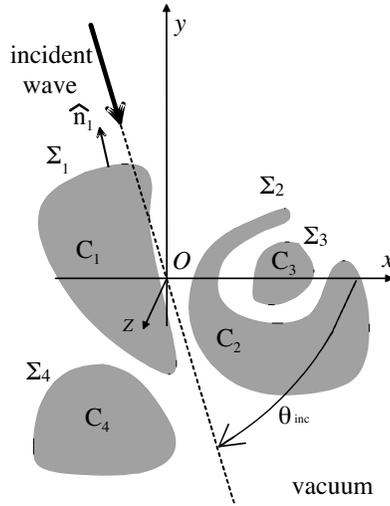


Figure 1. Presentation of the problem.

of $\exp(-i\omega t)$, the amplitude of the incident wave is given by:

$$\mathbf{E}^{\text{inc,ext}} = \hat{z}E^{\text{inc,ext}} = \hat{z} \exp\left(ik_0(x \cos(\theta^{\text{inc}}) + y \sin(\theta^{\text{inc}}))\right) \quad (1)$$

and we designate by \hat{x} , \hat{y} , \hat{z} the unit vectors along the coordinate axes.

The objects can be inhomogeneous and non isotropic, but for the sake of simplicity it is assumed that they are not magnetic (the extension to magnetic materials is straightforward). In the same way, no hypothesis is made about the shapes and locations of the objects, by contrast with the SMM in which it must be assumed that for each object, a circle can be drawn which contains it entirely, in such a way that two arbitrary circles have no intersection. The problem is to determine the field at any point of space. It is worth noting that in the present paper, we are not concerned by the calculation of the scattering properties of each individual cylinder. In other words, it is assumed that the classical problem of scattering from a single cylinder has been solved. The SOM provides the rigorous solution of the problem of scattering from a set of N objects as soon as the problem of scattering by each cylinder has been solved. Of course, the solution of the problem of scattering by a single cylinder depends on the material inside the cylinder. Here, we only suppose that this material is such that the polarization of the total field remains the same as the polarization of the incident field. When the material is homogeneous and isotropic, numerous classical methods allow one to solve the problem of scattering

from a single cylinder [15–19]. Of course, these methods are able, in theory, to solve the problem of scattering by N objects. However, they do not take advantage of the fact that the objects are separated and, consequently, that the preliminary solution of the problem of scattering from each object considered as alone in the space is an essential step in the solution of the N objects multiscattering problem.

3. BASIC PRINCIPLES OF THE SMM, DECOMPOSITION OF THE FIELD IN THE VICINITY OF A CYLINDER INTO A TOTAL INCIDENT FIELD AND A SCATTERED FIELD USING FOURIER-BESSEL EXPANSIONS

First, let us recall the basic principles of the SMM. In outline, this method takes into account separately the specific scattering properties of each object (which have been computed in a preliminary step) and the coupling phenomena between them. One of its advantages is to be accessible to postgraduate students. Moreover, the numerical implementation does not present major difficulty. In contrast with other classical methods like FDTD or finite element method, it becomes much simpler in the case of 2D photonic crystals with circular cross sections, or 3D photonic crystals formed by spherical inclusions.

The SMM is based on the expansion of the field in Fourier-Bessel series inside and outside the cylinders. This expansion can be obtained using a local system of polar coordinates (r_j, θ_j) with an origin located at a point O_j placed inside each cylinder and more or less in the middle. For simplicity, we consider in Figure 2 the case of 2 cylinders.

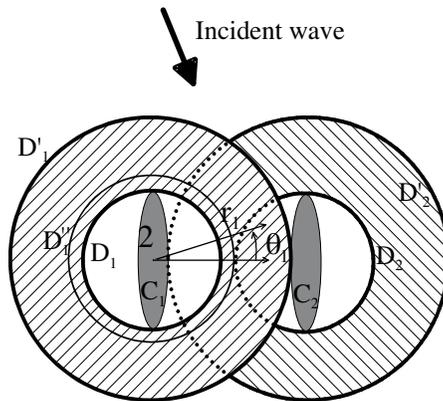


Figure 2. Domains of validity of the Fourier-Bessel expansions.

Obviously, the electric field is, for a given value of r_j , a periodic function of θ_j with period 2π and thus, it can be represented by a Fourier series:

$$E(r_j, \theta_j) = \sum_{m \in \mathbb{Z}} E_m(r_j) \exp(im\theta_j) \quad (2)$$

Outside the cylinders, the total field obeys the equation:

$$\nabla^2 E + k_0^2 E = 0 \quad (3)$$

Using the expression of the scalar Laplacian in polar coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r_j^2} + \frac{1}{r_j} \frac{\partial V}{\partial r_j} + \frac{1}{r_j^2} \frac{\partial^2 V}{\partial \theta_j^2} \quad (4)$$

then reporting the expression of the field given by Eq. (2) in Eq. (3), a straightforward calculation shows that the Fourier coefficients satisfy the following equation :

$$\forall m, \quad \frac{d^2 E_m}{dr_j^2} + \frac{1}{r_j} \frac{\partial E_m}{\partial r_j} + \left(k_0^2 - \frac{m^2}{r_j^2} \right) E_m = 0 \quad (5)$$

provided that the value of r_j is such that the circle of radius r_j centred at the origin of the local system of coordinates is entirely placed in vacuum. In other words, Eq. (5) is valid in the ring surrounding the j th cylinder and extending until the nearest point of the closest cylinder (hatched regions in Figure 2). Defining $r'_j = k_0 r_j$, this equation becomes:

$$\forall m, \quad \frac{d^2 E_m}{dr_j'^2} + \frac{1}{r'_j} \frac{\partial E_m}{\partial r'_j} + \left(1 - \frac{m^2}{r_j'^2} \right) E_m = 0. \quad (6)$$

Thus, E_m satisfies the Bessel equation, whose general solution can be given by:

$$E_m = a_{j,m} J_m(k_0 r_j) + b_{j,m} H_m^{(1)}(k_0 r_j). \quad (7)$$

with J_m and $H_m^{(1)}$ Bessel function of the first kind and Hankel function of the first kind respectively.

Introducing the expression of E_m in Eq. (2), we are led to an expression of the total field in the cylinder in the form of a Fourier-Bessel expansion:

$$E(r_j, \theta_j) = \sum_{m \in \mathbb{Z}} a_{j,m} J_m(k_0 r_j) \exp(im\theta_j) + \sum_{m \in \mathbb{Z}} b_{j,m} H_m^{(1)}(k_0 r_j) \exp(im\theta_j). \quad (8)$$

Now, it turns out that the properties of Bessel functions allow us to give a physical meaning to each of the terms placed in the right-hand side of Eq. (8). Indeed, from an intuitive physical viewpoint, the field in the ring surrounding the j th cylinder can be written in three parts that can be interpreted as:

- the incident plane wave (source) given by Eq. (1), called external incident field,
- the fields scattered by all the other cylinders towards the j th cylinder, which behave for this cylinder as incident fields. These fields add to the incident external field to constitute the total incident field on the j th cylinder,
- the field scattered by the j th cylinder which, according to the radiation condition, must propagate away from the cylinder.

In order to identify in Eq. (8) each physical component of the field, let us notice first that obviously, the total incident field on the j th cylinder has been generated by sources located outside this cylinder (at infinity for the incident plane wave and inside the other cylinders for the complementary part). Thus, it must satisfy inside this cylinder a Helmholtz equation and cannot have any singularity in this domain. This property allows us to deduce that the total incident field is contained inside the first sum of Eq. (8). Consequently the second sum represents a field scattered by the j th cylinder. The question which arises is to know whether this second sum represents the totality or only a part of the field scattered by the j th cylinder. In the first case the first sum represents the total incident field on this cylinder (and only the total incident field on this cylinder). In the second case, the first sum represents not only the total incident field on the cylinder, but also a part of the scattered field. Causality properties provide the answer to this question. The field scattered by the j th cylinder must propagate away from it. It emerges that the only Fourier-Bessel function that satisfies this condition is that containing the Hankel functions of the first kind [20]. In conclusion, the first sum in Eq. (8) does represent the total incident field on this cylinder whilst the second sum represents the field scattered by the j th cylinder. It can be shown that the domain of validity of the series representing the field scattered by the j th cylinder (second sum in the right-hand member of Eq. (8)) is not limited to the ring around C_2 . In fact, this domain extends from D_j to infinity.

In order to determine the coefficients $a_{j,n}$ and $b_{j,n}$ of Eq. (8), two series of matrix equations are used. The first one uses the local scattering matrix relation for each cylinder, which states that the coefficients $b_{j,n}$ of the field scattered by the j th cylinder deduces linearly from the coefficients $a_{j,n}$ of the total field incident on the same

cylinder:

$$\mathbf{b}_j = \mathbf{S}_j \mathbf{a}_j \quad (9)$$

In this equation, we have introduced the infinite column matrices \mathbf{a}_j and \mathbf{b}_j with components $a_{j,m}$ and $b_{j,m}$ and \mathbf{S}_j square matrix of infinite dimension. It must be noticed that \mathbf{S}_j , called scattering matrix of the j th cylinder, depends on the geometrical and electromagnetic characteristics of the j th cylinder only. In other words, \mathbf{a}_j and \mathbf{b}_j depend on the locations and shapes of the cylinders surrounding the j th cylinder, but it is not so for the linear relationship between these two column matrices. From a computational point of view, this feature means that the calculation of \mathbf{S}_j can be achieved using a classical method of scattering by solving a problem of scattering by a single cylinder. This fundamental property, which contributes to the strong stability of the method, is a direct consequence of the use of the concepts of total incident field on a cylinder and field scattered by this cylinder. On the other hand, the classical concepts of incident and scattered fields, which do not include the coupling phenomena between the cylinders, do not lead to a local relationship between the Fourier-Bessel coefficients and in that case the linear relationship involves the entire set of Fourier-Bessel coefficients of all the objects.

The second set of N equations is obtained through a second causality property: the total incident field on the j th cylinder can be interpreted as the sum of the external incident field and of the fields scattered by the other cylinders towards the j th cylinder,

$$\mathbf{a}_j = \mathbf{Q}_j + \sum_{l \neq j} \mathbf{T}_{l \rightarrow j} \mathbf{b}_l \quad (10)$$

In this equation, the infinite column matrix \mathbf{Q}_j , contribution of the external incident field, can be calculated in closed form [6], as well as the square matrices of infinite dimension $\mathbf{T}_{l \rightarrow j}$ [6], using the Graf formula [20]. It is worth noting that the matrix $\mathbf{T}_{l \rightarrow j}$, called coupling matrix from cylinder l to cylinder j , does not depend on the materials inside the N cylinders, but only on the shapes of surfaces S_j .

Multiplying Eq. (10) by \mathbf{S}_j and using Eq. (9) to eliminate \mathbf{a}_j in the left-hand member leads to a set of N matrix equations with unknowns \mathbf{b}_j , $j = (1, N)$:

$$\mathbf{b}_j = \mathbf{S}_j \mathbf{Q}_j + \sum_{l \neq j} \mathbf{S}_j \mathbf{T}_{l \rightarrow j} \mathbf{b}_l \quad (11)$$

This set of equations can be solved by truncating the column matrices \mathbf{a}_j and \mathbf{b}_j .

The limitation of this method is imposed by the calculation of matrices $\mathbf{T}_{l \rightarrow j}$. For simplicity, let us consider the case $N = 2$

represented in Figure 2. In order to find the matrix $T_{2 \rightarrow 1}$, we have to find the expression of the field scattered by cylinder 2 (second sum in the right-hand side of Eq. (8) with $j = 2$) in the form of an incident field on cylinder 1 (first sum in the right-hand side of Eq. (8) with $j = 1$). It has been seen that the expression of the field scattered by cylinder 2 converges from D_2 to infinity. Thus it converges inside the circle D_1'' centered on O_1 and tangent to D_2 . Inside D_1'' , it satisfies the Helmholtz equation (3) and does not have any singularity, thus it can be expressed in the form of a Fourier-Bessel series (with Bessel functions of the first kind only) in the local coordinates system linked to C_1 . In practice, this calculation can be made in closed form using Graf's formula. Now, this series must converge at any point of the profile C_1 , which shows that the circle D_1'' of convergence of the series must include entirely C_1 . We deduce the condition of validity of the method: two arbitrary circles D_j and D_l should not intersect each other. For ellipsoids of Figure 2, this condition is not fulfilled as soon as the distance between the objects is smaller than the large axis. The same remark holds true in Figure 1, since one object is included in an open cavity of another one, as it happens for example in SRRs.

For this reason, the new theory renounces to the use of Fourier-Bessel series of the field. On the other hand, it is still based on the notions of total incident field and scattered field in the vicinity of one object. By contrast with the SMM, these fields are defined until the surface of the scatterers, thanks to the use of Calderon projectors. It is worth noting that the equations take the same form in the new theory (SOM) as in the SMM, the matrices being replaced by operators.

4. GENERAL MATHEMATICAL DEFINITION OF THE FIELD SCATTERED BY A CYLINDER AND THE TOTAL INCIDENT FIELD IN THE VICINITY OF ITS SURFACE

4.1. Separation between Total Incident Field and Scattered Field

The aim of this section is to show that, in the vicinity of the surface of the j th cylinder, it is possible to define a total incident field $E_j^{\text{inc,total}}$, composed of the incident plane wave $E^{\text{inc,ext}}$ and a field $E_j^{\text{inc,cyl}}$ generated by the other cylinders:

$$E_j^{\text{inc,total}} = E^{\text{inc,ext}} + E_j^{\text{inc,cyl}} \quad (12)$$

The total field in the vicinity of the j th cylinder is the sum of the total

incident field and the field E_j^{scatt} scattered by the j th cylinder:

$$E = E_j^{\text{inc,total}} + E_j^{\text{scatt}} = E_j^{\text{inc,ext}} + E_j^{\text{inc,cyl}} + E_j^{\text{scatt}} \quad (13)$$

In order to find the integral expression of each component of the field in Eq. (13), let us define the function $U(x, y)$ at any point of space by:

$$U(x, y) = \begin{cases} E(x, y) & \text{outside the cylinders} \\ 0 & \text{inside the cylinders} \end{cases} \quad (14)$$

From Maxwell's equations, it can be shown easily that the field in vacuum, outside the cylinders, satisfies the Helmholtz equation in vacuum:

$$\nabla^2 E + k_0^2 E = 0 \quad (15)$$

thus, the function $U(x, y)$ satisfies at any point of space the Helmholtz equation in the sense of functions:

$$\nabla^2 U + k_0^2 U = 0 \quad (16)$$

Furthermore, the incident field satisfies the same equation:

$$\nabla^2 E^{\text{inc,ext}} + k_0^2 E^{\text{inc,ext}} = 0 \quad (17)$$

Thus, if we define at any point of space the function U' defined by:

$$U' = U - E^{\text{inc,ext}} \quad (18)$$

it turns out that U' satisfies in the sense of functions the equation:

$$\nabla^2 U' + k_0^2 U' = 0 \quad (19)$$

Since U' satisfies a radiation condition at infinity, it can be deduced from the theory of distributions (see Appendix A, Eq. (A5)) that it can be expressed in an integral form from the values $\psi_j(M')$ and $\varphi_j(M')$ (with $M' \in \Sigma_j$) of its jump and the jump of its normal derivative on the set of surfaces Σ_j :

$$U'(P) = \sum_{j=1, N} \int_{\Sigma_j} \left(G_{k_0}(P, M') \varphi_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} \psi_j(M') \right) dM' \quad (20)$$

with ψ_j and φ_j being functions defined on Σ_j as the jumps of U' and its normal derivative $\frac{\partial U'}{\partial n_j}$ respectively, \hat{n}_j unit normal to Σ_j oriented towards the exterior of Σ_j . The expressions of $G_{k_0}(P, M')$

and $\frac{\partial G_{k_0}(P, M')}{\partial n_j(M')}$ are given by Eqs. (A6) and (A7). Let us notice that the jumps ψ_j and φ_j on Σ_j of U' and its normal derivative $\frac{\partial U'}{\partial n_j}$ are identical to the jumps on Σ_j of U and its normal derivative $\frac{\partial U}{\partial n_j}$, thus identical to the limit values E^+ and $\frac{\partial E^+}{\partial n_j}$ on Σ_j (outside C_j) of the total field and its normal derivative.

Finally, from Eqs. (18) and (20), U can be written in the form:

$$U(P) = E^{\text{inc,ext}}(P) + \sum_{j=1, N} \int_{\Sigma_j} \left[G_{k_0}(P, M') \varphi_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} \psi_j(M') \right] dM' \quad (21)$$

Eq. (21) allows us to write the field given by Eq. (13) in three parts. Let us consider a point P outside C_j in the vicinity of the surface Σ_j of the j th cylinder. From Eq. (13), it emerges that the integral in Eq. (21) represents the sum of the field E_j^{scatt} scattered by the j th cylinder and the field $E_j^{\text{inc,cyl}}$ generated by the other cylinders. The separation between these two fields is straightforward and it turns out finally that

$$E_j^{\text{scatt}}(P) = \int_{\Sigma_j} \left(G_{k_0}(P, M') \varphi_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} \psi_j(M') \right) dM' \quad (22)$$

$$E_j^{\text{inc,cyl}}(P) = \sum_{l \neq j} \int_{\Sigma_l} \left(G_{k_0}(P, M') \varphi_l(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_l(M')} \psi_l(M') \right) dM' \quad (23)$$

Eqs. (22) and (23) allow us to distinguish, in the vicinity of the j th cylinder a total incident field $E_j^{\text{inc,total}} = E_j^{\text{inc,ext}} + E_j^{\text{inc,cyl}}$ from the field E_j^{scatt} scattered by the j th cylinder. However, the identification leads to the calculation of N integrals.

4.2. Field Scattered by a Cylinder and Total Incident Field in Its Vicinity in Terms of Integrals over the Surface of That Cylinder

Let us show that, in fact, the calculation of the total incident field $E_j^{\text{inc,total}} = E_j^{\text{inc,ext}} + E_j^{\text{inc,cyl}}$ in the vicinity of the j th cylinder can be achieved through a single integral solely over Σ_j , just like the calculation of the scattered field E_j^{scatt} . With this aim, let us recall

that U vanishes inside C_j . Thus, from Eq. (21), if $P \in C_j$:

$$E^{\text{inc,ext}}(P) + \sum_{j=1, N} \int_{\Sigma_j} \left(G_{k_0}(P, M') \varphi_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} \psi_j(M') \right) dM' = 0 \quad (24)$$

This equation can be written in the form: if $P \in C_j$

$$\begin{aligned} E^{\text{inc,ext}}(P) + \sum_{l \neq j} \int_{\Sigma_l} \left(G_{k_0}(P, M') \varphi_l(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_l(M')} \psi_l(M') \right) dM' \\ = - \int_{\Sigma_j} \left(G_{k_0}(P, M') \varphi_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} \psi_j(M') \right) dM' \end{aligned} \quad (25)$$

The right-hand side of Eq. (25) is formally minus the value of the field scattered by the j th cylinder given in Eq. (22), but now the point P is located inside C_j , and not outside. It is worth noticing that the value of this expression, as well as the value of its normal derivative are discontinuous across Σ_j , and thus its value of this expression inside C_j should not be considered as the analytic continuation of the scattered field E_j^{scatt} (here, we deal with the notion of analytic function of two variables satisfying an elliptic partial differential equation [21]). It is not so for the left-hand member of Eq. (25) since in this case the integrals have discontinuities on the surfaces Σ_l of the other cylinders, but not on Σ_j . Now, the left-hand side of Eq. (25) is the expression of the total incident field in the vicinity of Σ_j (see Eq. (23)), therefore, the right-hand side represents the analytic continuation inside C_j of the total incident defined in Eq. (23) and in the following, it will be called total incident $E_j^{\text{inc,total}}$ field as well. Thus, from Eq. (25):

If $P \in C_j$,

$$E^{\text{inc,total}}(P) = - \int_{\Sigma_j} \left(G_{k_0}(P, M') \varphi_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} \psi_j(M') \right) dM' \quad (26)$$

Finally, Eqs. (22) and (26) can be gathered together in the following equation:

$$\begin{aligned} R_j(P) &= \int_{\Sigma_j} \left(G_{k_0}(P, M') \varphi_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} \psi_j(M') \right) dM' \\ &= \begin{cases} E_j^{\text{scatt}}(P) & \text{if } P \notin C_j \\ -E_j^{\text{inc,total}}(P) & \text{if } P \in C_j \end{cases} \end{aligned} \quad (27)$$

It is worth noticing that, of course, the range of validity of the expression of the scattered field and the total incident field extends outside the domain located at the vicinity of C_j . The field scattered by C_j can be calculated from Eq. (27) at any point outside C_j . The same remark holds for the total incident field on C_j . However, by contrast with the scattered field, the expression of the total incident field on C_j given by Eq. (27) has no physical meaning outside the circle D'_j . Indeed, a part of this field has been generated by the other cylinders. Inside D'_j , this field actually represents a field incident on C_j (we have seen in section 3 that between D'_j and D_j , this field can be expanded in Fourier-Bessel series containing Bessel functions of the first kind only). It is not so outside D'_j and, for example, at infinity, this field satisfies the radiation condition and propagates away from C_j .

The integral contained in Eq. (27) provides the values of the total incident field and the scattered field inside and outside C_j . On Σ_j , the surface that separates these two regions, the integral is discontinuous and does not represent any physical quantity.

The result given by Eq. (27) is very simple to express using the theory of distributions and the notion of Calderon projectors. We define the column matrices $F_j^{\text{inc,total}} = \begin{bmatrix} u_j \\ u'_j \end{bmatrix}$ and $F_j^{\text{scatt}} = \begin{bmatrix} v_j \\ v'_j \end{bmatrix}$ of size 2 containing the functions u_j and u'_j describing $E_j^{\text{inc,total}}(M)$ and its normal derivative $\frac{dE_j^{\text{inc,total}}}{dn_j}(M)$ on Σ_j (or the functions v_j and v'_j describing $E_j^{\text{scatt}}(M)$ and its normal derivative $\frac{dE_j^{\text{scatt}}}{dn_j}(M)$ on Σ_j). Since the left-hand side of Eq. (27) represents a function satisfying a Helmholtz equation in vacuum and a radiation condition at infinity we are led to this important conclusion (see Appendix A): the field F_j^{scatt} scattered by the j th cylinder and minus the total incident field $F_j^{\text{inc,total}}$ on this cylinder are the Calderon projectors of the total field F_j on Σ_j :

$$F_j^{\text{scatt}} = \mathbf{P}_{\Sigma_j, k_0}^+ F_j \quad (28)$$

$$-F_j^{\text{inc,total}} = \mathbf{P}_{\Sigma_j, k_0}^- F_j \quad (29)$$

Eq. (A21) of Appendix A allows us to give the expressions of u_j and v_j , which are the limits on both sides of Σ_j of $R_j(P)$:

$$-u_j = G_{\Sigma_j, k_0} \varphi_j - \left(\frac{I}{2} + \frac{\partial G_{\Sigma_j, k_0}}{\partial n} \right) \psi_j \quad (30)$$

$$v_j = G_{\Sigma_j, k_0} \varphi_j + \left(\frac{I}{2} - \frac{\partial G_{\Sigma_j, k_0}}{\partial n} \right) \psi_j \quad (31)$$

Finally, we notice that it is possible to express $E_j^{\text{inc, total}}(P)$ (if $P \in C_j$) and $E_j^{\text{scatt}}(P)$ (if $P \notin C_j$) from their limit values u_j , u'_j , v_j and v'_j on Σ_j using the Kirchhoff-Helmholtz equation (Eq. (A22) of Appendix A):

$$P \notin C_j$$

$$E_j^{\text{scatt}}(P) = \int_{\Sigma_j} \left(G_{k_0}(P, M') v'_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} v_j(M') \right) dM' \quad (32)$$

$$P \in C_j$$

$$-E_j^{\text{inc, total}}(P) = \int_{\Sigma_j} \left(G_{k_0}(P, M') u'_j(M') - \frac{\partial G_{k_0}(P, M')}{\partial n_j(M')} u_j(M') \right) dM' \quad (33)$$

5. SCATTERING OPERATOR METHOD

5.1. First Equation: Expression of the Coupling between Cylinders

In this section, we use the causality relation between the part $E_j^{\text{inc, cyl}}$ of the total incident field which has been radiated by the other cylinders towards the j th cylinder and the origin of this field: the fields E_j^{scatt} , $l \neq j$ scattered by the other cylinders. With this aim, we apply Eq. (32) to the $N - 1$ other cylinders and we calculate the values of these scattered fields on the surface S_j . We deduce:

$$E_j^{\text{inc, cyl}}(M) = \sum_{l \neq j} \int_{\Sigma_l} \left[G_{k_0}(M, M') v'_l(M') dM' - \frac{\partial G_{k_0}(M, M')}{\partial n_l(M')} v_l(M') dM' \right] \quad (34)$$

with $M \in \Sigma_j$ and $M' \in \Sigma_l$. In order to obtain the total incident field on C_j , we add to the right-hand member of Eq. (34) the value $Q_j(M)$ of the incident plane wave $E_j^{\text{inc, ext}}$ on Σ_j :

$$E_j^{\text{inc, total}}(M) = Q_j(M) + \sum_{l \neq j} \int_{\Sigma_l} \left(G_{k_0}(M, M') v'_l(M') - \frac{\partial G_{k_0}(M, M')}{\partial n_l(M')} v_l(M') \right) dM' \quad (35)$$

$$Q_j(M) = E^{\text{in,ext}}(M) \quad (36)$$

In operator notation, this equation can be written

$$u_j = Q_j + \sum_{l \neq j} \left(G_{\Sigma_l \rightarrow \Sigma_j, k_0} v'_l - \frac{\partial G_{\Sigma_l \rightarrow \Sigma_j, k_0}}{\partial n_l} v_l \right) \quad (37)$$

where the operators $G_{\Sigma_l \rightarrow \Sigma_j, k_0}$ and $\frac{\partial G_{\Sigma_l \rightarrow \Sigma_j, k_0}}{\partial n_l}$ denote operators acting on functions defined on Σ_l , the result being a function defined on Σ_j :
if $M \in \Sigma_j$,

$$G_{\Sigma_l \rightarrow \Sigma_j, k_0} v'_l = \int_{\Sigma_l} G_{k_0}(M, M') v'_l(M') dM' \quad (38)$$

$$\frac{\partial G_{\Sigma_l \rightarrow \Sigma_j, k_0}}{\partial n_l} v_l = \int_{\Sigma_l} \frac{\partial G_{k_0}(M, M')}{\partial n_l(M')} v_l(M') dM' \quad (39)$$

In order to eliminate v'_l in Eq. (37), it suffices to remember that v_l and v'_l are the limit values on Σ_l of the field E_l^{scatt} scattered by the l th object. It is well known that if a function satisfies a Helmholtz equation outside C_j and a radiation condition at infinity, one can impose either its limit value on Σ_j (Dirichlet problem) or the value of its normal derivative on Σ_j (Neumann problem), but not both [22]. The linear relationship between v_l and v'_l is given by Eq. (A24)

$$v'_l = Z_{\Sigma_l, k_0}^+ v_l \quad (40)$$

The operator Z_{Σ_l, k_0}^+ is given by Eq. (A25):

$$Z_{\Sigma_l, k_0}^+ = G_{\Sigma_l, k_0}^{-1} \left(\frac{I}{2} + \frac{\partial G_{\Sigma_l, k_0}}{\partial n_l} \right) \quad (41)$$

the operators in the right-hand side of Eq. (41) being given by Eqs. (A6) and (A7). Finally, Eqs. (37) and (41) yield:

$$u_j = Q_j + \sum_{l \neq j} T_{l \rightarrow j} v_l \quad (42)$$

with

$$T_{l \rightarrow j} = G_{\Sigma_l \rightarrow \Sigma_j, k_0} Z_{\Sigma_l, k_0}^+ - \frac{\partial G_{\Sigma_l \rightarrow \Sigma_j, k_0}}{\partial n_l} \quad (43)$$

5.2. Second Equation: Use of the Scattering Operator between the Field Scattered by a Cylinder and the Total Incident Field That Illuminates It

The field E_j^{scatt} scattered by the j th cylinder and the total incident field $E_j^{\text{inc,total}}$ that illuminates it are linked by a linear relationship. Thus, v_j can be deduced from u_j by

$$v_j = S_j u_j \tag{44}$$

In this equation, the scattering operator S_j associates a function defined on Σ_j with another function defined on Σ_j . It is crucial to notice that the scattering operator of a given cylinder depends on the characteristics of this cylinder only. Thus, the calculation of the scattering operators of the N cylinders requires N independent calculations. When the cylinders are identical, the scattering operators are the same and a single calculation is needed. Of course, the complexity of the calculation of the scattering operators depends strongly on the complexity of the shape and electromagnetic characteristics of the cylinders. For example, in the case in which a cylinder is perfectly conducting, the total electric field vanishes on the surface, which entails that the scattering operator is equal to $-I$, with I the identity operator. In the case of circular cylinders, the scattering operator can be obtained in closed form.

5.3. Final Equation

Eqs. (42) and (44) provide us with a doubly-infinite set of linear equations between the unknown functions v_j and u_j . In order to obtain a single series of equations, it suffices to multiply both members of Eq. (42) by S_j , then to use Eq. (44) to eliminate u_j . We thus obtain:

$$v_j = S_j Q_j + \sum_{l \neq j} S_j T_{l \rightarrow j} v_l \tag{45}$$

In an explicit form, this equation can be written:

$$\begin{bmatrix} I & -S_1 T_{2 \rightarrow 1} & -S_1 T_{3 \rightarrow 1} & \cdots & \cdots \\ -S_2 T_{1 \rightarrow 2} & I & -S_2 T_{3 \rightarrow 2} & \cdots & \cdots \\ -S_3 T_{1 \rightarrow 3} & -S_3 T_{2 \rightarrow 3} & I & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdots \\ \cdots \end{bmatrix} = \begin{bmatrix} S_1 Q_1 \\ S_2 Q_2 \\ S_3 Q_3 \\ \cdots \\ \cdots \end{bmatrix} \tag{46}$$

This equation takes exactly the same form as the final equation of the SMM.

5.4. Calculation of the Diagram of Diffraction at Infinity

The scattered field at infinity can be obtained by calculating the asymptotic behavior at infinity of the scattered field given by equation (32). With this aim, we use the asymptotic expression of the Hankel functions at infinity:

$$\begin{aligned} H_0^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} \exp\left(-i\frac{\pi}{4}\right) \exp(iz) \\ H_1^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} \exp\left(-i\frac{3\pi}{4}\right) \exp(iz) \end{aligned} \quad (47)$$

and the asymptotic expression of PM :

$$PM' \sim r - \mathbf{u}(\theta) \cdot \mathbf{OM}' \quad (48)$$

r and θ being the polar coordinates of P and $\mathbf{u}(\theta) = \frac{\mathbf{OP}}{OP}$ the unit vector in the direction of P . Finally, we obtain:

$$E_j^{\text{scatt}}(P) \sim g(\theta) \frac{\exp(ik_0 r)}{\sqrt{r}} \quad (49)$$

$$\begin{aligned} g(\theta) = \sum_{j=1, N} \int_{S_j} &\left[\frac{\exp\left(-i\frac{\pi}{4}\right)}{2\sqrt{\lambda_0}} \mathbf{u}(P) \mathbf{n}_j(M') v_j(M') - \frac{\exp\left(i\frac{\pi}{4}\right) \sqrt{\lambda_0}}{4\pi} v'_j(M') \right] \\ &\times \exp(-ik_0 \mathbf{u}(P) \cdot \mathbf{OM}') dM' \end{aligned} \quad (50)$$

Let us recall that in the case of lossless materials, and when the incident field is a plane wave, the Energy Balance Criterion (EBC) requires:

$$\int_{\theta=0}^{2\pi} |g(\theta)|^2 d\theta + 2\sqrt{\lambda_0} \operatorname{Re} \left[\exp\left(i\frac{\pi}{4}\right) g(\theta^{\text{inc}}) \right] = 0 \quad (51)$$

5.5. Comparison with the Classical Integral Theory

The problem of scattering from N objects can be solved using the classical integral theory of scattering. In that case, the unknowns are, in general, the values of the field and its normal derivative on the objects. The big difference with the SOM is that the physical notions of input (total incident field on a cylinder) and output (field scattered by the same cylinder) is not used anymore. The first equation is

obtained by writing that the value of the field and its normal derivative on a cylinder can be deduced from the values of the field and its normal derivative on the entire set of cylinders, using the Kirchhoff-Helmholtz equation (Eq. (A22) of Appendix A). By contrast with the first equation of the SOM, this equation needs an integration over the entire set of cylinders, including the cylinder on which the field and its normal derivative are calculated. It is well known that one of the major difficulties of the integral theory comes from the singularities of the integrals when the point of calculation identifies with a point of integration. Thanks to the linear relation between the incident field on a cylinder and the field scattered by the other cylinders, this difficulty never happens in the first equation of the SOM (see Eq. (37)) where it suffices to evaluate the integral on the $(N - 1)$ cylinders where the field is not calculated. This difference is quite important since, by contrast with the second one, the first equation is that in which a large number of cylinders are involved, and thus it can be conjectured that it is the major origin of numerical errors.

As regards the second equation, the classical theory of scattering leads to a local relation as in the SOM: it is obtained by writing the relation between the field and its normal derivative at the surface of a cylinder, imposed by the material inside this cylinder. It is well known that this relation may present instabilities in some cases, due to internal resonances. The corresponding equation in the SOM links the values of the incident and scattered fields on the surface of the cylinders. The scattering operator, i.e., the operator which associates these functions, satisfies strong mathematical properties, for instance unitarity in some cases. Thus it can be conjectured that the scattering operators in the SOM are better conditioned than the operators deduced from compatibility between the field and its normal derivative in the classical integral theory. In conclusion, it can be conjectured that the SOM is, like the SMM, more stable than the classical integral theory.

5.6. Numerical Implementation

The numerical implementation of Eq. (46) requires the transformation of this equation into a matrix equation of finite size. One of the general ways for solving this problem is to expand the functions v_j , u_j and Q_j on a system of M_j finite elements, the variable being the curvilinear abscissa on the j th cylinder [18]. In that case, functions v_j , u_j and Q_j become column matrices of dimension M_j while the operators S_j and $T_{l \rightarrow j}$ reduce to matrices of dimensions $M_j \times M_j$ and $M_l \times M_l$ respectively. Finally, the size of the linear system to be solved is equal to $\sum_{j=1, N} M_j$. In this paper, the examples of numerical calculations are given on elliptic objects. The profiles of such objects are given by

parametric equations

$$x = l \cos(u), \quad y = h \cos(u)$$

The discretisation of the profile has been made by using equally spaced values of the parameter u . For circular cylinders, this choice corresponds to equally spaced values of the curvilinear abscissa. Of course, for more complicated profiles, or even for very flat ellipses, best adapted discretization process should be adopted in order to sample the profile more adequately, for example by choosing equally spaced values of the curvilinear abscissa.

When there exists, for each cylinder, a local polar coordinate system (r_j, θ_j) such that the profile Σ_j can be given by a function $r_j = f_j(\theta_j)$, it may be beneficial to represent the functions v_j , u_j and Q_j by Fourier series in θ_j . Truncating these Fourier series between coefficients $-P_j$ and $+P_j$, these functions become column matrices of dimension $2P_j+1$ while the operators S_j and $T_{l \rightarrow j}$ reduce to matrices of dimensions $(2P_j+1)^2$ and $(2P_l+1) \times (2P_j+1)$ respectively. Finally, the size of the linear system to be solved is equal to $(N + 2 \sum_{j=1, N} P_j)^2$. In the case when the shape of the cylinders is circular, the total incident field on a cylinder and the field scattered by the cylinder, as well as the total field inside the cylinder, can be represented by Fourier-Bessel series [6], in such a way that the calculation of the scattering operator can be achieved in closed form. In that case, the scattering operator method described in this paper reduces to the scattering matrix method. The strong advantage of the SOM lies on the fact that it can deal with any set of cylinders, whatever the shapes and the locations of the cylinders.

5.7. Numerical Validation

In order to check the validity of the method, we have first compared our results with those published in [6] on a very simple case. Figure 6 of this article shows the intensity scattered at infinity by a couple of dielectric cylinders of index 1.5 with diameter 6 cm, the centers being at a distance 9 cm from each other, illuminated with a s -polarized incident plane wave with a wave-vector tilted 45° on the symmetry planes of the device ($\theta^{inc} = -45^\circ$). The curve obtained from the SOM cannot be distinguished from that of Figure 6, obtained from the SMM. On this example, we have checked the precision of the results from three criteria: convergence of the results as the number of points $M = M_1 = M_2$ on each cylinder is increased, Energy Balance Criterion (EBC, see Eq. (51)) and reciprocity theorem. Let us recall that the reciprocity theorem states that the amplitude at infinity $g(\theta)$ obtained

Table 1. Convergence, energy balance and reciprocity criteria for the problem of scattering by a couple of dielectric circular cylinders (Figure 3).

Number M of discretization points on each cylinder	25	50	75	100	125	150
$ g(\theta) $ (case 1)	0.930	0.825	0.822	0.823	0.824	0.824
$ g(\theta) $ (case 2)	0.930	0.825	0.822	0.823	0.824	0.824
Arg $g(\theta)$ (case 1)	-149.8	-146.7	-148.5	-149.3	-149.8	-150.0
Arg $g(\theta)$ (case 2)	-149.8	-146.7	-148.5	-149.3	-149.8	-150.0
Precision on EBC (case 1)	0.07	0.005	0.002	0.0006	0.0003	0.0002
Precision on EBC (case 2)	0.06	0.002	0.0002	0.0002	0.0002	0.0002

in the direction of scattering given by the angle θ , when the set of cylinders is illuminated by a plane wave with angle of incidence θ^{inc} , remains unchanged when the incidence angle and the scattering angle become equal to $\theta + 180^\circ$ and $\theta + 180^\circ$ respectively, or in other words when the directions of propagation of the incident and scattered fields are reversed [23]. Cases 1 and 2 are defined in Figure 3. It can be seen in Table 1 that the convergence of the results to within 1% in relative value is obtained for $|g(\theta)|$ for $M = 50$ while the same precision needs $M = 75$ for $(g(\theta))$. On the other hand, the EBC is satisfied to within $5 \cdot 10^{-3}$ as soon as $M > 50$. The reciprocity theorem is perfectly satisfied for the first 4 digits whatever M may be. These results show that the EBC and the reciprocity theorem cannot give an estimate of the precision of the results. The computation time for $M = 150$ is of the order of 1 second on a desktop.

A second validation test has been made on a set of 4 elliptic perfectly conducting cylinders illuminated with a s -polarized plane wave. Cases 1 and 2 for the reciprocity theorem are defined in Figure 4 and the results are given in Table 2. The conclusions are almost the same as in Table 1, even though a precision of 1% on the results needs slightly greater values of M . It is worth noting that these results could not be obtained from the SMM.

In order to show a second example of the capability of the SOM to deal with crystals which cannot be analysed using the SMM, we have investigated the properties in transmission of the photonic crystal shown in Figure 5. The dashed circles at the top right of the figure

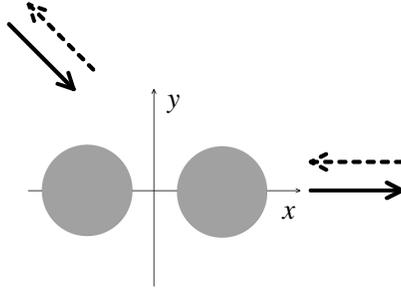


Figure 3. The two cases in the reciprocity theorem for the set of two dielectric circular cylinders. Solid and dashed arrows represent cases 1 and 2 respectively.

Table 2. Convergence, energy balance and reciprocity criteria for the problem of scattering by a couple of dielectric circular cylinders (Figure 4).

Number M of discretization points on each cylinder	25	50	75	100	125	150
$ g(\theta) $ (case 1)	1.057	1.133	1.155	1.164	1.168	1.170
$ g(\theta) $ (case 2)	1.058	1.133	1.155	1.164	1.168	1.170
$\text{Arg } g(\theta)$ (case 1)	22.04	27.68	28.45	28.73	28.83	28.90
$\text{Arg } g(\theta)$ (case 2)	22.09	27.69	28.46	28.73	28.83	28.90
Precision on EBC (case 1)	0.005	0.002	0.0008	0.0004	0.0003	0.0002
Precision on EBC (case 2)	0.0009	0.0002	0.00009	0.00004	0.00003	0.00002

show that the calculation cannot be achieved using the SMM since they intersect each other. Figure 6 shows the value of the modulus of the field at the point of observation versus the wavelength in both cases, the amplitude of the incident wave being equal to unity. The calculations are made for 48 values of the wavelength and need a computation time of less than one hour on a desktop working at 1.4 GHz with 512 MB of RAM.

The physical intuition inclines one to conjecture that the transmitted field should be much larger in case 2 than in case 1. This rule is true in general, but for greater wavelengths it is reversed. This surprising result can be explained in the following way: when the wavelength is much lower than the period of the crystal, the rules of geometrical optics give a good approximation of scattering phenomena.

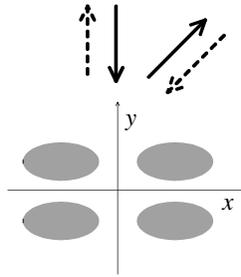


Figure 4. The two cases in the reciprocity theorem for the set of two perfectly conducting elliptic cylinders. The vertical and horizontal axes of the ellipses are equal to 1 and 2 respectively. The distances between the centers of the ellipses in the vertical and horizontal directions are equal to 3 and 1.5 respectively. The wavelength in vacuum is equal to 2.5 and the incident light is s -polarized. Solid and dashed arrows represent cases 1 and 2 respectively.

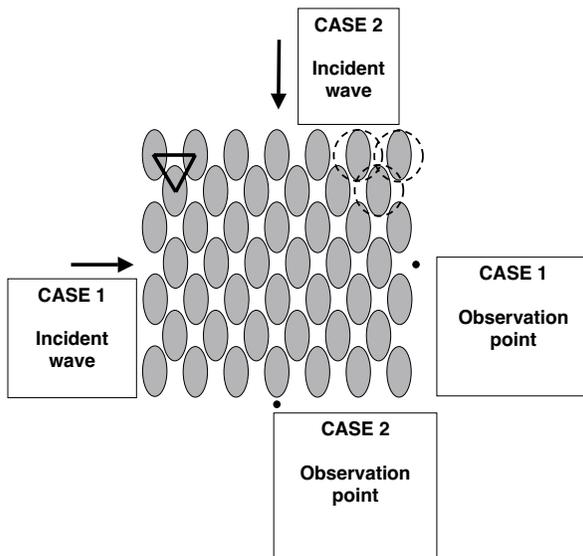


Figure 5. Photonic crystal made of 46 elliptic cylinders. The ellipses are perfectly conducting, their centers being located at the edges of equilateral triangles of side unity. The vertical and horizontal axes of the ellipses are equal to 1.2 and 0.4 respectively. The crystal is illuminated by a p -polarized incident plane wave propagating horizontally (case 1) or vertically (case 2). The field is calculated in both cases at a point located in the shadow region of the crystal.

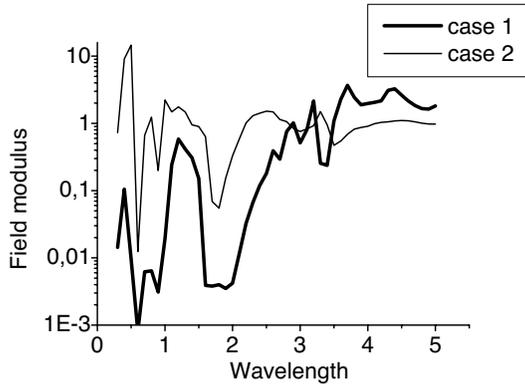


Figure 6. Modulus of the field at the observation point versus the wavelength.

As the wavelength reach values of the same order or even greater than the period, we are in the resonance regime and these rules do not apply anymore.

6. CONCLUSION

It has been shown that it is possible to generalize the SMM to an arbitrary case of scattering from a set of cylinders of arbitrary shapes located arbitrarily. The new method can be implemented on a simple desktop and provides a remarkable precision with moderate computation times. We are actually implementing the method to the swiss rolls (split ring resonators) used in the context of metamaterials to generate magnetic resonances. The generalization to 3D problems of scattering by N objects does not present difficulties from a theoretical point of view, but of course it leads to a significant increase of the complexity of the equations and of the numerical implementation. The generalization to the 2D case in off-plane mountings should be very interesting since it would enable one to deal with microstructured optical fibers with non-circular holes.

The generalization to many other kinds of structures can be envisaged, for instance the scattering by a stack of rough surfaces. In that case, the SOM permits one to interpret the total field as the sum of incident and scattered fields at the vicinity of each interface. From that point of view, the definition of these fields can be interpreted as a generalization of the Rayleigh hypothesis [24], which represents the scattered field by plane wave expansions up to the interfaces. The

SOM provides a means to calculate the rigorous value of the scattered field on the interface in cases where the Rayleigh expansion does not converge.

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APPENDIX A. THE THEORY OF DISTRIBUTIONS IN OUTLINE, CALDERON PROJECTORS

A.1. Definitions

In this appendix, we present the notion of Calderon projector from the theory of distributions. Then we establish the Kirchhoff-Helmholtz equation and we give some remarks about its use in this paper. The reader interested in more details on the theory of distributions can refer to [25].

A.2. Convolution and Integral Expression of a Function Satisfying a Helmholtz Equation in the Sense of Functions

Let us consider the case of a function u satisfying the Helmholtz equation in the sense of functions, the jumps of the function $u(x, y)$ and its normal derivative on a contour Σ being respectively equal to $[u]_{\Sigma}$ and $\left[\frac{\partial u}{\partial n}\right]_{\Sigma}$. In addition, it is assumed that this function satisfies the radiation condition at infinity. This function satisfies in the sense of distributions a Helmholtz equation with a right-hand side:

$$\nabla^2 u + k^2 u = \left[\frac{\partial u}{\partial n}\right]_{\Sigma} \delta_{\Sigma} + \nabla \cdot (\mathbf{n}[u]_{\Sigma} \delta_{\Sigma}) \quad (\text{A1})$$

In order to express u in an integral form, we define the elementary solution of the Helmholtz equation g_k in a material of wavenumber k , which must satisfy the radiation condition at infinity and the following Helmholtz equation:

$$\nabla^2 g_k + k^2 g_k = \delta_{0,0} \quad (\text{A2})$$

It is very easy to show that u is given by the convolution product:

$$u = g_k * \left(\left[\frac{\partial u}{\partial n}\right]_{\Sigma} \delta_{\Sigma} + \nabla \cdot (\mathbf{n}[u]_{\Sigma} \delta_{\Sigma}) \right) \quad (\text{A3})$$

Let us recall that the elementary solution of Helmholtz equation satisfying a radiation condition at infinity is a locally integrable function given by:

$$g_k = -\frac{i}{4}H_0^{(1)}(kr) \quad (\text{A4})$$

with r the distance to the origin of coordinates and $H_0^{(1)}$ the Hankel function of the first kind and zero th order.

Tedious algebraic calculations show that the convolution products can be expressed in the following form:

$$u(P) = \int_{\Sigma} \left(G_k(P, M') \left[\frac{\partial u}{\partial n} \right]_{\Sigma} (M') - \frac{dG_k(P, M')}{dn(M')} [u]_{\Sigma}(M') \right) dM' \quad (\text{A5})$$

$$G_k(P, M') = g_k(PM') = -\frac{i}{4}H_0^{(1)}(kPM') \quad (\text{A6})$$

$$\begin{aligned} \frac{\partial G_k(P, M')}{dn(M')} &= -\frac{i}{4} \mathbf{n}(M') \cdot \nabla_{M'}(g_k(PM')) \\ &= \frac{ki}{4} \mathbf{n}(M') \cdot \frac{\mathbf{PM}'}{PM'} H_1^{(1)}(kPM') \end{aligned} \quad (\text{A7})$$

A.3. Calderon Projectors

Eq. (A5) allows us to express the limit values u^{\pm} and $\frac{\partial u^{\pm}}{\partial n}$ of u on both sides of Σ , the sign $+$ and $-$ assigned to the sides of the surface and the direction of the normal are such that the normal points from the $-$ to the $+$ side:

$$\nabla^2 u + k^2 u = \left[\frac{\partial u}{\partial n} \right]_{\Sigma} \delta_{\Sigma} + \nabla \cdot (\mathbf{n}[u]_{\Sigma} \delta_{\Sigma}) \quad (\text{A8})$$

$$u^{\pm}(M) = \lim_{P \rightarrow M_{\pm}} \int_{\Sigma} \left(G_k(P, M') \left[\frac{\partial u}{\partial n} \right]_{\Sigma} (M') - \frac{\partial G_k(P, M')}{\partial n(M')} [u]_{\Sigma}(M') \right) dM' \quad (\text{A9})$$

In these equations the symbol $P \rightarrow M_{\pm}$ means that the point P of space tends to the point M of Σ either from the side of the normal (M_+) or from the other side (M_-). The limit values $\frac{\partial u^{\pm}}{\partial n}$ could be calculated as well by differentiating Eq. (A5) and taking its limits as $P \rightarrow M_{\pm}$.

We define the column matrices

$$F = \begin{pmatrix} [u]_{\Sigma} \\ \left[\frac{\partial u}{\partial n} \right]_{\Sigma} \end{pmatrix}, \quad F^{\pm} = \begin{pmatrix} \pm u^{\pm} \\ \pm \frac{\partial u^{\pm}}{\partial n} \end{pmatrix}.$$

The Calderon projectors $\mathbf{P}_{\Sigma,k}^{\pm}$ are defined by [26]

$$F^{\pm} = \mathbf{P}_{\Sigma,k}^{\pm} F \quad (\text{A10})$$

Thus, $\mathbf{P}_{\Sigma,k}^{\pm}$ are matrix operators of size 2×2 which associate the limits on both sides of the contour Σ of a function F satisfying the Helmholtz equation with wavenumber k in the sense of functions and its normal derivative to the jumps on Σ of the same quantities. It can be noticed that, in general, the Calderon projectors are defined through the use of the electric field and the tangential component of the magnetic field (instead of the normal derivative of the electric field). In fact, the two definitions are equivalent in our case since the normal derivative of the electric field is proportional to the tangential component of the magnetic field.

Obviously, since $F^+ + F^- = F$, we have:

$$\mathbf{P}_{\Sigma,k}^+ + \mathbf{P}_{\Sigma,k}^- = \mathbf{I} \quad (\text{A11})$$

Furthermore, it can be shown that Calderon projectors satisfy the following relations:

$$\mathbf{P}_{\Sigma,k}^+ F^+ = F^+, \quad \mathbf{P}_{\Sigma,k}^- F^- = F^-, \quad \mathbf{P}_{\Sigma,k}^+ F^- = \mathbf{P}_{\Sigma,k}^- F^+ = 0 \quad (\text{A12})$$

which implies:

$$\mathbf{P}_{\Sigma,k}^+ \mathbf{P}_{\Sigma,k}^+ = \mathbf{P}_{\Sigma,k}^+, \quad \mathbf{P}_{\Sigma,k}^- \mathbf{P}_{\Sigma,k}^- = \mathbf{P}_{\Sigma,k}^-, \quad \mathbf{P}_{\Sigma,k}^+ \mathbf{P}_{\Sigma,k}^- = \mathbf{P}_{\Sigma,k}^- \mathbf{P}_{\Sigma,k}^+ = 0 \quad (\text{A13})$$

Eq. (A13) shows that $\mathbf{P}_{\Sigma,k}^{\pm}$ are idempotent operators, or in other words, projectors.

Now, in order to deduce the expression of Calderon projectors, we will eliminate the limits in equation (A9). Let us define:

$$J(P) = \int_{\Sigma} \left(G_k(P, M') \left[\frac{\partial u}{\partial n} \right]_{\Sigma} (M') - \frac{\partial G_k(P, M')}{\partial n(M')} [u]_{\Sigma}(M') \right) dM' \quad (\text{A14})$$

in such a way that Eq. (A9) becomes:

$$u^{\pm}(M) = \lim_{P \rightarrow M_{\pm}} J(P) \quad (\text{A15})$$

Now, from the definition of the jump of u on Σ ,

$$[u]_{\Sigma}(M) = u^+(M) - u^-(M) \quad (\text{A16})$$

therefore

$$\lim_{P \rightarrow M_+} J(P) - \lim_{P \rightarrow M_-} J(P) = [u]_{\Sigma} \quad (\text{A17})$$

Furthermore, it can be shown [19] that the value $J(M)$ of the integral on a point M of Σ is the average of the limit values of on both sides of Σ :

$$\lim_{P \rightarrow M_+} J(P) + \lim_{P \rightarrow M_-} J(P) = 2J(M) \quad (\text{A18})$$

From Eqs. (A15), (A16), (A17) and (A18), it can be deduced that:

$$u^{\pm}(M) = J(M) \pm \frac{[u]_{\Sigma}(M)}{2} \quad (\text{A19})$$

and from Eq. (A14), we get:

$$\begin{aligned} u^{\pm}(M) &= \int_{\Sigma} \left(G_k(M, M') \left[\frac{\partial u}{\partial n} \right]_{\Sigma}(M') - \frac{\partial G_k(M, M')}{\partial n(M')} [u]_{\Sigma}(M') \right) dM' \\ &\quad \pm \frac{[u]_{\Sigma}(M)}{2} \end{aligned} \quad (\text{A20})$$

In order to write this equation in a more condensed form, it is convenient to introduce the notation in terms of operators:

$$\begin{aligned} u^{\pm} &= G_{\Sigma,k} \left[\frac{\partial u}{\partial n} \right]_{\Sigma} - \frac{\partial G_{\Sigma,k}}{\partial n} [u]_{\Sigma} \pm \frac{[u]_{\Sigma}}{2} \\ &= G_{\Sigma,k} \left[\frac{\partial u}{\partial n} \right]_{\Sigma} + \left(\pm \frac{I}{2} - \frac{\partial G_{\Sigma,k}}{\partial n} \right) [u]_{\Sigma} \end{aligned} \quad (\text{A21})$$

In this equation, $G_{\Sigma,k}$ and $\frac{\partial G_{\Sigma,k}}{\partial n}$ are considered as operators which, acting on functions defined on Σ , give another function defined on Σ , I being the identity operator.

A.4. Kirchhoff-Helmholtz Equation

From Eq. (A5), it is possible to deduce the so-called Kirchhoff-Helmholtz equation, which is a basis of the theory of Green functions.

Let us suppose that u vanishes inside C and satisfies a Helmholtz equation outside (with a radiation condition at infinity). Of course, it satisfies the Helmholtz equation, including inside C , thus Eq. (A5)

enables one to have the expression of u everywhere from the jumps $[u]_\Sigma$ and $\left[\frac{\partial u}{\partial n}\right]_\Sigma$ of and $\frac{\partial u}{\partial n}$ on Σ , which in that case reduce to u^+ and $\frac{\partial u^+}{\partial n}$. Thus we get:

$$u(P) = \int_{\Sigma} \left(G_k(P, M') \frac{\partial u^+}{\partial n}(M') - \frac{dG_k(P, M')}{dn(M')} u^+(M') \right) dM' \quad (\text{A22})$$

This equation, generally called Kirchhoff-Helmholtz equation, permits one to deduce the value of a function u satisfying a Helmholtz equation outside C , from its value and the value of its normal derivative on Σ . From the theory of distributions, it is fundamental to notice that in that case, there exists a relationship between u^+ and $\frac{\partial u^+}{\partial n}$. Indeed, if u^+ and $\frac{\partial u^+}{\partial n}$ are chosen arbitrarily, these two functions will be equal to the jumps of u and $\frac{\partial u}{\partial n}$ on Σ , and not to their limit values outside Σ . The relationship between u^+ and $\frac{\partial u^+}{\partial n}$ is straightforward: we must write either that the limit of u outside Σ is equal to u^+ , or that its limit inside Σ is equal to 0. In both cases, the jumps of u and $\frac{\partial u}{\partial n}$ on Σ will reduce to the limit values outside S , the resulting equation being:

$$G_{\Sigma,k} \frac{\partial u^+}{\partial n} - \left(\frac{I}{2} + \frac{\partial G_{\Sigma,k}}{\partial n} \right) u^+ = 0 \quad (\text{A23})$$

This equation of compatibility between the limit values on Σ of a function satisfying a Helmholtz equation outside C and a radiation condition at infinity can be written in the form:

$$\frac{\partial u^+}{\partial n} = Z_{\Sigma,k}^+ u^+ \quad (\text{A24})$$

with

$$Z_{\Sigma,k}^+ = G_{\Sigma,k}^{-1} \left(\frac{I}{2} + \frac{\partial G_{\Sigma,k}}{\partial n} \right) \quad (\text{A25})$$

The same calculation can be achieved to find the expression of a function satisfying a Helmholtz equation inside C . By considering now a function u equal to this function inside C and 0 outside, the expression of u is given by:

$$u(P) = - \int_{\Sigma} \left(G_k(P, M') \frac{\partial u^-}{\partial n}(M') - \frac{dG_k(P, M')}{dn(M')} u^-(M') \right) dM' \quad (\text{A26})$$

and the compatibility between $\frac{\partial u^-}{\partial n}$ and u^- leads to

$$G_{\Sigma,k} \frac{\partial u^-}{\partial n} + \left(\frac{I}{2} - \frac{\partial G_{\Sigma,k}}{\partial n} \right) u^- = 0 \quad (\text{A27})$$

or

$$\frac{\partial u^-}{\partial n} = Z_{\Sigma,k}^- u^- \quad (\text{A28})$$

with

$$Z_{\Sigma,k}^- = G_{\Sigma,k}^{-1} \left(-\frac{I}{2} + \frac{\partial G_{\Sigma,k}}{\partial n} \right) \quad (\text{A29})$$

REFERENCES

1. Joannopoulos, J. D., R. D. Meade, and J. N. Winn, *Photonic Crystals*, University Press, Princeton, 1995.
2. Reinex, A. and B. Jecko, "A new photonic band gap equivalent model using Finite Difference Time Domain method," *Annales des Télécommunications*, Vol. 51, 656–662, 1996.
3. Chew, W. C., L. Gürel, Y. M. Wang, G. Otto, R. L. Wagner, and Q. H. Liu, "A generalized recursive algorithm for wave-scattering solutions in two dimensions," *IEEE Trans. on Microwave Theory and Techniques*, Vol. 40, 716–723, 1992.
4. Defos du Rau, M., "Diffusion électromagnétique dépendante dans les milieux hétérogènes denses. Présentation d'un modèle mixte en vue de l'étude des matériaux hétérogènes," Ph.D. Thesis, Bordeaux University, France, 1997.
5. Elsherbeni, A. Z. and A. Kishk, "Modeling of cylindrical objects by circular dielectric or conducting cylinders," *IEEE Trans. on Antennas and Propagation*, Vol. 40, 96–99, 1992.
6. Felbacq, D., G. Tayeb, and D. Maystre, "Scattering by a random set of parallel cylinders," *J. Opt. Soc. Am. A*, Vol. 11, 2526–2538, 1994.
7. Nicorovici, N. A., R. C. McPhedran, and L. C. Botten, "Photonic band gaps for arrays of perfectly conducting cylinders," *Phys. Rev. E*, Vol. 52, 1135–1145, 1995.
8. Pendry, J. B., A. J. Holden, D. J. Robbins, and W. J. Stewart, "Magnetism from conductors and enhanced nonlinear phenomena," *IEEE Trans. Microwaves Theory Tech.*, Vol. 47, 2075–2084, 1999.
9. Pendry, J. B., A. J. Holden, W. J. Stewart, and I. Youngs, "Extremely low frequency plasmons in metallic mesostructures," *Phys. Rev. Lett.*, Vol. 76, 4773–4776, 1996.

10. Shelby, R. A., D. R. Smith, and S. Schultz, "Experimental verification of a negative index of refraction," *Science*, Vol. 292, 77–79, 2001.
11. Veselago, V. G., "The electrodynamics of substances with simultaneously negative values of ε and μ ," *Sov. Phys. Usp.*, Vol. 10, 509–514, 1968.
12. Pendry, J. B., "Negative refraction makes a perfect lens," *Phys. Rev. Lett.*, Vol. 86, 3966–3969, 2000.
13. Garcia, N. and M. Nieto-Vesperinas, "Left-handed materials do not make a perfect lens," *Phys. Rev. Lett.*, Vol. 88, 207403, 1–4, 2002.
14. Maystre, D. and S. Enoch, "Perfect lenses made with left handed materials: Alice's mirror," *J. Opt. Soc. Am. A*, Vol. 21, 122–131, 2004.
15. Chew, W. C., J. M. Jin, E. Michielson, and J. Song (eds.), *Fast and Efficient Algorithms in Computational Electromagnetics*, Artech House, Boston, 2001.
16. Taflov, A., *Computational Electrodynamics: The finite Difference Time Domain Method*, Artech House, Boston, 1995.
17. Hafner, C., *The Generalized Multipole Technique for Computational Electromagnetics*, Artech House, Boston, 1990.
18. Silvester, P. P. and G. Pelosi, *Finite Elements for Wave Electromagnetics*, IEEE Press, New York, 1994.
19. Maystre, D. and P. Vincent, "Diffraction d'une onde électromagnétique plane par un objet cylindrique non infiniment conducteur de section arbitraire," *Opt. Commun.*, Vol. 5, 327–330, 1972.
20. Abramovitz, M. and I. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1970.
21. Courant, R. and D. Hilbert, *Methods of Mathematical Physics*, Vol. 2, Interscience, New-York, 1962.
22. Kong, J. A., *Electromagnetic Wave Theory*, EMW Publishing, Cambridge, 2000.
23. Roger, A., "Reciprocity theorem applied to the computation of functional derivatives of the scattering matrix," *Electromagnetics*, Vol. 2, 69–83, 1982.
24. Maystre, D. and M. Cadilhac, "Singularities of the continuation of the fields and validity of Rayleigh's hypothesis," *Journal of Mathematical Physics*, Vol. 26, 2201–2204, 1985.
25. Schwartz, L., *Mathematics for the Physical Sciences*, Addison-Wesley, Reading, Mass., 1966.

26. Cessenat, M., “The use of the calderon projectors and the capacity operators in scattering,” *Scattering in Volumes and Surfaces*, M. Nieto-Vesperinas and J. C. Dainty (eds.), 255–268, Elsevier, Amsterdam, 1990.