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2D MAGNETIC PHOTONIC CRYSTALS WITH SQUARE LATTICE-GROUP THEORETICAL STANDPOINT

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Abstract—We consider possible magnetic symmetries of two-dimensional square lattices with circular ferrite rods magnetized by a uniform dc magnetic field. These structures can be used as tunable and nonreciprocal photonic crystals. Classification of eigenmodes in such crystals is defined on the basis of magnetic group theory and the theory of (co)representations. Some general electromagnetic properties of the magnetic crystals such as change in the basic domain of the Brillouin zone, change of symmetry in limiting cases, bidirectionality and nonreciprocity, symmetry relations for the waves and lifting of eigenwave degeneracies by dc magnetic field are also discussed.

1. INTRODUCTION

The theory of groups was used for investigation of electronic band structure of crystals since 30th of the last century. One of the first pioneering works in this field was [1] which had a significant impact on the solid state physics. Later, the theory of magnetic groups was applied to magnetic electronic crystals, and summary of this theory can be found in [2].

Approximately 15 years ago, the idea of using dielectric periodic structures for controlling the frequencies and directions of electromagnetic wave propagation attracted much attention of scientists around the world. These structures were called photonic band gap materials or photonic (electromagnetic) crystals. For investigation of physical properties of nonmagnetic photonic crystals, the theory of groups was used in [3, 4] and in some other publications.

In the last years, magnetic photonic crystals were investigated intensively because they allow tunability of some crystal parameters and nonreciprocity (see for example, [5, 6]).

Several new physical effects in magnetic crystals were discovered lately. One example is the effect of "the frozen mode" [20]. Theoretically, with properly chosen parameters of a magnetic crystal, one of its branches in a spectral band can possess a stationary inflection point. This point defines a frequency with zero group velocity of the corresponding Bloch wave for one direction in the crystal (this wave was called "the frozen mode"). For the opposite direction, the Bloch wave of the same frequency can propagate freely in the crystal.

Another example is the magneto-optic effect similar to the Faraday effect but for the propagation of waves perpendicular to the dc external magnetic field. This effect is intrinsic to magnetic photonic crystals and can be explained by the multiple scattering of waves and the symmetry [21].

Still another example is the effect of opening up of multiple band gaps due to the applied dc magnetic field discovered in semiconductor-dielectric photonic crystals [9].

Studying 3D simple cubic lattice of small magnetized ferrite spheres, the authors of [22] discovered an electrically controlled stop band which appears for one of the two circularly polarized eigenwaves. This stop band is below the usual lattice band gap.

As in case of electronic crystals, a natural development of the general photonic crystal theory can be made applying the theory of magnetic groups to magnetic photonic crystals. There is a principal difference between the electronic and electromagnetic waves in crystals. Electronic waves are described by scalar (neglecting electron spin) Schrödinger's equation whilst in the case of electromagnetic waves, one deals with vector Maxwell's equations [19]. The symmetry properties of the scalar and vector quantities are different. Therefore we can not transfer directly the results of the electronic wave symmetry theory to the photonic crystals.

From the point of view of symmetry, any photonic crystal is a periodic structure, i.e., it possesses a discrete translational symmetry. Besides, we can consider also geometrical symmetry of dielectric elements and their material (for example, anisotropy), and geometrical symmetry of the crystal unit cells. The symmetry of possible external perturbations (such as static electric or dc magnetic fields, heating) should also be taken into account.

Still another important physical symmetry is defined by the Time reversal. Literally, the Time reversal operator T denotes the change of the sign of Time t, i.e., $t \to -t$. In our practical application of the operator T of course, there is nothing of science fictions where one can travel from the present in the past and vice versa. In the magnetic group theory, the operator T simply reverses direction of motion. In the time domain, as a result, it changes the signs of the quantities which are odd in Time: the velocity, the wave vector, the magnetic field produced by moving charges, etc. In the frequency domain, it also complex transposes all quantities [7].

For the crystals with losses, the so-called *restricted* Time reversal operator $\mathcal{T}$ is a useful tool. The operator $\mathcal{T}$ does not complex conjugate the quantities describing losses [7]. This preserves the passive or active nature of media after application of the operator and consequently, preserves the damping or growing character of electromagnetic waves which propagate in the crystal. In the limiting case of crystals without losses, the operator $\mathcal{T}$ is equivalent to T. Notice that in accordance with our definitions of the operators $\mathcal{T}$ and T, invariance of a crystal with respect to $\mathcal{T}$ means reciprocity of the crystal, and invariance with respect to the operator T means both reciprocity and the absence of losses. For our purposes in this paper, the operator $\mathcal{T}$ is more appropriate, in particular, for discussion of reciprocity and bidirectionality in lossy crystals.

Maxwell's equations without sources and the corresponding constitutive relations for vacuum are invariant with respect to the transformation $t \to -t$. It means that the equations are invariant both to the operator $\mathcal{T}$ and to T (the classical vacuum is both reciprocal and lossless). Maxwell's equations in vacuum are also invariant with respect to all the possible rotations, reflections and Space inversion [8]. Thus, the solutions of Maxwell's equations do not depend on the possible rotations (i.e., on direction in Space), reflections and Space inversion.

If we consider the electromagnetic waves in a medium, the equations (constitutive relations) which describe physical properties of the medium should be complemented to Maxwell's equations. Exactly symmetry of these equations that defines some general properties of electromagnetic waves in the medium.

Thus, the rotation-reflection and Time reversal symmetry of the solutions of Maxwell's equations in a medium is defined by the symmetry of the constitutive relations for the medium. In particular, the results of application of the operator $\mathcal{T}$ to the constitutive relations depend on the physical properties of this medium. For a *nonmagnetic* medium, the constitutive relations remain invariant with respect to $\mathcal{T}$, and this means physically that the medium is reciprocal.

Magnetic media are not invariant with respect to $\mathcal{T}$. In this case, the Time reversal $\mathcal{T}$ applied to the constitutive relations changes them. The solutions of Maxwell's equations correspond to the Time-reversed (transformed) medium with the magnetic field reversed.

Some magnetic media can be invariant with respect to the operator $\mathcal{T}$ combined with geometrical rotations-reflections and translations.

In this case, the magnetic medium does not change its properties under the combined operators, and Maxwell's equations (without sources) with the constitutive relations are also invariant with respect to these operators. Though such a medium is nonreciprocal, we can use the combined operators containing $\mathcal{T}$ for investigation of physical properties of electromagnetic waves in the medium.

We shall analyze in this paper electromagnetic properties of magnetic photonic crystals using the theory of magnetic groups and the theory of (co)representations. Notice that the magnetic group theory includes nonmagnetic groups as a particular case.

In this paper, we shall not be concerned with numerical values of the solutions of Maxwell's equations. Our aim is to present some general concepts which can be useful in theoretical investigations of magnetic photonic crystals. In order to illustrate these concepts we shall consider a simple example of the two-dimensional (2D) square magnetic crystal lattice of circular ferrite rods. Some numerical results for magnetic crystals can be found in [9–12, 25].

Notice that the group theory is used not only to obtain a deep insight into the problems and to reduce the "band indexing difficulties". It is also useful to improve the efficiency of numerical computations.

This paper is addressed first of all to those who are not familiar with the theory of magnetic theory. In particular, we have tried to give a detailed description of the magnetic groups, the representations, the basic domain of the Brillouin zone, the star of the wave vector, etc.

The organization of the paper is as follows. Sec. 2 is devoted to the symmetry description of 2D square magnetic lattices. We consider in Sec. 3 symmetry of the wave vector in the magnetic lattices. In Sec. 4 we discuss briefly four limiting cases of these crystals. Some general symmetry properties of the crystals such as reduction of the vector wave equation to a scalar form, change of symmetry in limiting cases, symmetry relations for the waves with the opposite sign of the wave vector **k**, bidirectionality and nonreciprocity of crystals and lifting of the eigenwave degeneracies by dc magnetic field are discussed in Sec. 5. Finally, concluding remarks of Sec. 6 summarize the general symmetry properties of magnetic photonic crystals under consideration.

2. SYMMETRY DESCRIPTION OF 2D SQUARE MAGNETIC LATTICES

2.1. Possible Symmetries of Magnetic Lattices

We begin with a general description of the magnetic symmetries for 2D square lattices. The analyzed structure is shown in Fig. 1a. The

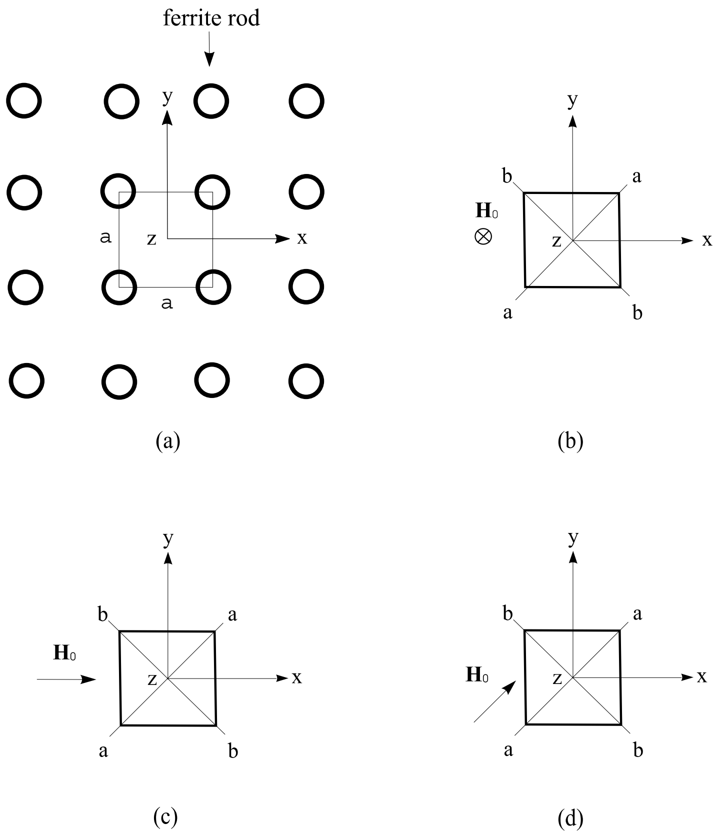

Figure 1. (a) 2D square lattice of circular cross-section ferrite rods, (b) the unit cell magnetized by $\mathbf{H}_0 \parallel z$, (c) the unit cell magnetized by $\mathbf{H}_0 \parallel x$, (d) the unit cell magnetized by $\mathbf{H}_0 \parallel (a - a)$.

uniform in z-direction circular ferrite rods are oriented along the z-axis. They form a square lattice in the plane $x0y$. The permeability of the magnetized ferrite rods is a tensor of the second rank $\bar{\boldsymbol{\mu}}(\mathbf{r})$ and the permittivity is a scalar $\epsilon(\mathbf{r})$. The space between the rods is filled with a dielectric with a scalar permeability μ_0 and a scalar permittivity ϵ_0. Both the ferrite and the dielectric are in general lossy (notice that considering Brillouin zones below, we shall assume the material parameters and consequently, the wave vector $\mathbf{k}$ to be real). Without dc magnetic field, one can consider the ferrite rods as dielectric ones described by a scalar permeability $\mu(\mathbf{r})$.

The square unit sell of the lattice has the period a in both the x- and the y-direction (Fig. 1a). Therefore, translational symmetry of the lattice is described by the two elementary lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\mathbf{a}_1 = \begin{pmatrix} \mathrm{a} \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ \mathrm{a} \end{pmatrix}. \tag{1}$$

The permeability $\bar{\boldsymbol{\mu}}(\mathbf{r})$ and the permittivity $\epsilon(\mathbf{r})$ are periodic functions with respect to translations in the lattice, that is $\bar{\boldsymbol{\mu}}(\mathbf{r}+\mathbf{a}) = \bar{\boldsymbol{\mu}}(\mathbf{r})$ and $\epsilon(\mathbf{r}+\mathbf{a}) = \epsilon(\mathbf{r})$ where $\mathbf{a} = m\mathbf{a}_1 + n\mathbf{a}_2$, m and n are integers.

The unit cell of the lattice shown in Fig. 1a is a square possessing the geometrical symmetry C_{4v} (in Schönflies notations [2]). The group C_{4v} contains the following 8 elements of symmetry:

- e is the identity element,
- C_2 is a rotation by π around the z-axis,
- C_4 and C_4^{-1} are the rotations around the z-axis by $\pi/2$ and by $-\pi/2$, respectively,
- σ_x and σ_y are the reflections in the planes $x = 0$ and $y = 0$, respectively,
- $\sigma_{(a-a)}$ and $\sigma_{(b-b)}$ are the reflections in the planes which pass through the axis z and the line $(a-a)$ and $(b-b)$, respectively.

The discussed geometrical group C_{4v} does not completely describe the physical properties of the nonmagnetic crystal. The full symmetry of this nonmagnetic crystal includes also the Time reversal operator $\mathcal{T}$ and the product of $\mathcal{T}$ with all the geometrical elements of the group C_{4v}, i.e., the full magnetic group can be denoted as $C_{4v} + \mathcal{T}C_{4v}$. This group contains 16 elements. The elements of geometric symmetry (i.e., planes and axes) combined with the Time reversal will be called further as the *antielements*, i.e., the *antiplanes* and *antiaxes*. In the absence of an external dc bias magnetic field, the crystal is invariant under the operation of the Time reversal because $\mathcal{T}$ is an element of the symmetry group of the crystal.

Magnetization of the crystal by a dc magnetic field changes the symmetry of the crystal. A useful method to define all the possible magnetic symmetries of our crystal is the group decomposition. In our case of the 2D square lattice, the group decomposition tree is shown in Fig. 2. The group decomposition includes all the possible subgroups of the group C_{4v}. They are C_4, C_{2v}, C_2, C_s, C_1. The highest group C_{4v} is at the top of the tree, and all its subgroups are below. At the bottom, the lowest group C_1 which contains only the identity element e.

The magnetization can result in magnetic groups which contain only geometrical elements of symmetry and do not contain the Time reversal operator at all. These groups (denoted by bold letters) are

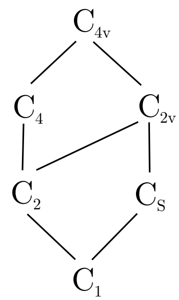

Figure 2. Subgroup decomposition of the point group C_{4v}.

$\boldsymbol{C}_{4v}$, $\boldsymbol{C}_4$, $\boldsymbol{C}_{2v}$, $\boldsymbol{C}_2$, $\boldsymbol{C}_s$ and $\boldsymbol{C}_1$. We do not consider these groups because they correspond to some special nonuniform configurations of dc magnetic field.

Fig. 2 allows one also to define magnetic groups of another type which do not contain the Time reversal on its own but they contain only the Time reversal operator in combination with certain geometrical elements of symmetry. These groups are $C_{4v}(C_4)$, $C_{4v}(C_{2v})$, $C_4(C_2)$, $C_{2v}(C_2)$, $C_{2v}(C_s)$, $C_2(C_1)$, and $C_s(C_1)$. In the notation $G(H)$, G is a group and H in the brackets is its subgroup. The subgroup H contains the elements $\mathcal{R}_1$ without operator $\mathcal{T}$ (these elements are called unitary), and the rest of the elements of G are combinations of some other geometrical operators $\mathcal{R}_2$ with the Time reversal $\mathcal{T}$ (the elements $\mathcal{T}\mathcal{R}_2$ are called nonunitary). The number of elements with and without the Time reversal is equal. Below, we shall consider some of these groups. The described above magnetic groups comprise all the possible magnetic symmetries of our photonic crystal which in the nonmagnetic state has the point symmetry $C_{4v} + \mathcal{T}C_{4v}$.

As was mentioned already, some of the magnetic symmetries in Fig. 2 require for their implementation a nonuniform magnetization. In what follows, we shall restrict ourselves by the magnetic structures which are magnetized by a uniform dc magnetic field.

2.2. Symmetry of Magnetic Lattice with Uniform dc Magnetic Field

A uniform dc magnetic field $\mathbf{H}_0$ of an arbitrary orientation with respect to the coordinate system xyz is applied to the crystal. The uniform dc magnetic field is an axial odd in Time vector with the symmetry $D_{\infty h}(C_{\infty h})$. The group $D_{\infty h}(C_{\infty h})$ contains all the rotations about the vector $\mathbf{H}_0$, the two-fold rotations about the axis normal to $\mathbf{H}_0$

combined with $\mathcal{T}$, and it has also the product of Space inversion with all the above operations. In accordance with Curie's principle of symmetry superposition [13], the magnetic group of the crystal is defined by the elements of symmetry which are common for the point group $C_{4v} + \mathcal{T}C_{4v}$ of the nonmagnetic square lattice and the magnetic group $D_{\infty h}(C_{\infty h})$ of the dc magnetic field $\mathbf{H}_0$.

For an arbitrary orientation of $\mathbf{H}_0$ with respect to the axis z in Fig. 1a, the resulting group of symmetry of the magnetic crystal contains only the identity element e. This group is trivial and gives no information about the properties of the crystal. We shall consider 3 nontrivial cases. One of them corresponds to $\mathbf{H}_0 \parallel z$, the second one is $\mathbf{H}_0 \perp z$ and in the third case the field $\mathbf{H}_0$ is in one of the planes $x0z$, $y0z$, $(a-a)0z$ or $(b-b)0z$ but it is neither parallel to the axis z nor perpendicular to it.

I. The field $\mathbf{H}_0$ is directed along the axis z (i.e., $\mathbf{H}_0 \parallel z$, Fig. 1b). The resulting group of symmetry of the system "2D square lattice + dc magnetic field" is $C_{4v}(C_4)$ which contains the following 8 elements:

- e is the identity element,
- C_2 is a rotation by π around the z-axis,
- C_4 and C_4^{-1} are rotations around the z-axis by $\pi/2$ and by by $-\pi/2$, respectively,
- $\mathcal{T}\sigma_x$ and $\mathcal{T}\sigma_y$ are the antireflections in the plane $x = 0$ and in the plane $y = 0$, respectively,
- $\mathcal{T}\sigma_{(a-a)}$ and $\mathcal{T}\sigma_{(b-b)}$ are the antireflections in the planes which pass through the axis z and the line $(a-a)$ and $(b-b)$, respectively.

II. The field $\mathbf{H}_0$ lies in the plane $x0y$. Here, we can consider the following 3 subcases:

a) $\mathbf{H}_0 \parallel x$ (Fig. 1c) or $\mathbf{H}_0 \parallel y$. One can come from $\mathbf{H}_0 \parallel x$ to $\mathbf{H}_0 \parallel y$ by simple rotation of the coordinate system around the z-axis by $\pi/2$. Therefore, the cases $\mathbf{H}_0 \parallel x$ and $\mathbf{H}_0 \parallel y$ are physically equivalent.

b) $\mathbf{H}_0 \parallel (a-a)$ (Fig. 1d) or $\mathbf{H}_0 \parallel (b-b)$. Also, one can come from $\mathbf{H}_0 \parallel (a-a)$ to $\mathbf{H}_0 \parallel (b-b)$ by rotation of the coordinate system around the z-axis by $\pi/2$. Therefore, the cases $\mathbf{H}_0 \parallel (a-a)$ and $\mathbf{H}_0 \parallel (b-b)$ are also physically equivalent.

In cases IIa and IIb, the resulting group of symmetry is $C_{2v}(C_s)$ which consists of the following 4 elements:

- e is the identity element,
- $\mathcal{T}C_2$ is an antirotation by π around the z-axis,
- σ_1 is the reflection in the plane which is perpendicular to $\mathbf{H}_0$,

- $\mathcal{T}\sigma_2$ is the antireflection in the plane which is parallel to $\mathbf{H}_0$.

From the group-theoretical point of view, the cases IIa and IIb are indistinguishable, because they are described by the same group of symmetry. However, they lead to different forms of the permeability tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$.

c) For any other orientation of $\mathbf{H}_0 \perp z$, the group of symmetry is lower, and it is $C_2(C_1)$ with the following 2 elements of symmetry:

- e is the identity element,
- $\mathcal{T}C_2$ is an antirotation by π around the z-axis.

III. The field $\mathbf{H}_0$ is in one of the planes $x0z$, $y0z$, $(a-a)0z$ or $(b-b)0z$ but it is neither parallel to the axis z nor perpendicular to it.

The group of symmetry in this case is $C_s(C_1)$ with the following 2 elements of symmetry:

- e is the identity element,
- $\mathcal{T}\sigma$ is an antiplane which is defined by the axis z and the vector $\mathbf{H}_0$.

As expected, all the above groups of symmetry can be obtained from the tree of Fig. 2. In the following, we shall consider the magnetic lattices with the symmetries $C_{4v}(C_4)$, $C_{2v}(C_s)$, $C_2(C_1)$ and $C_s(C_1)$. All these groups are subgroups of the group of symmetry $C_{4v} + \mathcal{T}C_{4v}$ of our crystal in nonmagnetic state.

Comparing the content of the group $C_{4v} + \mathcal{T}C_{4v}$ and those of the groups $C_{4v}(C_4)$, $C_{2v}(C_s)$, $C_2(C_1)$ and $C_s(C_1)$ we see that application of a dc magnetic field leads to a general reduction of symmetry of the nonmagnetic lattice.

It should be noted at this point that for simplicity, we consider 2D crystals. But the real 3D magnetic crystals can contain also the plane σ_z or antiplane of symmetry $\mathcal{T}\sigma_z$, the axis C_2 or antiaxis $\mathcal{T}C_2$ lying in the plane $x0y$ which can give an important information. This will be discussed in Sec. 5.2.

The final remark of this section is as follows. We need to consider different symmetries of the magnetic structure under investigation because, for example, in physical experiments, a small deviation of the applied dc magnetic field orientation from the chosen one can lead to qualitative change of the basic domain, of the band structure, of the field structure of eigenwaves, etc. Thus, the knowledge of the possible change of symmetry and the consequences of this due to the change of the orientation of the dc magnetic field can be useful in interpreting the obtained experimental results. Besides, different orientations of the dc magnetic field can lead to different physical effects which can be used in electromagnetic devices.

2.3. Permeability Tensors

The structure of the permeability tensors of the ferrite is defined by the symmetry of the uniform dc magnetic field $\mathbf{H}_0$ which is $D_{\infty h}(C_{\infty h})$, and by the orientation of $\mathbf{H}_0$ with respect to the coordinate axes. These tensors have been calculated by the known methods [13] and some of them are given in Table A1 of Appendix A. The tensors are invariant with respect to all the operations of the corresponding groups of symmetry.

Notice that the less symmetry (i.e., the less the number of the elements which the group has) the more parameters the corresponding tensor has. Namely, the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$ for a nonmagnetized ferrite degenerates to a scalar with 1 complex parameter. The tensors $\bar{\boldsymbol{\mu}}(\mathbf{r})$ in Table A1 for the orientations of $\mathbf{H}_0$ parallel the axis z have 3 parameters. The tensors $\bar{\boldsymbol{\mu}}(\mathbf{r})$ for the orientation of $\mathbf{H}_0$ parallel to the lines $(a-a)$ have 4 parameters and for an arbitrary orientation of the dc magnetic field in the plane $x0y$, the tensor has 6 parameters. For a general orientation of $\mathbf{H}_0$ with respect to the coordinate system xyz, the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$ has all 9 complex parameters. Thus, with reducing the symmetry of the crystal, the electrodynamic calculations become more involved.

Notice also that in spite of different forms and different number of parameters, all the tensors for the homogeneous ferrite for different directions of magnetization in Table A1 have the same number of *independent* parameters. The simplest form of the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$ has 3 parameters. All the other forms can be reduced to the simplest one by a rotation of the coordinate system.

A remark concerning the symmetry of magnetization in ferrite elements should be made. The numerical values of the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$ depend on magnetization of the ferrite. For ellipsoids in a uniform external dc magnetic filed $\mathbf{H}_0$, the magnetization in the saturated regime is uniform. Infinitely long cylinders are a particular case of ellipsoids, therefore the magnetization of them is also uniform. Thus, in this case the parameters of $\bar{\boldsymbol{\mu}}(\mathbf{r})$ are constant inside the ferrite elements. In a general case of a nonuniform dc magnetic field and/or of ferrite elements of an arbitrary shape, the magnetization is nonuniform. The parameters of the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$ will depend on $\mathbf{r}$ inside the ferrite but the structure of the tensor defined above (i.e., equality of some of the elements to zero or equality of some of them to each other) depends only on the symmetry. It should be stressed also that all the following discussions of this paper which are based on the geometry only are applicable to the crystals with nonuniform magnetization of the ferrite elements as well.

3. SYMMETRY OF THE WAVE VECTOR IN MAGNETIC LATTICES

3.1. Brillouin Zone and Basic Domain of the Magnetic Crystal

In Sec. 2.2 above, we have studied the magnetic symmetry of the square, that is the point symmetry of the crystals. In the Seitz notations [2], a space group operation is denoted by $\{\mathcal{R}|\mathbf{a}\}$ where $\mathcal{R}$ is a point group operation and $\mathbf{a}$ is a translation operation of the lattice. Our crystal is described by a symmorphic space group because it does not contain screw axes and glide planes. In the symmorphic groups, one can always write any element of the group $\{\mathcal{R}|\mathbf{a}\}$ as a the product $\{\mathcal{R}|\mathbf{a}\} = \{\mathcal{R}|\mathbf{0}\}\{e|\mathbf{a}\}$, $\mathbf{0}$ is the zero vector.

The irreducible representations (IRREPs) of the space groups are defined by $\exp\{i\mathbf{k}\cdot\mathbf{a}\}\cdot\Gamma(\mathcal{R})$ where $\mathbf{k}$ is the wave vector, $\Gamma(\mathcal{R})$ is the rotation-reflection part and $\exp\{i\mathbf{k}\cdot\mathbf{a}\}$ is the translational part of the IRREPs. Our aim in this paper is the magnetic point symmetry therefore we shall not discuss the irreducible representations of the translational part of the crystal space groups.

The reciprocal lattice of the crystal is defined by the two primitive vectors $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$\mathbf{b}_1 = \begin{pmatrix} 2\pi/a \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 2\pi/a \end{pmatrix}, \tag{2}$$

and the reciprocal lattice vector $\mathbf{G}$ is

$$\mathbf{G} = n_1\mathbf{b}_1 + n_2\mathbf{b}_2, \tag{3}$$

where n_1 and n_2 are arbitrary integers.

Before discussing the symmetry of the wave vector $\mathbf{k}$, we should define the Brillouin zone (BZ) for the above cases of magnetic structures. First of all, the shape of the BZ zone does not coincide in general, with the shape of the unit cell of a lattice. However, in the case of the nonmagnetic square unit cell, the BZ has also the square shape.

Besides, in general, a dc magnetic field can change the size and even the shape of the BZ. But in our case of the *uniform* dc magnetic field, the unit cell and consequently, the BZ are not changed because the translational symmetry of the crystal is unchanged after biasing by such a dc magnetic field. Thus, in spite of different magnetic symmetries, the BZ of the photonic crystal with and without magnetization are exactly the same. Therefore, for all our symmetries $C_{4v}(C_4), C_{2v}(C_s)$, $C_2(C_1)$ and $C_s(C_1)$ we shall investigate the square BZ which is identical to the BZ of the nonmagnetic lattice (Fig. 3).

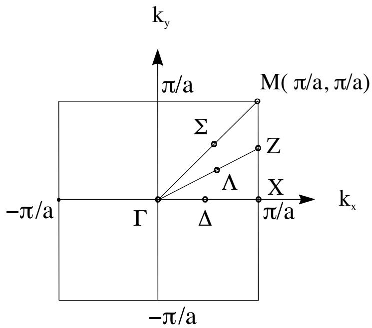

Figure 3. The reduced Brillouin zone for the 2D square lattice of circular cross-section ferrite rods.

In band calculations, we can usually restrict ourselves to a single *basic* domain of the BZ. This allows one to reduce the burden of numerical calculations. The basic domain for nonmagnetic crystals is defined by the smallest part of the BZ from which the whole BZ can be obtained by applying all the operators of the point group [2]. The basic domain for the nonmagnetic square lattice is the triangle ΓMX shown in Fig. 3. It is one-eights of the area of the whole BZ.

One of the consequences of the reduction of symmetry of crystal due to the applied dc magnetic field is an increase of the basic domain size. In order to define the basic domain in our magnetic 2D crystals, we can use the notion of the star of the wave vector $\mathbf{k}$. A set of distinct vectors $\mathbf{k}$ constructed by applying all the operators of the crystal point group to a vector $\mathbf{k}$ in a general position inside the BZ (for example, the vector Λ in Fig. 3 which is not a point or line of symmetry) is called the star of $\mathbf{k}$. The area occupied by the basic domain is equal to the whole area of the BZ divided by the number of the vectors in the star. The number of the vectors in the star for the groups $C_{4v} + \mathcal{T}C_{4v}$ and $C_{4v}(C_4)$ is 8. The star and the basic domain for these groups are shown in Fig. 4a.

For the group $C_{2v}(C_s)$, the star consists of 2 vectors. The star and the basic domain for this group are shown in Fig. 4b. The basic domain is a half of the whole BZ. In the case of $C_s(C_1)$, the operator $\mathcal{T}\sigma$ transforms $\mathbf{k}$ in $-\mathbf{k}$ so that the number of the vectors in the star is 2. Therefore, the basic domain for the group $C_s(C_1)$ coincides with that of the group $C_{2v}(C_s)$. For the group $C_2(C_1)$, the star consists of a single vector because the operator $\mathcal{T}C_2$ transforms $\mathbf{k}$ into itself, and the basic domain is the whole BZ (Fig. 4c). A more detailed discussion of the problem of the basic domain in magnetic electronic crystals can be found in [14].

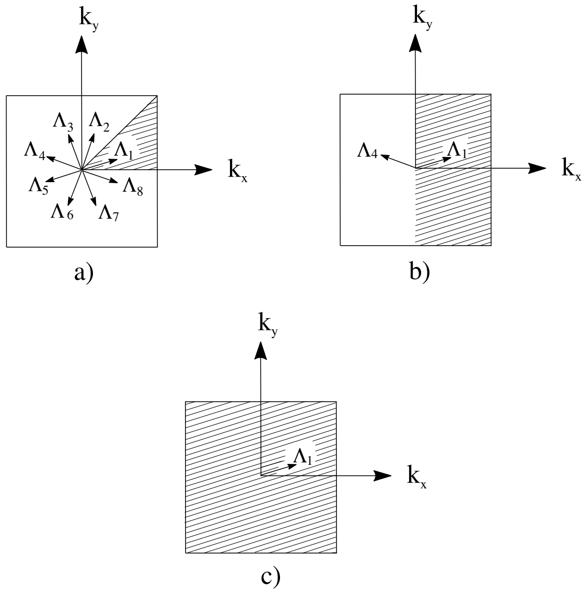

Figure 4. The basic domains (dashed areas) and the stars for the magnetic crystals with different symmetries: (a) $C_{4v} + \mathcal{T}C_{4v}$ and $C_{4v}(C_4)$, (b) $C_{2v}(C_s)$ and $C_s(C_1)$, (c) $C_2(C_1)$.

3.2. Group of Symmetry of the Wave Vector

Now, let us apply to the symmetry of the wave vector $\mathbf{k}$. In the theory of electronic waves in crystals, the symmetry group of $\mathbf{k}$ is called the little group. In the theory of magnetic crystals, it is called the magnetic little group. We shall denote the magnetic little group for a given $\mathbf{k}$ as $\mathbf{M}^{\mathbf{k}}$.

There is a general symmetry property of the wave vector $\mathbf{k}$ in crystals. The groups of the wave vector $\mathbf{k}$ for different points and lines of symmetry of a given crystal are subgroups of the symmetry group of the crystal as a whole. In order to clarify this property, let us denote a magnetic group of symmetry of a crystal as $G_1(H_1)$. At any symmetric point or line of the BZ with a lower symmetry, the group of the vector $\mathbf{k}$ denoted as $G_2(H_2)$ will be a subgroup of $G_1(H_1)$. Moreover, the group H_2 is a subgroup of H_1. These subgroup relations are shown pictorially in Table 1 for the groups $C_{4v} + \mathcal{T}C_{4v}$ and $C_{4v}(C_4)$, and in Table 2 for the groups $C_{4v}(C_4)$ and $C_{2v}(C_2)$. These groups are met in our discussion (see for example, Table 4).

In order to define the group $\mathbf{M}^{\mathbf{k}}$ for electromagnetic waves in

Table 1. Subgroup relations for the group $C_{4v}+\mathcal{T}C_{4v}$ and its subgroup $C_{4v}(C_4)$.

Nonunitary group		Unitary subgroup
$C_{4v}+\mathcal{T}C_{4v}$	$\longrightarrow$	C_{4v}
$\downarrow$		$\downarrow$
$C_{4v}(C_4)$	$\longrightarrow$	C_4

Table 2. Subgroup relations for the group $C_{4v}(C_4)$ and its subgroup $C_{2v}(C_2)$.

Nonunitary group		Unitary subgroup
$C_{4v}(C_4)$	$\longrightarrow$	C_4
$\downarrow$		$\downarrow$
$C_{2v}(C_2)$	$\longrightarrow$	C_2

magnetic photonic crystals, one should consider all the constituents of the physical problem from the point of view of magnetic symmetry. The wave vector $\mathbf{k}$ in free space is a polar odd in Time vector with the symmetry $D_{\infty h}(C_{\infty v})$. The group $D_{\infty h}(C_{\infty v})$ contains the axis of an infinite order C_∞ coinciding with $\mathbf{k}$, an infinite number of planes of symmetry σ_v passing through this axis, an antiplane $\mathcal{T}\sigma_h$ which is perpendicular to the principal axis, an infinite number of the two-fold antiaxis $\mathcal{T}C_2$ lying in the antiplane $\mathcal{T}\sigma_h$, and also the anticenter $\mathcal{T}i$.

The symmetry of $\mathbf{k}$ in free space does not depend on its orientation in Space. In a magnetic lattice, the group of symmetry of $\mathbf{k}$ (i.e., the little group $\mathbf{M}^{\mathbf{k}}$) is defined by the "environment", i.e., by the symmetry of the lattice and by the symmetry of the magnetic field $\mathbf{H}_0$. $\mathbf{M}^{\mathbf{k}}$ depends also on the orientation of the vector $\mathbf{k}$ and its size. The point Γ ($\mathbf{k}=0$) of the centre of the BZ has the symmetry of the crystal as a whole.

The Time reversal operator $\mathcal{T}$ as an element of the group of symmetry of a nonmagnetic crystal, sends $\mathbf{k}$ into $-\mathbf{k}$, i.e.,

$$\mathcal{T}\mathbf{k} = -\mathbf{k}. \tag{4}$$

In the cases of magnetic crystals, the Time reversal $\mathcal{T}$ does not exist in "pure" form, but it can enter in the group in the combined operations (a geometrical operation + Time reversal). Let us denote any operator of geometrical symmetry as $\mathcal{R}_1$ and an operator of combined symmetry as $\mathcal{T}\mathcal{R}_2$. The magnetic little group $\mathbf{M}^{\mathbf{k}}$ consists of those geometrical operators $\mathcal{R}_1$ which transform the wave vector $\mathbf{k}$ into itself or into $\mathbf{k}+\mathbf{G}_1$ [2]:

$$\mathcal{R}_1\mathbf{k} = \mathbf{k} \quad or \quad \mathcal{R}_1\mathbf{k} = \mathbf{k} + \mathbf{G_1}, \tag{5}$$

and also of the combined operators $\mathcal{T}\mathcal{R}_2$ with $\mathcal{R}_2$ which transform $\mathbf{k}$ into $-\mathbf{k}$ or into $-\mathbf{k}+\mathbf{G}_2$:

$$\mathcal{R}_2\mathbf{k} = -\mathbf{k} \quad \mathbf{or} \quad \mathcal{R}_\mathbf{2}\mathbf{k} = -\mathbf{k} + \mathbf{G_2}. \tag{6}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are primitive translations of the reciprocal lattice.

We shall denote a general point Λ of the BZ by $\pi/a(\alpha,\,\beta)$ which means that $\mathbf{k} = \pi/a(\alpha\mathbf{e}_x + \beta\mathbf{e}_y)$, where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors in the x and y directions, respectively.

Nonmagnetic case. In Table 3, we give a description of the little groups for the nonmagnetic crystal described by the group C_{4v}. This Table can serve as a reference for the magnetic symmetries discussed below.

Table 3. Little groups and their elements for points and lines of symmetry for square nonmagnetic lattice.

Symmetry symbol	Representative wave vector $\mathbf{k}$	Little group	Order of the group	Elements of the group
Γ	$\pi/a(0,0)$	C_{4v}	8	e, C_4, C_4^{-1}, C_2, σ_x, σ_y, $\sigma_{(a-a)}$, $\sigma_{(b-b)}$
M	$\pi/a(1,1)$	C_{4v}	8	e, C_4, C_4^{-1}, C_2, σ_x, σ_y, $\sigma_{(a-a)}$, $\sigma_{(b-b)}$
X	$\pi/a(1,0)$	C_{2v}	4	e, C_2, σ_x, σ_y
Z	$\pi/a(1,\alpha)$	C_s	2	e, σ_x
Σ	$\pi/a(\alpha,\alpha)$	C_s	2	e, $\sigma_{(a-a)}$
Δ	$\pi/a(\alpha,0)$	C_s	2	e, σ_y
Λ	$\pi/a(\alpha,\beta)$	C_1	1	e

The points Γ and M of the BZ (Fig. 3) have the symmetry C_{4v}. When we depart from the point Γ in the direction of M, we are on the line denoted Σ with the coordinates of the wave vector $(\alpha,\ \alpha,)$,

$0 < \alpha < \sqrt{2}\pi/a$. The group of symmetry of the wave vector on the line Σ is C_s which is a subgroup of C_{4v}. The group C_s contains the elements e and $\sigma_{(a-a)}$. Analogous examination can be made for other points and lines of the BZ.

Now we apply to the magnetic crystals.

$\mathbf{H}_0 \parallel z$. This magnetization reduces the symmetry of the crystal from $C_{4v} + \mathcal{T}C_{4v}$ to $C_{4v}(C_4)$. For the points Γ and M of the BZ (Fig. 3), the wave vectors have the symmetry $C_{4v}(C_4)$ (Table 4). The symmetry of the point X is $C_{2v}(C_2)$. The symmetry of the vectors Δ and Z is $C_s(C_1)$. The wave vector Λ in a general point of the BZ has no symmetry.

Table 4. Little groups and their elements for points and lines of symmetry for square magnetic lattice with dc magnetic field $\mathbf{H}_0 \parallel z$, the crystal group is $C_{4v}(C_4)$.

Symmetry symbol	Representative wave vector $\mathbf{k}$	Little group	Order of the group	Elements of the group
Γ, M	$\pi/a(0,0), \pi/a(1,1)$	$C_{4v}(C_4)$	8	e, C_4, C_4^{-1}, C_2, $\mathcal{T}\sigma_x$, $\mathcal{T}\sigma_y$, $\mathcal{T}\sigma_{(a-a)}$, $\mathcal{T}\sigma_{(b-b)}$
X	$\pi/a(1,0)$	$C_{2v}(C_2)$	4	e, C_2, $\mathcal{T}\sigma_x$, $\mathcal{T}\sigma_y$
Z	$\pi/a(1,\beta)$	$C_s(C_1)$	2	e, $\mathcal{T}\sigma_y$
Δ	$\pi/a(\alpha,0)$	$C_s(C_1)$	2	e, $\mathcal{T}\sigma_x$
Σ, Λ	$\pi/a(\alpha,\alpha), \pi/a(\alpha,\beta)$	C_1	1	e

$\mathbf{H}_0 \parallel x$. When the magnetic field is along the axis x, the magnetic group of the crystal is $C_{2v}(C_s)$ (Table 5). We can note that the wave vectors in the points Γ, M, X and Z for the magnetization along the axis x have the symmetry $C_{2v}(C_s)$, but the vectors Σ, Δ and Λ have the lower symmetry $C_2(C_1)$.

$\mathbf{H}_0 \parallel (a-a)$. This magnetic crystal is also described by the magnetic group $C_{2v}(C_s)$ (Table 6). Comparing Table 5 and Table 6 we can see a difference in the above two cases with the same symmetry $C_{2v}(C_s)$ but with the different orientation of $\mathbf{H}_0$ in the plane $x0y$.

Magnetic crystal with an arbitrary orientation of $\mathbf{H}_0 \perp z$ *in the plane* $x0y$. For an arbitrary orientation of $\mathbf{H}_0$ in the plane $x0y$, the group of symmetry is $C_2(C_1)$ which consists only of 2 elements, the unity element e and antirotation by π around the z axis $\mathcal{T}C_2$. All the vectors in the BZ have this symmetry (Table 7).

Table 5. Little groups and their elements for points and lines of symmetry for square magnetic lattice with dc magnetic field $\mathbf{H}_0 \parallel x$, the crystal group is $C_{2v}(C_s)$.

Symmetry symbol	Little group	Order of the group	Elements of the group
Γ, M, X, Z	$C_{2v}(C_s)$	4	e, σ_x, $\mathcal{T}C_2$, $\mathcal{T}\sigma_y$
Σ, Δ, Λ	$C_2(C_1)$	2	e, $\mathcal{T}C_2$

Table 6. Little groups and their elements for points and lines of symmetry for square magnetic lattice with dc magnetic field $\mathbf{H}_0 \parallel (a-a)$, the crystal group is $C_{2v}(C_s)$.

Symmetry symbol	Little group	Order of the group	Elements of the group
Γ, M	$C_{2v}(C_s)$	4	e, $\sigma_{(b-b)}$, $\mathcal{T}C_2$, $\mathcal{T}\sigma_{(a-a)}$
X, Z, Σ, Δ, Λ	$C_2(C_1)$	2	e, $\mathcal{T}C_2$

Table 7. Little groups and their elements for points and lines of symmetry for square magnetic lattice with arbitrary orientation of dc magnetic field $\mathbf{H}_0 \perp z$, the crystal group is $C_2(C_1)$.

Symmetry symbol	Little group	Order of the group	Elements of the group
$\Gamma, M, X, Z, \Sigma, \Delta, \Lambda$	$C_2(C_1)$	2	e, $\mathcal{T}C_2$

Magnetic crystal with an arbitrary orientation of $\mathbf{H}_0$ *in one of the planes* $x0z$, $y0z$, $(a-a)0z$ *or* $(b-b)0z$. In all these cases, the group of symmetry is $C_s(C_1)$ with 2 elements: e and the antiplane $\mathcal{T}\sigma$ which is defined by $\mathbf{H}_0$ and the axis z. Magnetic little groups for $\mathbf{H}_0 \parallel x0z$ are written in Table 8.

Table 8. Little groups and their elements for points and lines of symmetry for square magnetic lattice with dc magnetic field $\mathbf{H}_0 \parallel x0z$, the crystal group is $C_s(C_1)$.

Symmetry symbol	Little group	Order of the group	Elements of the group
Γ, M, X, Z	$C_s(C_1)$	2	e, $T\sigma_y$
Σ, Δ, Λ	C_1	1	e

4. LIMITING CASES

One can consider 4 limiting cases of our problem. The first one is the *material* limiting case. When the dc magnetic field goes to zero, the permeability tensor (the material parameter) of the ferrite rods reduces to a scalar, and as a result, Eqs. (7) and (8) of Sec. 5.1 can be reduced to the scalar form. Obviously, this nonmagnetic case is much simpler than the magnetic one.

Besides, two *geometric* limiting cases can be considered also. Directing the fractal volume occupied by the ferrite to 1, one comes to the limit of the homogeneous ferrite. On the opposite extreme of the zero fractal ferrite volume, one has the homogeneous dielectric medium. The discrete symmetry (periodicity) of the crystal is transformed in both cases in continuous symmetry of the homogeneous media.

Still another limiting case is the long-wavelength approximation (i.e., for $\mathbf{k} \to 0$). In this approximation, the photonic crystal behaves like a homogeneous media. The resulting symmetry of the medium is defined by symmetry of the ferrite rods, of the crystal lattice and of the applied field $\mathbf{H}_0$. In this case, the effective scalar permittivity and the effective second-rank tensor describing the magnetic properties of the media can be calculated. The structure of the tensor can be defined using group-theoretical methods and the numerical values of the parameters can be calculated by the methods of homogenization.

All the above limits lead to a change of the crystal symmetry. In the first (material) limit, the periodicity is preserved but the point group is transformed from a magnetic case to the nonmagnetic one. In the two geometric limits, the periodicity is disappeared and the crystal transforms into a homogeneous material with electromagnetic properties which do not depend on the position but can depend on the direction. The symmetry group becomes K_h for the dielectric isotropic medium, and $D_{\infty h}(C_{\infty h})$ for the magnetic one with the principal axis $C_\infty \parallel \mathbf{H}_0$.

In the long-wavelength approximation, the symmetry group C_{4v} is transformed in $C_{\infty v}$ [13], and the resulting magnetic group of symmetry will depend on the mutual orientation of the axis C_∞ and the vector $\mathbf{H}_0$. For $\mathbf{H}_0 \parallel C_\infty \parallel z$, the group $C_{4v}(C_4)$ of the magnetic crystal is transformed in $C_{\infty v}(C_\infty)$, for $\mathbf{H}_0 \perp C_\infty \parallel z$, we obtain the group $C_{2v}(C_s)$. For any other orientation of $\mathbf{H}_0$, the group is $C_s(C_1)$.

The nonmagnetic homogeneous case is trivial one with the linear solution $\omega = vk$ where v is the wave velocity. The case of magnetic homogeneous media is more complex but the corresponding solutions are also well known [16]. This limit can be used in the analysis of magnetic crystals.

The above limiting cases can serve as references in numerical calculations and measurements.

5. SYMMETRY PROPERTIES OF EIGENWAVES

5.1. Wave Equations

We shall consider Maxwell's equations in the frequency domain. The vector wave equations for the electric displacement $\mathbf{D}(\mathbf{r})$ and the magnetic induction $\mathbf{B}(\mathbf{r})$ in the photonic ferrite crystal with the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$ and the scalar $\epsilon(\mathbf{r})$ have the following form:

$$\mathcal{L}_D\mathbf{D}(\mathbf{r})\frac{\omega^2}{c^2}\mathbf{D}(\mathbf{r}), \tag{7}$$

$$\mathcal{L}_B\mathbf{B}(\mathbf{r})\frac{\omega^2}{c^2}\mathbf{B}(\mathbf{r}), \tag{8}$$

where the differential operators $\mathcal{L}_D$ and $\mathcal{L}_B$ are

$$\mathcal{L}_D\boldsymbol{\nabla}\times\left\{\bar{\boldsymbol{\mu}}^{-1}(\mathbf{r})\boldsymbol{\nabla}\times[\epsilon^{-1}(\mathbf{r})\cdot\quad]\right\}, \tag{9}$$

$$\mathcal{L}_B\boldsymbol{\nabla}\times\left\{\epsilon^{-1}(\mathbf{r})\boldsymbol{\nabla}\times[\bar{\boldsymbol{\mu}}^{-1}(\mathbf{r})\cdot\quad]\right\}, \tag{10}$$

$\bar{\boldsymbol{\mu}}^{-1}(\mathbf{r})$ is the tensor inverse to $\bar{\boldsymbol{\mu}}(\mathbf{r})$, c is the light velocity in vacuum. Eqs. (7) and (8) should be solved together with the equations

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 0, \tag{11}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{12}$$

and the vectors $\mathbf{E}$ and $\mathbf{H}$ can be defined by the constitutive relations

$$\mathbf{D} = \epsilon(\mathbf{r})\mathbf{E}, \quad \mathbf{B} = \bar{\boldsymbol{\mu}}(\mathbf{r})\mathbf{H}. \tag{13}$$

The Space and Time reversal symmetry properties of the vector $\mathbf{D}$ and $\mathbf{E}$ are the same. They are even in Time polar vectors connected by the scalar $\epsilon(\mathbf{r})$. The Space and Time reversal symmetry properties of the vector $\mathbf{B}$ and $\mathbf{H}$ are also the same. They are odd in Time axial vectors related by the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$. The tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$ has the symmetry of the magnetized ferrite medium, because its structure was calculated using this symmetry (see Table A1). Thus, the symmetry properties of the crystals can be discussed in terms of $\mathbf{D}$ and $\mathbf{B}$ vectors or equivalently, in terms of $\mathbf{E}$ and $\mathbf{H}$ vectors.

Following [15], we write the vector $\mathbf{B}$ as a plane wave

$$\mathbf{B} = e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}), \tag{14}$$

where $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$ is a periodic function with the period $\mathbf{a}$, n is a band index. The quantity $\exp(i\mathbf{k}\cdot\mathbf{r})$ changes the sign of its exponent under Time reversal (this corresponds to changing the direction of propagation). Taking into account this circumstance, we can discuss the symmetry properties of the vector $\mathbf{B}$ in terms of the vector $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$.

Substituting $\mathbf{B}$ in (8) by the expression of (14), we obtain the eigenvalue equation which contains the wave vector $\mathbf{k}$:

$$(i\mathbf{k}+\boldsymbol{\nabla})\times\left\{\epsilon^{-1}(\mathbf{r})(i\mathbf{k}+\boldsymbol{\nabla})\times[\bar{\boldsymbol{\mu}}^{-1}(\mathbf{r})\cdot\mathbf{u}_{\mathbf{k}n}(\mathbf{r})]\right\}\frac{\omega_n^2}{c^2}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}), \tag{15}$$

or

$$\mathcal{L}_u\mathbf{u}_{\mathbf{k}n}(\mathbf{r})\frac{\omega_n^2}{c^2}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}). \tag{16}$$

Analogously, writing the vector $\mathbf{D}$ as

$$\mathbf{D} = e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{v}_{\mathbf{k}n}(\mathbf{r}), \tag{17}$$

we can obtain the following wave equation:

$$(i\mathbf{k}+\boldsymbol{\nabla})\times\left\{\bar{\boldsymbol{\mu}}^{\mathbf{-1}}(\mathbf{r})\cdot(i\mathbf{k}+\boldsymbol{\nabla})\times[\epsilon^{-1}(\mathbf{r})\mathbf{v}_{\mathbf{k}n}(\mathbf{r})]\right\}\frac{\omega_n^2}{c^2}\mathbf{v}_{\mathbf{k}n}(\mathbf{r}), \tag{18}$$

or

$$\mathcal{L}_v\mathbf{v}_{\mathbf{k}n}(\mathbf{r})\frac{\omega_n^2}{c^2}\mathbf{v}_{\mathbf{k}n}(\mathbf{r}). \tag{19}$$

The symmetry properties of the vector $\mathbf{D}$ can be discussed in terms of the vector $\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ taking into account the above mentioned property of the exponent $e^{i(\mathbf{k}\cdot\mathbf{r})}$. Eqs. (15) and (18) can be solved numerically to obtain the band diagrams $\omega_n(\mathbf{k})$.

5.2. Reduction of the Vector Wave Equations to a Scalar Form

We consider propagation of waves in the plane xoy. For this problem in 2D *nonmagnetic* square lattices, the general vector wave equation can be reduced to two independent scalar equations [4]. This property of the wave equation is a consequence of symmetry. Namely, it is stipulated by the presence of the symmetry plane σ_z [15]. Notice that σ_z is not an element of the groups of symmetry of our idealized 2D problem. It is an element of the 3D problem. But the real problem is always 3D, therefore we can include σ_z in our investigation as well. Thus, the operator σ_z transforms the wave vector $\mathbf{k}$ into itself, that is the element σ_z is in the little group. Using the transformation properties of this operator, we can easily find that the E-polarized modes can have only the components E_z, H_x, and H_y, and the H-polarized modes can have the components H_z, E_x, and E_y.

In one of the limiting cases of our problem, namely in a homogeneous infinite ferrite medium, the reduction of the vector wave equation to two independent scalar differential equations of the second order is possible for two particular cases of the wave vector orientation with respect to the dc magnetic field: when $\mathbf{k}$ is parallel to the dc magnetic field $\mathbf{H}_0$ ($\mathbf{k} \parallel \mathbf{H}_0$), and when $\mathbf{k}$ is perpendicular to $\mathbf{H}_0$ ($\mathbf{k} \perp \mathbf{H}_0$) [16].

For our magnetic photonic crystals and $\mathbf{k}$ lying in the plane xoy, the reduction of the vector wave equation to 2 scalar equations is possible for the orientation of the dc magnetic field parallel to the axis z, i.e., for $\mathbf{H}_0 \parallel z$, because the plane of symmetry σ_z in this case exists as well.

If $\mathbf{H}_0 \parallel xoy$, the plane σ_z is transformed into the antiplane $\mathcal{T}\sigma_z$. Even when ($\mathbf{k} \parallel \mathbf{H}_0$), a reduction of the vector wave equation to 2 scalar equations is impossible. The operator $\mathcal{T}\sigma_z$ transforms $+\mathbf{k}$ into $-\mathbf{k}$, therefore in general $\mathcal{T}\sigma_z$ is not an element of a little group. The presence of this operator defines the relation $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$, i.e., bidirectionality for any direction of $\mathbf{k}$ in the plane $x0y$. Bidirectionality is discussed below in Sec. 5.4.

5.3. Symmetry of Eigenmodes

All the operators $\mathcal{R}_1$ and $\mathcal{T}\mathcal{R}_2$ of the symmetry group of the magnetic crystal commute with the tensor $\bar{\boldsymbol{\mu}}(\mathbf{r})$, and this is reflected in its simplified structure. It is not difficult to show that Eqs. (16) and (19) are invariant with respect to the symmetry operators $\mathcal{R}_1$ and $\mathcal{T}\mathcal{R}_2$.

Thus, for the operators $\mathcal{TR}_2$ for example we have

$$\mathcal{L}_u \mathcal{T}\mathcal{R}_2 \mathbf{u}_{\mathbf{k}n}(\mathbf{r}) \frac{\omega_n^2(\mathbf{k})}{c^2} \mathcal{T}\mathcal{R}_2 \mathbf{u}_{\mathbf{k}n}(\mathbf{r}), \tag{20}$$

$$\mathcal{L}_v \mathcal{T}\mathcal{R}_2 \mathbf{v}_{\mathbf{k}n}(\mathbf{r}) \frac{\omega_n^2(\mathbf{k})}{c^2} \mathcal{T}\mathcal{R}_2 \mathbf{v}_{\mathbf{k}n}(\mathbf{r}). \tag{21}$$

Notice that in practical calculations, the operator $\mathcal{R}_1$ or $\mathcal{R}_2$ should be replaced by the corresponding 3×3 matrix representation [13].

The transformed vector $\mathbf{k}$ is defined by $\mathcal{R}_1\mathbf{k}$ or $\mathcal{T}\mathcal{R}_2\mathbf{k}$. The transformed field structure is calculated by $\mathcal{R}_1\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$ or $\mathcal{T}\mathcal{R}_2\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$ and $\mathcal{R}_1\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ or $\mathcal{T}\mathcal{R}_2\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$. Using the elements of magnetic little groups one can find some restrictions on the structure of eigenwaves. One example of such restrictions was discussed in Section 5.2.

One important consequence of the symmetry is as follows: the eigenmodes of the operators $\mathcal{L}_u$ and $\mathcal{L}_v$, i.e., the functions $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$ and $\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ form the basis for the IRREPs. In other words, the dispersion characteristics $\omega_n(\mathbf{k})$ of the magnetic crystal have the full symmetry of the point magnetic group of the crystal. Thus, we can use the IRREPs of the magnetic little groups to classify the eigenmodes. In fact, in most cases for the correct classification of the eigenmodes, it is sufficient to use only the unitary subgroups of the corresponding magnetic little groups. The peculiarities of the IRREPs of the point magnetic groups are discussed in [2].

The IRREPs of the point nonmagnetic groups can be found in many books on group theory (see, for example [2]). For the reader convenience, we present in Appendix B the IRREPs of some of the groups which are met in our discussion. From the Tables of Appendix B, one can see also the content of these groups. The IRREPs of the group C_{4v} are given in Table B1. The upper line of this Table shows 8 elements of the group. The left column of this Table gives 5 IRREPs according to 5 classes existing in this group. There are 4 one-dimensional representations $\Gamma(\mathcal{R})$ denoted as A_1, A_2, B_1, B_2 and 1 two-dimensional representation E. Therefore, for the nonmagnetic crystal the eigenwaves with the wave vectors Γ and M are doubly degenerate. Table B2 contains the IRREPs of the group C_4 and Table B3 shows the IRREPs of the group C_s, Table B4 gives the IRREPs of C_2, and Table B5 of C_1. Notice that all the IRREPs of the groups C_4, C_s and C_2 and C_1 are one-dimensional.

5.4. Bidirectionality of Crystals

The problem of bidirectionality of electromagnetic waves in homogeneous media is discussed in [13] and in electromagnetic waveguides in [18]. We can also apply the notion of bidirectionality to photonic crystals. We call a given magnetic photonic crystal bidirectional for a given direction $\mathbf{k}$ if there is an operator $\mathcal{R}_3$ or $\mathcal{T}\mathcal{R}_4$ or such that

$$\mathcal{R}_3\mathbf{k} = -\mathbf{k}, \tag{22}$$

or

$$\mathcal{T}\mathcal{R}_4\mathbf{k} = -\mathbf{k}, \tag{23}$$

With this condition, for any branch of the dispersion characteristic $\omega_n(\mathbf{k})$ with the vector $\mathbf{k}$ there exists another branch $\omega_m(-\mathbf{k})$ with the vector $-\mathbf{k}$ such that

$$\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k}). \tag{24}$$

We use in (24) different subindexes n and m, because, in general, the structure of the electromagnetic field of the eigenwaves corresponding to $\mathbf{k}$ and $-\mathbf{k}$ is different.

The following symmetry elements change the sign of the vector $\mathbf{k}$ defining bidirectionality in our photonic crystals:

- reflection in a plane for the direction of propagation normal to the plane, i.e., σ_x, σ_y, $\sigma_{(a-a)}$, $\sigma_{(b-b)}$,
- rotation about an axis through π for the directions of propagation perpendicular to this axis, i.e., C_2,
- reflection in an antiplane for the directions of propagation parallel to the plane, i.e., $\mathcal{T}\sigma_x$, $\mathcal{T}\sigma_y$, $\mathcal{T}\sigma_{(a-a)}$, $\mathcal{T}\sigma_{(b-b)}$, and also $\mathcal{T}\sigma_z$.

Which orientations of dc magnetic field lead to bidirectionality in our case of the 2D square lattice? For the field $\mathbf{H}_0 \parallel z$, we have the axis $C_2 \parallel z$. For the case $\mathbf{H}_0 \parallel x0y$, there is the antiplane $\mathcal{T}\sigma_z$. In both cases, bidirectionality exists for any orientation of $\mathbf{k}$ in the plane $x0y$. Magnetic crystals with an arbitrary orientation of $\mathbf{H}_0$ in the plane $x0z$, $y0z$, $(a-a)0z$ or $(b-b)0z$ are described by the group of symmetry which contains an antiplane $\mathcal{T}\sigma$. If the vector $\mathbf{k}$ is parallel to the antiplane where $\mathbf{H}_0$ lies, the crystal is bidirectional for this direction.

The symmetry elements of the crystal magnetic group which change the sign of $\mathbf{k}$ can be used to relate the structure of electromagnetic waves propagating in the opposite directions. Let us consider the case of $\mathbf{H}_0 \parallel xoy$. If an eigenwave corresponding to the branch $\omega_n(\mathbf{k})$ has the fields $\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ and $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$, the eigenwave corresponding to the branch $\omega_m(-\mathbf{k})$ can be found applying the operator $\mathcal{T}\sigma_z$ to the fields $\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ and $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$.

In a nonmagnetic crystal, the notion of bidirectionality is related to the notion of equivalent directions. All the physical properties of the crystal along the equivalent directions (not necessarily opposite) are the same. This is stipulated by the presence of some elements of symmetry: axes, planes and the center. However, existence of these symmetry elements in magnetic crystals does not always lead to equivalence of the directions. For example, a plane of symmetry in a nonmagnetic crystal defines equivalent directions normal to the plane. But the plane of symmetry which is perpendicular to a dc magnetic field does not define equivalent directions. For the opposite directions normal to this plane, the circularly (elliptically) polarized eigenwaves of the same handedness have different propagation constants, and this property defines the well-known nonreciprocal Faraday effect.

The dispersion characteristics of a bidirectional crystal for a given direction $\mathbf{k}$ are symmetric with respect to the sign of $\mathbf{k}$. However, it does not always mean that the crystal is reciprocal. The condition $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$ is not a suffucient condition for nonreciprocity, that is the bidirectional crystal can be both reciprocal and nonreciprocal.

5.5. Nonreciprocity of Crystals

Now, let us apply to the problem of reciprocity of the crystals. Different definitions of reciprocity are used in electromagnetic theory [24]. We adopt here the notion of reciprocity related to the restricted Time reversal symmetry [7]. The presence of the Time reversal $\mathcal{T}$ in the group $C_{4v} + \mathcal{T}C_{4v}$ for the nonmagnetic crystal leads to reciprocity and to a general relation $\omega_n(\mathbf{k})\omega_m(-\mathbf{k})$ for dispersion characteristics for any direction of $\mathbf{k}$. This symmetry of the dispersion characteristics allows one to reduce the band calculations by half.

This is not the case for magnetic crystals because the Time reversal is not present in the magnetic little group. The nonreciprocity of a magnetic crystal can manifest itself in difference of the wave vectors (phase, velocity difference), polarization or amplitude for the electromagnetic waves propagating in opposite directions. For magnetic crystals, in general, the dispersion characteristics are different for $\mathbf{k}$ and $-\mathbf{k}$, i.e., $\omega_n(\mathbf{k}) \neq \omega_m(-\mathbf{k})$ and one has to calculate both $\omega_n(\mathbf{k})$ and $\omega_m(-\mathbf{k})$. But for some directions in magnetic crystals, we can have $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$ and this is the case of bidirectionality descussed above.

Formally, we could consider reciprocity as a particular case of bidirectionality with $\mathcal{R}_4 = e$ in (23) (e is the identity element of the group), but one should remember that in magnetic crystals, the Time reversal operator $\mathcal{T}$ in its "pure" form is not present.

5.6. Lifting of Degeneracy by dc Magnetic Field

In Fig. 5, the dispersion diagram for the nonmagnetic crystal for the E polarization is shown [4]. The points Γ and M are denoted by circles. With dc magnetic field applied to the crystal, we can expect these eigenwaves with symmetry-induced degeneracies to be split by symmetry reduction into 2 different nondegenerate eigenwaves. This splitting can be predicted without numerical calculations by inspection of the irreducible representation Tables.

Notice that the effect of the degeneracy lifting can not be investigated in the simplified description of the magnetic (or semiconductor) 2D crystals with the tensor $\bar{\boldsymbol{\mu}}$ (or $\bar{\boldsymbol{\epsilon}}$) which does not contain off-diagonal elements (this approximation was used for example, in [10, 12]).

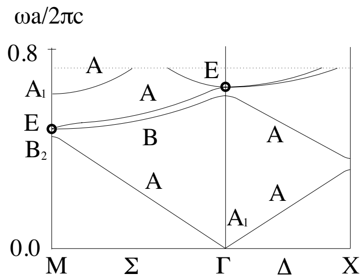

Figure 5. Dispersion diagrams for nonmagnetic crystal with with symmetry C_{4v} for E-polarization, adapted from [4]. The points marked by the circles correspond to the doubly degenerate representation E which can split by dc magnetic field.

6. CONCLUSIONS

We can enumerate the main symmetry effects of magnetization in photonic crystals:

1) magnetization leads to a general reduction of crystal symmetry, and the reduced magnetic symmetry depends on the orientation of dc magnetic field $\mathbf{H}_0$;
2) nonuniform magnetization can increase the basic domain and change the shape of the BZ;
3) a general lifting of degeneracies of eigenmodes occurs caused by the absence of the Time reversal in the magnetic symmetry group;
4) the crystal becomes nonreciprocal;

5) the scalar permeability of the ferrite becomes a tensor and the form of the tensor depends on the orientation of $\mathbf{H}_0$;
6) the symmetry of crystals in limiting cases is changed;
7) reduction of the vector wave equation to 2 scalar ones is possible only in the case of $\mathbf{H}_0 \parallel z$;
8) one can predict a splitting of degenerate eigenmodes due to magnetization in some points of the BZ;
9) some elements of symmetry can exist in magnetic photonic crystals which define bidirectionality,
10) magnetic symmetry imposes certain restrictions on the electromagnetic field of the eigenmodes.

Notice that most of these effects are also appear in magnetic homogeneous media. It should be also mentioned that in some cases of magnetic crystals, an additional degeneracy of eigenmodes can arise which is not predicted by the unitary subgroup, but this effect does not occur in the magnetic crystals with *uniform* magnetization which is the case of our present investigation.

The main results of our paper are as follows. We have defined all the possible magnetic groups of 2D magnetic photonic crystals with square lattice magnetized by a uniform dc magnetic field and magnetic little groups for points and lines of symmetry of the BZs. Using symmetry arguments we have shown that the basic domains of the BZs of the crystals depend on the orientation of dc magnetic field. We have discussed the crystal symmetry changing in possible limiting cases. We have also defined that the reduction of the vector wave equations to a scalar form is possible only in the case $\mathbf{H}_0 \parallel z \perp \mathbf{k}$. We have shown how to defined restrictions on the structure of eigenmode electromagnetic fields stipulated by magnetic symmetry. Also, we have discussed the problems of nonreciprocity and bidirectionality in the crystals and lifting of their degeneracy by applied dc magnetic field.

We have considered in this paper the case of uniform magnetization of the ferrite elements. But the group theoretical methods are also valid for crystals with any geometry of the ferrite elements and for nonunoform magnetization. It should only be remembered that the resulting group of crystal symmetry will depend on the shape of the ferrite elements and on the geometry of external dc magnetic field.

With small modifications (it concerns the wave equations), the symmetry analysis of this paper can be applied also to the photonic crystals with magnetized semiconductor elements.

The theoretical results of our paper which are free of approximations can provide a more profound insight to the problems of magnetic crystals. These results can be used to improve computational efficiency. They can also serve for checking numerical calculations and as a reference for experiments.

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APPENDIX A. PERMEABILITY TENSORS

Table A1. Permeability tensors for different orientations of dc magnetic field.

Orientation of $\mathbf{H}_0$	$\mathbf{H}_0 \parallel z$	$\mathbf{H}_0 \parallel (a-a)$	arbitrary in plane x0y
$\bar{\boldsymbol{\mu}}(\mathbf{r})$	$\begin{pmatrix} \mu_{11}(\mathbf{r}) & \mu_{12}(\mathbf{r}) & 0 \\ -\mu_{12}(\mathbf{r}) & \mu_{11}(\mathbf{r}) & 0 \\ 0 & 0 & \mu_{33}(\mathbf{r}) \end{pmatrix}$	$\begin{pmatrix} \mu_{11}(\mathbf{r}) & \mu_{12}(\mathbf{r}) & \mu_{13}(\mathbf{r}) \\ \mu_{12}(\mathbf{r}) & \mu_{11}(\mathbf{r}) & -\mu_{13}(\mathbf{r}) \\ -\mu_{13}(\mathbf{r}) & \mu_{13}(\mathbf{r}) & \mu_{33}(\mathbf{r}) \end{pmatrix}$	$\begin{pmatrix} \mu_{11}(\mathbf{r}) & \mu_{12}(\mathbf{r}) & \mu_{13}(\mathbf{r}) \\ \mu_{12}(\mathbf{r}) & \mu_{22}(\mathbf{r}) & \mu_{23}(\mathbf{r}) \\ -\mu_{13}(\mathbf{r}) & -\mu_{23}(\mathbf{r}) & \mu_{33}(\mathbf{r}) \end{pmatrix}$
Number of parameters	3	4	6

APPENDIX B. IRREDUCIBLE REPRESENTATIONS OF SOME POINT GROUPS

Table B1. Irreducible representations of the group C_{4v}.

$\boxed{C_{4v}}$	e	C_2	C_4^+, C_4^-	σ_x, σ_y	$\sigma_{(a-a)}$, $\sigma_{(b-b)}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
B_1	1	1	−1	1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

Table B2. Irreducible representations of the group C_4.

$\boxed{C_4}$	e	C_2	C_4^+	C_4^-
A	1	1	1	1
B	1	1	-1	-1
${}^1E$	1	-1	$-i$	i
${}^2E$	1	-1	i	$-i$

Table B3. Irreducible representations of the group C_s.

$\boxed{C_s}$	e	σ
A'	1	1
A''	1	-1

Table B4. Irreducible representations of the group C_2.

$\boxed{C_2}$	e	C_2
A	1	1
B	1	-1

Table B5. Irreducible representations of the group C_1.

$\boxed{C_1}$	e
A	1

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