

## **DETECTION AND LOCALIZATION OF RF RADAR PULSES IN NOISE ENVIRONMENTS USING WAVELET PACKET TRANSFORM AND HIGHER ORDER STATISTICS**

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**Abstract**—Weak signal detection and localization are basic and important problems in radar systems. Radar performance can be improved by increasing the receiver output signal-to-noise ratio (SNR). Localizing the received signal is an important task in the detection of signal in noise. Distorting the localization of the received signal can leads to incorrect target range measurements. In this paper an algorithm is described for extracting and localizing an RF radar pulse from a noisy background. The algorithm combines two powerful tools: the wavelet packet analysis and higher-order-statistics (HOS). The use of the proposed technique makes detection and localization of RF radar pulses possible in very low signal-to-noise ratio conditions, which leads to a reduction of the required microwave power or alternatively extending the detection range of radar systems.

### **1. INTRODUCTION**

Radar is an instrument that radiates electromagnetic waves is space and detects the presence and location of objects from the reflected waves. Pulse radar that uses pulses as the radar signal is being used in aviation control, weather forecasting, and ships. Radar differs from general communication systems in that the receiver margin of

the received signal is not considered and the signal buried in noise is used up to the level where its detection becomes impossible. The strength of the received signal by the radar varies with the distance from radar to the target and is also dependent on the target radar cross-section. The detectable radar range is given as a function of the SNR of the receiver output through the radar equation [8, 9]. In radar systems, weak signal detection is a basic and important problem. Solution of this problem increases the possibility of detecting smaller objects from great distances. Improvement of receiver output SNR is traditionally accomplished with pulse integration where the received signal consists of a number of pulse repetition intervals (PRI) before or after detection. However, pulse integration needs a number of pulses to improve the received SNR. For radar with fast scanning feature the required number of pulses for one object may not be adequate, to perform pulse integration.

Wavelet analysis [1] and higher-order statistics [2] are two of the most successful tools in the field of signal processing in the last twenty years. We propose combining both techniques and show how such a combination can improve the quality of RF-pulse detection and localization in noisy environment.

The problem addressed here concerns the denoising and localization of received RF radar pulse immersed in noise. In [3] the noise was removed using a non-linear time-frequency filter, which is based on the discrete windowed Fourier transform. It's known that the wavelet transform gives better localization in the time-frequency domain than the discrete windowed Fourier transform, [1]. This means localizing the detected pulse in time domain cannot be achieved by using windowed Fourier transform, as we will realize in the following sections. In the proposed work we will use the wavelet packet transform for denoising the RF radar pulses. Wavelet denoising techniques was previously used in electromagnetic waves radar [7, 10, 11]. The noise is removed by thresholding the wavelet transform coefficients of the received RF radar pulses. In the previous work the threshold was selected based on the estimated value of the noise. Such a selection of a threshold can lead to losing the received pulse in very low SNR radar returns. In this paper the threshold level selection is based on the higher-order-statistics (HOS) of the coefficients. Using a threshold, that is based on higher-order-statistics, proved to be more efficient than the usual way of wavelet thresholding [4], which is based on the estimate of the amount of noise, especially at very low signal-to-noise ratio conditions. By combining the wavelet packet and higher-order-statistics correct detection and localization of the RF radar pulse with a signal-to-noise ratio of down to  $-24$  dB can be achieved, leading to

a pronounced improvement in the performance of the radar.

The organization of the paper will be as follow. Section 2 will give a short introduction to wavelet and wavelet packet transform, and will show how to generate the wavelet packet coefficients of the received RF pulse. Section 3 will give a short introduction to higher order statistics and how to use the properties of HOS in threshold selection for signal denoising. Section 4 will describe the proposed technique. Section 5 shows some simulation results for the proposed technique. Section 6 shows a performance comparison between the proposed technique and the conventional windowed Fourier in localizing the received RF pulse in time domain. Section 7 is a conclusion.

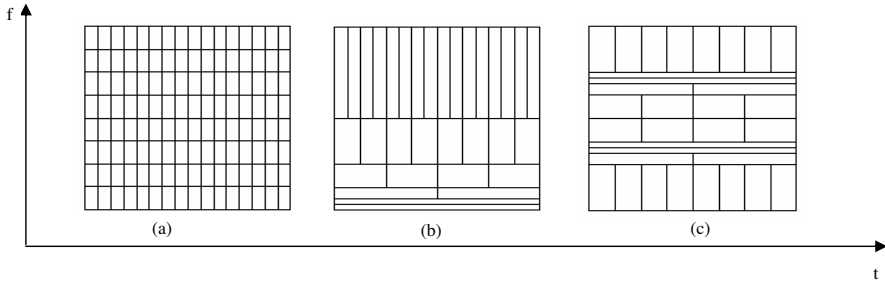
## 2. WAVELET AND WAVELET PACKET TRANSFORM

Time-frequency representations show the distribution of the energy of a signal in the time-frequency plane; in such a way that relevant information can be extracted to achieve good detection. The results generally depend on the method used as a time-frequency representation.

For example, discrete windowed Fourier transform tile the time-frequency plane in regular cells all of which have the same uncertainties. Discrete wavelet bases tile the time-frequency plane more naturally. A low frequency needs to be observed for a long time to be correctly estimated whereas a high frequency can rapidly change at any time. Hence, time-frequency localization naturally depends on the 'observation scale'. It is possible, using adapted wavelet transform, to obtain adapted tiling in the time-frequency plane, which is automatically generated based on the signal observation. On the other hand, the time-frequency plane tiling, using wavelet packet transform, corresponds to a complete set of admissible wavelets constituting a Hilbert space. The signal is projected on each element of this space producing decomposition coefficients. Figure 1 shows the tiling of the time frequency plane for windowed Fourier transform (a), wavelet transform (b), and wavelet packet transform (c). The difference between wavelet transform and wavelet packet transform will be indicated in the next subsections.

### 2.1. Wavelet Transform

Wavelet analysis is perhaps best viewed in the context of multiresolution analysis as developed by Malat [12]. There are two functions to be consider in such an analysis: the scaling function,  $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$ , and the mother wavelet,  $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t -$



**Figure 1.** Tiling the time-frequency plane (a) Windowed Fourier transform (b) Wavelet transform (c) Wavelet packet transform.

$k$ ), where  $j$  and  $k$  are integers representing the scale factor and the translation factor respectively. In the time-scale (or time-frequency) joint representation the horizontal stripes of the wavelet transform coefficients are the correlations between the signal and the wavelets at given scale  $j$ . When the scale is small the wavelet is concentrated in time, and the wavelet analysis has a detailed view of the signal. When the scale increases the wavelet spreads out in time, and the wavelet analysis takes into account the long-time behavior of the signal. We define  $V_j$  to be the subspace spanned by  $\phi_{j,k}(t)$ , and require a nesting of the spanned spaces,  $\dots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots \subset L^2$ . The space spanned by the mother wavelet at scale  $j$  is denoted  $W_j$  and constitutes the orthogonal complement of  $\phi_{j-1,k}$  in  $\phi_{j,k}$ . If we choose a coarsest scale  $j_0$  then we get that  $L^2 = V_{j_0} \cup W_{j_0} \cup W_{j_0+1} \cup \dots$ . A function  $f(t)$  then can be represented as

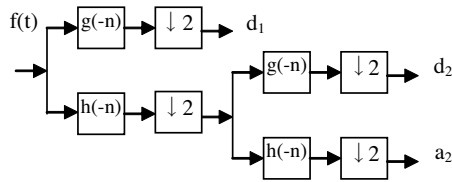
$$f(t) = \sum_{k=-\infty}^{\infty} a_{j_0}(k)\phi_{j_0,k}(t) + \sum_{j=j_0}^{\infty} \sum_{k=-\infty}^{\infty} d(j,k)\psi_{j,k}(t). \quad (1)$$

The first summation in (1) provides us with a coarse approximation to  $f(t)$ , which is the projection of  $f(t)$  onto  $V_{j_0}$ . The second summation for each  $j$  provides finer details and is the projection of  $f(t)$  onto the  $W_j$  spaces.

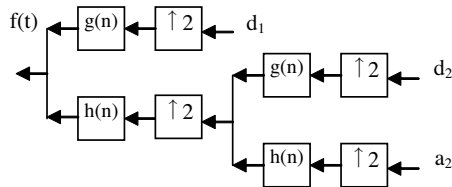
In practice, the wavelet approximation coefficients  $a_{j_0}(k)$  and the detail coefficients  $d(j,k)$  are computed using Mallat's fast algorithm [12] which involves the following filtering operations:

$$a_j(k) = \sum_m h(m-2k)a_{j+1}(m) \quad (2)$$

$$d_j(k) = \sum_m g(m-2k)a_{j+1}(m) \quad (3)$$



**Figure 2.** Filter bank implementation of discrete wavelet decomposition.



**Figure 3.** Filter bank implementation of discrete wavelet reconstruction.

where  $h(n)$  and  $g(n)$  are referred to as the scaling filter and wavelet filter, respectively. Equations (2) and (3) show how the discrete wavelet transform (DWT) is performed: By convolving the coefficients at scale  $j$  with the time reversed filter coefficients  $h(-n)$  and  $g(-n)$  and then down sampling to get the coefficients at scale  $j - 1$ . Figures 2 and 3 show a filter bank implementation for the decomposition and reconstruction of the wavelet transform. These filter structures are known in terms of subband coding as 2-band perfect reconstruction quadrature mirror filters (PR QMF). The PR QMF subband coding scheme depicted in Figs. 2 and 3 adhere to a dyadic tree structure, which splits only the lower half of the signal spectrum at each successive level. When the subband coding tree has been fully traversed, the approximation coefficients are produced at the final tree split, with the detail coefficients being produced at each tree split. A thorough treatment of wavelet decompositions as they relate to subband coding can be found in [13].

## 2.2. Wavelet Packet Transform

The wavelet packet transform has a number of applications. One of these involves the calculation of the best basis, which is a minimal representation of the data relative to a particular cost function. The best basis is used in applications that include noise reduction and in

data compression. One step in the wavelet transform calculates a low pass (scaling function) result and a high pass (wavelet function) result. The low pass result is a smoother version of the original signal. The low pass result recursively becomes the input to the next wavelet step, which calculates another low and high pass result, until only a single low pass ( $2^{\circ}$ ) result is calculated. The wavelet transform applies the wavelet transform step to the low pass result. The wavelet packet transform applies the transform step to both the low pass and the high pass result.

Assume that the received signal  $y(n)$  is given by:

$$y(n) = x(n) + z(n) \quad (4)$$

where  $x(n)$  is the received RF radar pulse,  $z(n)$  is a white Gaussian noise and  $n = 1, 2, \dots, N$ . Using the wavelet packet transform one can have a two dimension time-frequency wavelet packet coefficients as follow:

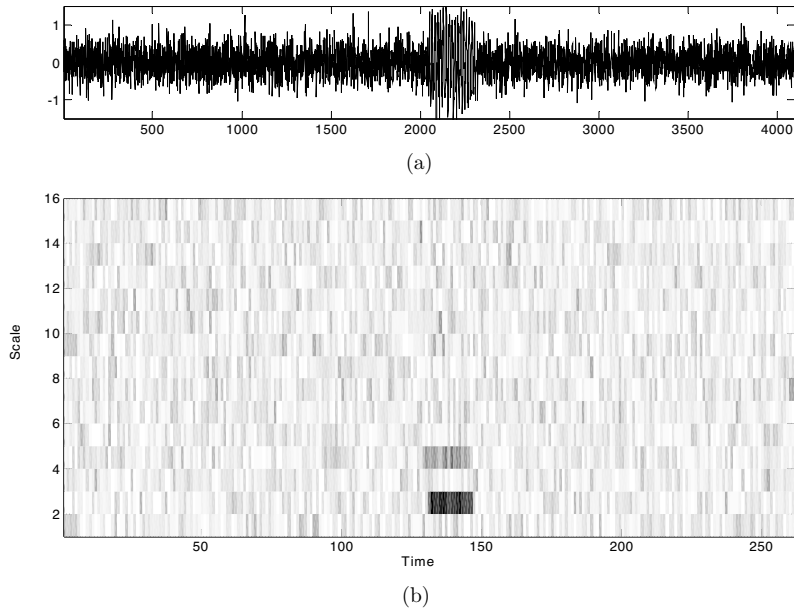
$$WP_{j,s}^y(i) = WP_{j,s}^x(i) + WP_{j,s}^z(i) \quad (5)$$

where  $WP_{j,s}^y(i)$ ,  $WP_{j,s}^x(i)$ ,  $WP_{j,s}^z(i)$  are the wavelet packet coefficients of  $y$ ,  $x$  and  $z$  respectively,  $j = 1, 2, \dots, J$  while  $J$  is the number of decomposition levels and  $s = 1, 2, \dots, 2^j$ , is the number of scales and  $i = 1, 2, \dots, M$ , with  $M = N/2^j$  and  $N$  is the length of the signal. As an example, Fig. 4(a) shows the received signal  $y(n)$  in the time domain with  $N = 4096$  sample, SNR =  $-3$  dB. Figure 4(b) shows the wavelet packet coefficients of  $y(n)$ ,  $WP_{j,s}^y(i)$ , using Daubechies wavelet [15] of order 4 and number of levels  $J = 4$ .

The order of the Daubechies wavelet controls the number of vanishing moments, which is related to the regularity of the wavelet. Increasing the order of the Daubechies wavelet increases its regularity. On the other hand this will reduce the localization of the wavelet. Consequently, a tradeoff between the regularity and the localization of the wavelet should take place. Daubechies wavelets of order 4, 8 and 16 have been tested in different situations and the order which results in the best results has been chosen.

### 2.3. Wavelet and Wavelet Packet Denoising

The concept of denoising in wavelet and wavelet packet transform is the same. Donoho [14] used the same approach to wavelet-based denoising. The idea is that only large wavelet coefficients contribute to the signal, and hence to obtain the estimated value of  $x$  one needs to keep only those coefficients whose magnitudes are greater than a certain hard threshold with value  $\lambda$ . In recognizing that each wavelet coefficient



**Figure 4.** (a) The received noisy radar pulse  $y(n)$ ,  $\text{SNR} = -3$  dB. (b) The wavelet packet coefficients of  $y(n)$ ,  $J = 4$ .

contains a signal and noise portion, it is desirable to try removing the noisy portion. Soft thresholding like hard thresholding, aim to meet this objective by keeping only those coefficients whose magnitudes are greater than a certain level  $\lambda$ . However, the remaining coefficients are shrunk towards zero by an amount  $\lambda$  hence, soft thresholding is often referred to in wavelet literature as wavelet shrinkage.

In applying wavelet thresholding the choice of  $\lambda$  is critical. Choosing too large threshold results in oversmoothing, whereas choosing too small threshold results in noisy estimates. In previous works of wavelet denoising the selection of the threshold  $\lambda$  is based on an estimate for the amount of noise in the wavelet coefficients. Applying such a technique in very low SNR situation can lead to completely losing the signal, which is hidden in noise. In this paper a new technique for denoising based on wavelet packet transform is presented. To extract the RF radar pulse, a denoising procedure, which is based on setting Gaussian coefficients (of the wavelet packet transform of the received signal) to zero, is performed. A denoised signal is then reconstructed from the retained coefficients. The problem becomes now how to get a Gaussianity measure. Higher-order-statistics are traditionally used to accomplish this task.

### 3. HIGHER ORDER STATISTICS

One of the key motivations behind the use of cumulants in signal processing problems is their ability to suppress additive Gaussian noise [1]. This ability of noise suppression is based on the fact that the  $n$ th order cumulants of a Gaussian signal,  $Cum^n[z]$ , are equal to zero for  $n > 2$ .

In the case under study the noise samples are Gaussian distributed when observed for a sufficiently long time. On the other hand the signal samples are not Gaussian. We will apply the Gaussianity measure for the wavelet packet coefficients of the received signal,  $WP_{J,s}^y(i)$ . The presence of the signal will give non-Gaussian coefficients at some frequency bands where the radar pulse exists. On the other hand, Gaussian coefficients will represent noise only. The wavelet coefficients of Gaussian noise clearly remain Gaussian when applying the linear wavelet transform [5]. A good candidate from the higher order cumulants is the kurtosis, which is the normalized version of the fourth-order cumulant [1]. The Gaussian process has a kurtosis value that theoretically equal to zero. The fourth-order cumulant is computed by a statistical expectation as (assuming zero mean of the wavelet packet coefficients):

$$Cum^4(WP_{J,s}) = E[WP_{J,s}^4] - 3 \left( E[WP_{J,s}^2] \right)^2. \quad (6)$$

One should consider a normalized measure because the Gaussianity measure must not depend on the signal energy at each frequency band. The kurtosis is defined as  $K_4(WP_{J,s}) = Cum^4(WP_{J,s}) / (E[WP_{J,s}^2])^2$ . In practice we have a limited number of data samples. So we are not able to have an exact value of the kurtosis. Instead we have an estimate value using time average. The estimation of the kurtosis can be calculated as:

$$\hat{K}_4(WP_{J,s}) = M \frac{\sum_{i=1}^M WP_{J,s}^4(i)}{\left( \sum_{i=1}^M WP_{J,s}^2(i) \right)^2} - 3. \quad (7)$$

The estimated value is allowed to exist in a predetermined interval, which is conditioned by the probability properties of the estimator. Thus one needs to frame the estimator. By using the Bienayme-Tchebychev inequality, given a desired predetermined percentage, the estimator can be framed between two values depending on the first



statistics of the estimator. In the case where the  $M$  coefficients  $WP_{J,s}$  are white and Gaussian, bias and variance of the kurtosis estimator when computed using (7) are given by [6]:

$$B(\hat{K}_4) = -6/M, \quad Var(\hat{K}_4) = 24/M. \tag{8}$$

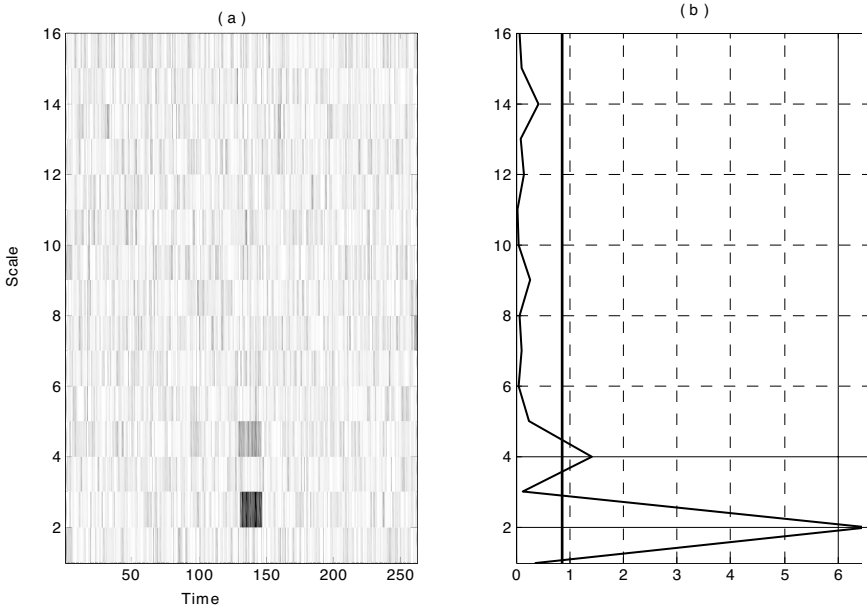
The Bienyme-Tchebychev inequality allows a Gaussian estimator to move between

$$\pm\sqrt{24/M}/\sqrt{1-\alpha} + 6/M \tag{9}$$

with an  $\alpha$  authorized confidence percentage value. The simple test for Gaussianity measure is that:

$$|\hat{K}_4| < \sqrt{24/M}/\sqrt{1-\alpha}. \tag{10}$$

Figure 5 shows the Gaussianity measure for the example of Fig. 4. Figure 5(a) is the wavelet packet coefficients and Fig. 5(b) shows the estimated kurtosis of the coefficients for each scale. The threshold is calculated using  $\alpha = 90\%$ , which was numerically found to be optimum). From Fig. 5 we notice that the coefficients at scale 2,4 belong to the signal. Whereas the coefficients of the other scales belong to the noise.



**Figure 5.** (a) The wavelet packet of  $y(n)$ . (b) The kurtosis of the wavelet packet.

#### 4. THE ALGORITHM

This algorithm is applied to denoising a received RF radar pulses. The algorithm is divided into two stages to decrease the computation complexity and hence increases the processing speed. The first stage is fundamental. The second stage is used for further improvement in SNR if necessary. The steps of the first stage of the algorithm are as listed below:

1. Compute the wavelet packet coefficients of the received signal  $WP_{J,s}^y(i)$  at level  $J$ , scale  $s = 1, 2, \dots, 2^J$ .
2. Estimate the kurtosis for the wavelet packet coefficients of each scale using (7), (force the mean value of the coefficients at each scale to be zero).
3. Apply the Gaussianity test of (10).
4. Set the Gaussian coefficients to zero.
5. Count the number of the remaining non-Gaussian scales. If the number of the non-Gaussian scales is greater than one go to the next stage. Otherwise go to the next step.
6. Reconstruct the signal from the retained coefficients.

The purpose of the next stage is to further improve the SNR. The steps of this stage of the algorithm are:

1. Apply a hard threshold to the remaining non-Gaussian coefficients scales. The threshold value is calculated using the following relation:

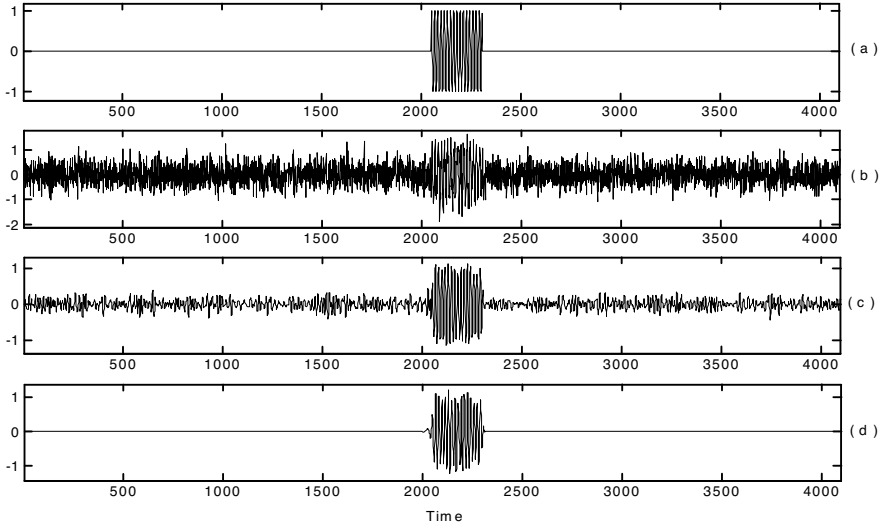
$$\lambda_s = \sigma_s \sqrt{2 \log(N)} \quad (11)$$

where  $\sigma_s = \text{Median}[|WP_{J,s}^y|]/0.6745$ , is the estimate of the noise at scale  $s$  [1].

2. Reconstruct the signal from the retained coefficients.

The number of decomposition levels  $J$  defines the finest bandwidth available for decomposition in the same time it also determines the number of coefficients within each band. Increasing the value of this parameter allows selecting and/or discriminating the frequency bands more precisely. A high value of  $J$  means however less number of coefficients within each band. This leads to insufficient estimation reliability of the kurtosis, which can lead to incorrect decision. The values of  $J$  of 4, 5, 6 and 7 have been tested in different situations and a tradeoff between the bandwidth precision and estimation reliability has been made.

Figure 6 shows the result of the first and second stage of the algorithm for the example in Fig. 4.

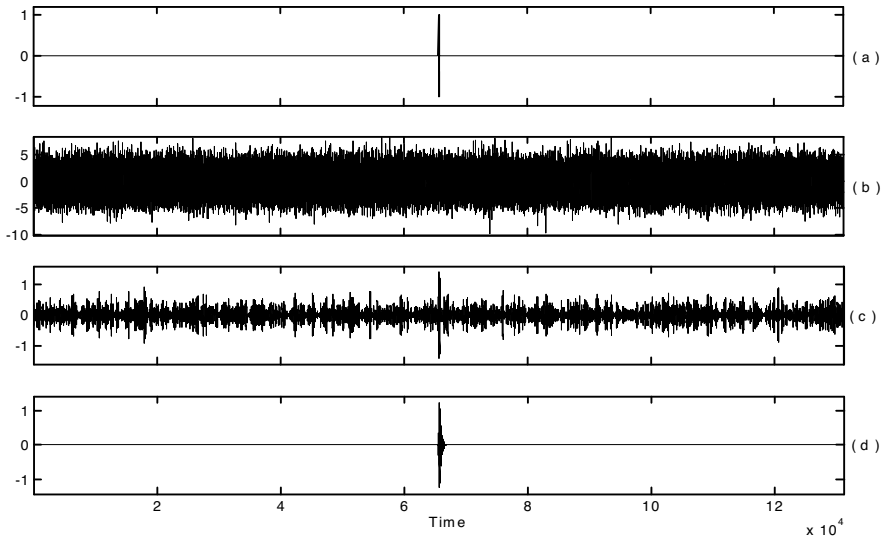


**Figure 6.** (a) The clean RF radar pulse  $x(n)$  (b) The received noisy RF radar pulse  $y(n)$ , SNR =  $-3$  dB (c) The result of the first stage (d) The result of the second stage.

## 5. SIMULATION AND RESULTS

The denoising of the received RF radar pulse is simulated in the presence of white Gaussian noise. The effect of signal parameter changes on the algorithm has been investigated. These parameters include the SNR and pulse repetition period (PRP) of the signal. The SNR is defined as the ratio of the signal power to the noise power in the entire period. Figure 7 shows another example for denoising the RF radar pulse at SNR =  $-18$  dB,  $N = 131072$ , which is equivalent to increasing the PRP. Using Daubechies wavelet of order 16 and number of levels  $J = 7$ . The threshold is calculated using  $\alpha = 90\%$ . From Fig. 7 it is clear that our proposed technique is still able to detect the radar pulse at SNR =  $-18$  dB.

As a measure for the quality of the algorithm we calculated the RMSE (Root Mean Square Error) between the clean and the denoised signal. We made a comparison between our algorithm and the wavelet denoising technique available in MATLAB software using soft threshold with 'heursure' threshold selection. Figure 8 shows a comparison for the RMSE between our proposed techniques (solid line) and that of the MATLAB (dashed line) with the same number of decomposition levels and the same mother wavelet (Daubechies order



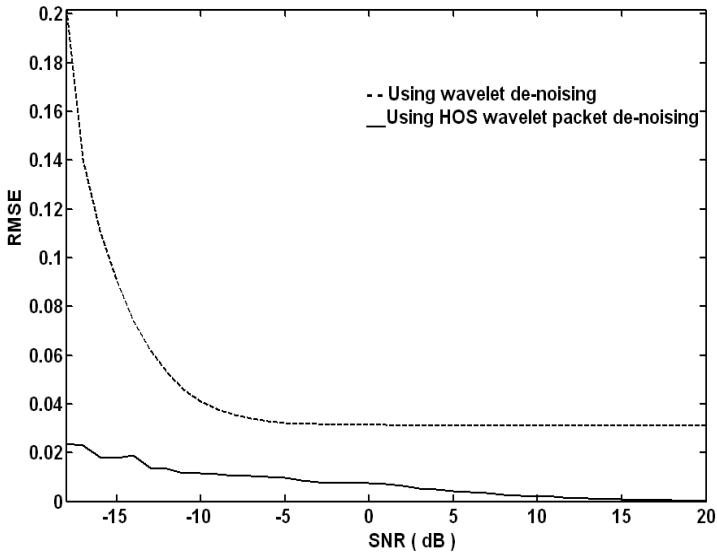
**Figure 7.** (a) Clean signal (b) Noisy signal  $\text{SNR} = -18 \text{ dB}$  (c) denoised signal after the first stage (d) denoised signal after the second stage.

16) for both techniques. It is clear from Fig. 8 that our proposed technique gives better results. The proposed technique gives very low RMSE value (maximum 0.02 at  $\text{SNR} = -18 \text{ dB}$ ). Where the wavelet denoising technique using ‘heursure’ threshold selection gives  $\text{RMSE} = 1.6$  at the same  $\text{SNR}$ .

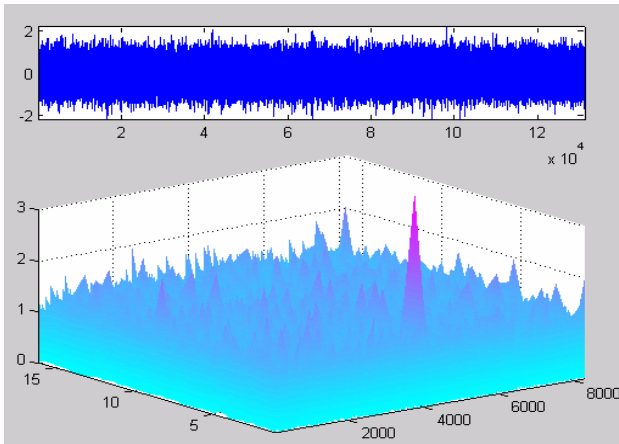
## 6. PULSE LOCALIZATION

In denoising the RF radar pulse, one of the important tasks is the localization of the received pulse in time domain. This means that our job is not only to see a clean signal in time domain but also undistorted information of the pulse (pulse width and position). Distorting the localization of the pulse in time domain will leads to an error in the radar ranging. As we mentioned in the previous sections that the wavelet transform gives better localization in the time-frequency domain than the discrete windowed Fourier transform. To confirm the ability of the proposed technique in localizing the received RF pulse in time domain we will present a comparison between the proposed denoising technique and denoising using windowed Fourier transform.

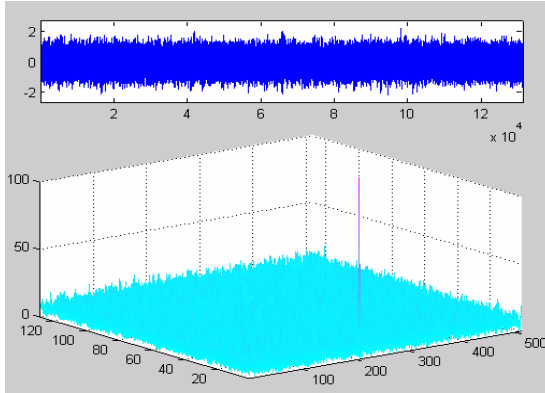
Figures 9 and 10 show a comparison between the wavelet packet transform and windowed Fourier transform for the received RF pulse



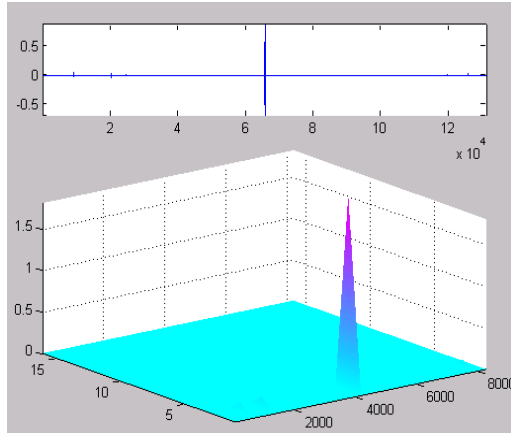
**Figure 8.** RMSE as a function in SNR for the proposed technique and the wavelet denoising in MATLAB.



**Figure 9.** Wavelet packet transform for noisy signal.

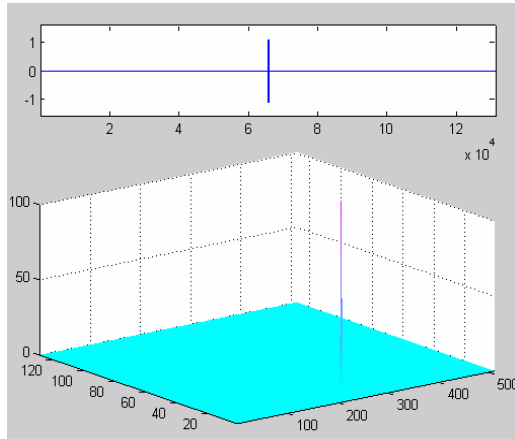


**Figure 10.** Windowed Fourier transform for noisy signal.

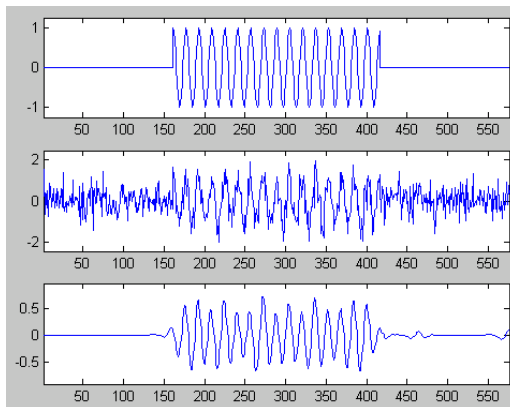


**Figure 11.** Wavelet packet transform for signal after denoising.

(in the upper panel of the figures) with a  $\text{SNR} = -24$  dB. Figure 9 is the wavelet packet transform and Fig. 10 is the windowed Fourier transform for the noisy signal. Figure 11 and 12 shows comparison between the wavelet packet transform and windowed Fourier transform for the same signal after denoising. Again Fig. 11 is wavelet packet transform of the signal after denoising and Fig. 12 is the windowed Fourier transform of the signal after denoising. From Figs. 9 to 12 its clear that the windowed Fourier transform for the RF pulse is able to well localize the signal in the frequency domain in comparison with the wavelet packet transform. The reason for this is that the sinusoidal kernel of the Fourier transform is highly correlated with the

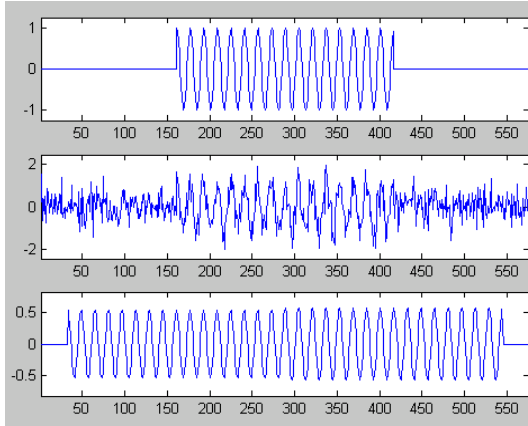


**Figure 12.** Windowed Fourier transform for signal after denoising.



**Figure 13.** Result for signal denoising using wavelet packet Transform.

carrier frequency of the received RF pulse. Figures 13 and 14 show a comparison between the denoised signals using the two techniques. Figure 13 shows the result using the proposed technique and Fig. 14 shows the result using windowed Fourier transform. In each figure the clean signal (upper panel), noisy signal (middle panel), and the denoised signal (lower panel) was presented. It is clear from Fig. 14 that the windowed Fourier transform is not able to localize the signal in time domain. In the other hand as shown in Fig. 13 the wavelet packet transform is able to will localize the signal in time domain.



**Figure 14.** Result for signal denoising using windowed Fourier transform.

## 7. CONCLUSION

An algorithm for denoising RF radar pulses has been described. The proposed algorithm combines two powerful tools; the wavelet packet transform and higher-order-statistics. The proposed algorithm is able to detect and well localize RF radar pulses without a prior knowledge of the pulse parameters (e.g., its frequency and duration). The proposed algorithm has been tested for SNR down to  $-24$  dB and proved to work successfully. Using such a technique in electromagnetic wave radar will lead to a reduction on the required microwave power supplied to the radar or extending the detection range of the radar.

## REFERENCES

1. Abbate, C. M. and P. K. Das, *Wavelets and Subbands Fundamentals and Applications*, Birkhäuser Bosten, 2002.
2. Nikias, L. and A. P. Petropulu, *Higher-Order Spectra Analysis: A nonlinear signal processing framework*, PTR Prentice Hall, New Jersey, 1993.
3. Sobhy, M. I., K. H. Moustafa, and M. Y. Makkey, "Real-time processing of noisy RF pulses," *Proceeding of the 32nd European Microwave Conference*, London, UK, September 2001.
4. Donoho, D. L., "Denoising by soft-thresholding," *IEEE Transactions on Information Theory*, Vol. 41, No. 3, 613–627, May 1995.



5. Childers, G., *Probability and Random Processes*, McGraw-Hill, Inc., 1997.
6. Ravier, P. and P. O. Amblard, "Wavelet packets and denoising based on higher-order-statistics for transient detection," *Signal Processing*, Vol. 81, No. 9, 1909–1926, August 2001.
7. Elsehely, E. and M. I. Sobhy, "Reduction of interference in microwave automotive radars," *IEEE, Microwave Theory and Techniques Society*, Boston, June 2000.
8. Skolnik, M. I., *Introduction to Radar Systems*, 3rd edition, McGraw-Hill, New York, 2001.
9. Yoshida, *Radar Technology*, I.E.C.E. Publ., 1984.
10. Huether, B. M., A. C. Gustafson, and R. P. Broussard, "Wavelet preprocessing for high range resolution radar classification," *IEEE Transaction on Aerospace and Electronic Systems*, Vol. 37, No. 4, 1321–1331, Oct. 2001.
11. Ehara, N., I. Sasase, and S. Mori, "Weak radar signal detection based on wavelet transform," *Electronics and Communications in Japan*, Part 3, Vol. 77, No. 8, 105–114, May 1994.
12. Malat, S. G., "A theory for multiresolution signal decomposition," *IEEE Transaction on Pattern Analysis and Machine Intelligence*, Vol. 11, No. 7, 674–693, 1989.
13. Akansu, A. N. and R. A. Haddad, *Multiresolution Signal Decomposition: Transforms, subbands, and wavelet*, Academic Press, New York, 1992.
14. Donoho, D. L. and I. Johnstone, "Ideal spatial adaptation by wavelet shrinkage," *Biometrika*, Vol. 81, 425–455, 1994.
15. Daubechies, I., *Ten Lectures on Wavelets*, SIAM, 1992.