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A TIME-DOMAIN THEORY OF WAVEGUIDE

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Abstract—A new time-domain theory for waveguides has been presented in the paper. The electromagnetic fields are first expanded by using the complete sets of vector modal functions derived from the transverse electric field. The expansion coefficients are then determined by solving inhomogeneous Klein-Gordon equation in terms of retarded Green's function. The theory has been validated by considering propagation problems excited by various excitation waveforms, which indicates that the higher order modes play a significant role in the field distributions excited by a wideband signal.

1. INTRODUCTION

In recent years the major advances made in ultrawideband techniques have made time-domain analysis of electromagnetic phenomena an important research field. A short pulse can be used to obtain high resolution and accuracy in radar and to increase information transmission rate in communication systems. Another important feature of a short pulse is that its rate of energy decay can be slowed down by decreasing the risetime of the pulse [1].

According to linear system theory and Fourier analysis, the response of the system to an arbitrary pulse can be obtained by superimposing its responses to all the real frequencies. In other words, the solution to a time domain problem can be expressed in terms of a time harmonic solution through the use of the Fourier transform. This process can be assisted by the fast Fourier transforms and has been used extensively in studying the transient response of linear systems. This procedure is, however, not always most effective and is not a trivial exercise since the harmonic problem must be solved for a large range of frequencies and only an approximate time harmonic solution valid over a finite frequency band can be obtained. Another reason is that the original excitation wave may not be Fourier transformable.

Thus we are forced to seek a solution in the time domain in some situations.

In high-speed circuits the signal frequency spectrum of a picosecond pulse may extend to terahertz regime and signal integrity problems may occur, which requires a deep understanding of the propagation characteristics of the transients in a waveguide. The waveguide transients have been studied by many authors [2–24]. One of the research topics is to determine the response of the waveguide to an arbitrary input signal. If the input signal of a waveguide in a single-mode operation is $x(t)$, then after traveling a distance z, the output from the waveguide is given by the Fourier integral

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j[\omega t - \beta z]} d\omega \tag{1}$$

where $X(\omega)$ is the Fourier transform of $x(t)$; β is the propagation constant $\beta = \sqrt{\omega^2 - \omega_c^2}$; and ω_c is the cut-off frequency of the propagating mode and $c = (\mu\varepsilon)^{-1/2}$ with μ and ε being the permeability and permitivity of the medium filling the waveguide respectively.

Several methods have been proposed to evaluate the Fourier integral in (1), such as the method of saddle point integration [2], method of stationary phase [3, 4], contour integration technique [5], and the quadratic approximation of the propagation constant around the carrier frequency [6–9]. A serious drawback to stationary phase is that it contradicts the physical realizability [10] as well as causality [13] (i.e., the response appears before the input signal is launched). A more rigorous approach is based on impulse response function for a lossless waveguide, which is defined as the inverse Fourier transform of the transfer function $e^{-j\beta z}$. The impulse response function can be expressed as an exact closed form and has been applied to study transient response of waveguide to various input signals [12–16].

The response given by (1) is, however, hardly realistic for describing the propagation of a very short pulse or an ultrawideband signal since it is based on an assumption that the waveguide is in a single-mode operation. This assumption is reasonable only for a narrow band signal but not valid for a short pulse, which covers a very wide range of frequency spectrum, and will excite a number of higher order modes in the waveguide. It should be mentioned that most authors have discussed transient responses for various input signals, such as a step function, a rectangular pulse or even a δ impulse on the basis of an assumption that the waveguide is in a single-mode operation. This kind of treatment has oversimplified the problem and the theoretical results obtained cannot accurately describe the real transient process in the waveguide.

In order to get the real picture of the transient process in an arbitrary waveguide, we must solve the time-domain Maxwell equations subject to initial conditions, boundary conditions and excitation conditions, and we should include the higher order mode effects. One such approach is based on the Green's function of Maxwell equations and has been discussed in [20]. Another approach is based on the field expansions in terms of the eigenfunctions in the waveguide [21–24], which are usually derived from the eigensolutions of the longitudinal fields. When these field expansions are introduced into homogeneous Maxwell equations one finds that the expansion coefficients satisfy the homogeneous Klein-Gordon equations. A wave splitting technique has been used to solve the homogeneous Klein-Gordon equations by introducing two Green's functions $G^{+}(z, t)$ and $G^{-}(z, t)$, which satisfy an integro-differential equation [21], and the excitation problem in a waveguide has been studied by introducing the time-domain vector mode functions, each of which is time dependent and satisfies the homogeneous Maxwell equations. The total fields generated by the source can then be expressed as an infinite sum of convolutions of expansion coefficients and the time-domain vector mode functions. The approach based on wave splitting seems rigorous but very complicated. In addition it is based on the field expansions in terms of the longitudinal field components. Consequently the theory is only suitable for a hollow waveguide and cannot be applied to transverse electromagnetic (TEM) transmission lines whose cross section is multiple connected.

However a hollow waveguide is not suitable for carrying a wideband signal since it blocks all the low frequency components below the cut-off frequency of dominant mode, which may cause severe distortion. For this reason, a hollow waveguide is only appropriate for narrow band applications where the frequency spectrum of the transmitted signal falls in between the cut-off frequency of the dominant mode and the cut-off frequency of the next higher order mode. On the other hand the cut-off frequency of the dominant mode of a TEM transmission line is zero. Therefore it is often used for wideband applications.

Hence it is necessary to develop a more general time-domain theory for the waveguide, which is suitable for hollow waveguides as well as TEM transmission lines. It is the purpose of present paper to achieve this goal. A more straightforward and concise time-domain approach for the analysis of waveguides has been presented in this paper. Instead of starting from eigenvalue theory of the longitudinal field components, we use the eigenvalue theory of the transverse electric field. The eigenfunctions of this transverse eigenvalue problem constitute a complete set and can be used to expand all field components in an arbitrary waveguide (hollow or multiple connected). A set of inhomogeneous Klein-Gordon equations for the field expansion coefficients can be obtained when these field expansions are substituted into Maxwell equations. As a result, the excitation problems may be reduced to the solution of these inhomogeneous scalar equations. On the contrary, all the previous publications deal with homogeneous Klein-Gordon equations and leave the excitation problem in the waveguide very complicated [21]. To solve the inhomogeneous Klein-Gordon equation, the Green's function method has been used. The new time-domain theory has been applied to study the transient process in both hollow waveguide and TEM transmission line with the effects of higher order mode being taken into account. It should be mentioned that, as far as the author can determine, there is no a single numerical example in previous publications that has really considered the higher order mode effects when dealing with the transients in a waveguide.

For the purpose of validating the new time-domain theory for the waveguide, a typical excitation problem by a sinusoidal wave, which is turned on at $t = 0$, has been investigated. Our theoretical prediction shows that the steady state response of the field distribution tends to the well-known result derived from time-harmonic theory as $t \to \infty$. Other excitation waveforms are also discussed, which indicates that the higher order modes play a significant role in the field distributions excited by a wideband signal. To further validate the theory three appendices have been attached. Appendix A gives a simple proof of the completeness of the transverse eigenvalue problem for the waveguide. Appendices B and C investigate the numerical examples studied in Section 4 by directly solving Maxwell equations subject to the initial and boundary conditions in the waveguide. It is shown that both approaches give the exactly same results.

2. FIELD EXPANSIONS BY EIGENFUNCTIONS

Consider a perfect conducting waveguide, which is uniform along z-axis and filled with homogeneous and isotropic medium. The cross-section of the waveguide is denoted by Ω and the boundary of Ω by Γ. Note that the cross-section Ω can be multiple connected. The transient electrical field in a source free region of the waveguide will satisfy the following equations

$$\begin{cases} \nabla^2 \boldsymbol{E}(\boldsymbol{r},t) - \partial^2 \boldsymbol{E}(\boldsymbol{r},t)/c^2\partial t^2 = 0, & \boldsymbol{r} \in \Omega \\ \nabla \cdot \boldsymbol{E}(\boldsymbol{r},t) = 0, & \boldsymbol{r} \in \Omega \\ \boldsymbol{u}_n \times \boldsymbol{E}(\boldsymbol{r},t) = 0, & \boldsymbol{r} \in \Gamma \end{cases} \tag{2}$$

where $\boldsymbol{u}_n$ is the outward normal to the boundary Γ. The solution can then be expressed as the sum of a transverse component and a longitudinal component, both of which are the separable functions of transverse coordinates and the longitudinal coordinate with time, i.e.,

$$\boldsymbol{E}(\boldsymbol{r},t) = (\boldsymbol{e}_t + \boldsymbol{u}_z e_z) u(z,t)$$

where $\boldsymbol{e}_t$ and e_z are functions of transverse coordinates only. For a hollow waveguide (i.e., the cross-section Ω is simple connected), the transverse fields may be expressed in terms of the longitudinal fields. Therefore the analysis for the hollow waveguide can be based on the longitudinal fields and this is the usual way to study the transient process in the hollow waveguide [21–24]. If the cross-section of the waveguide is multiple connected, such as multiple conductor transmission line for which the dominant mode has no longitudinal field components at all, the above analysis is no longer valid. To ensure the theory to be applicable to the general situations we may use the transverse fields. Inserting the above equation into (2) and taking the boundary conditions into account, we obtain

$$\begin{cases} \nabla \times \nabla \times \boldsymbol{e}_t - \nabla\nabla \cdot \boldsymbol{e}_t - k_c^2 \boldsymbol{e}_t = 0, & \boldsymbol{r} \in \Omega \\ \boldsymbol{u}_n \times \boldsymbol{e}_t = \nabla \cdot \boldsymbol{e}_t = 0, & \boldsymbol{r} \in \Gamma \end{cases}$$

where k_c^2 is the separation constant. The above system of equations constitutes an eigenvalue problem and has been studied by Kurokawa [25]. It has been be shown that the system of eigenfunctions $\{\boldsymbol{e}_{tn} | n = 1, 2, \ldots\}$ or vector mode functions is complete in the product space $L^2(\Omega) \times L^2(\Omega)$, where $L^2(\Omega)$ stands for the Hilbert space consisting of square-integrable real functions. Kurokawa's approach is rigorous enough for engineering purposes although he does not provide the proof of the existence of eigenfunctions of the above eigenvalue problem. In Appendix A of present paper, a more rigorous proof on the completeness of the eigenfunctions of the above eigenvalue problem has been presented.

The corresponding eigenvalues $\{k_{cn}^2 \geq 0,\ k_{cn+1} \geq k_{cn} | n = 1, 2, \ldots\}$ are the cut-off wavenumbers. These mode functions can be arranged to fall into three different categories (1) TEM mode, $\nabla \times \boldsymbol{e}_{tn} = 0$, $\nabla \cdot \boldsymbol{e}_{tn} = 0$; (2) TE mode, $\nabla \times \boldsymbol{e}_{tn} \neq 0$, $\nabla \cdot \boldsymbol{e}_{tn} = 0$;

and (3) TM mode, $\nabla \times \boldsymbol{e}_{tn} = 0,\ \nabla \cdot \boldsymbol{e}_{tn} \neq 0$. It should be noted that only TEM modes correspond to a zero cut-off wavenumber. The cut-off wavenumbers for TE and TM modes are always great than zero. From the complete set $\{\boldsymbol{e}_{tn}\}$ other three complete systems may be constructed

$$\begin{aligned} \{\boldsymbol{u}_z \times \boldsymbol{e}_{tn} | \boldsymbol{u}_n \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn} &= 0,\ \boldsymbol{r} \in \Gamma\} \\ \{\nabla \cdot \boldsymbol{e}_{tn}/k_{cn} | \nabla \cdot \boldsymbol{e}_{tn}/k_{cn} &= 0,\ \boldsymbol{r} \in \Gamma\} \\ \{\boldsymbol{u}_z \cdot (\nabla \times \boldsymbol{e}_{tn}/k_{cn}), \tilde{c} | \boldsymbol{u}_n \cdot \nabla[\boldsymbol{u}_z \cdot (\nabla \times \boldsymbol{e}_{tn}/k_{cn})] &= 0,\ \boldsymbol{r} \in \Gamma\} \end{aligned}$$

where $\tilde{c}$ is a constant. Hereafter we assume that the vector mode functions are orthonormal, i.e., $\int_\Omega \boldsymbol{e}_{tm} \cdot \boldsymbol{e}_{tn} d\Omega = \delta_{mn}$. If the set $\{\boldsymbol{e}_{tn}\}$ is orthonormal, then all the above three complete sets are also orthonormal. According to the boundary conditions that the vector mode functions must satisfy, $\{\boldsymbol{e}_{tn}\}$ are electric field-like and most appropriate for the expansion of the transverse electric field; $\{\boldsymbol{u}_z \times \boldsymbol{e}_{tn}\}$ are magnetic field-like and most appropriate for the expansion of the transverse magnetic field; $\{\nabla \cdot \boldsymbol{e}_{tn}/k_{cn}\}$ are electric field-like and most appropriate for the expansion of longitudinal electric field; $\{\nabla \times \boldsymbol{e}_{tn}/k_{cn}\}$ are magnetic field-like and most appropriate for the expansion of longitudinal magnetic field. Notice that $\nabla \times \boldsymbol{E}$ is magnetic field-like while $\nabla \times \boldsymbol{H}$ is electric field-like. It should be notified that all the modes and cut-off wavenumbers are independent of time and frequency and they only depend on the geometry of the waveguide. Therefore these modes can be used to expand the fields in both frequency and time domain. Following a similar procedure described in [25] the transient electromagnetic fields in the waveguide can be represented by

$$\boldsymbol{E} = \sum_{n=1}^{\infty} v_n \boldsymbol{e}_{tn} + \boldsymbol{u}_z \sum_{n=1}^{\infty} \left(\frac{\nabla \cdot \boldsymbol{e}_{tn}}{k_{cn}} \right) \overline{e}_{zn} \tag{3}$$

$$\boldsymbol{H} = \sum_{n=1}^{\infty} i_n \boldsymbol{u}_z \times \boldsymbol{e}_{tn} + \frac{\boldsymbol{u}_z}{\sqrt{\Omega}} \int_\Omega \frac{\boldsymbol{u}_z \cdot \boldsymbol{H}}{\sqrt{\Omega}} d\Omega + \sum_{n=1}^{\infty} \left(\frac{\nabla \times \boldsymbol{e}_{tn}}{k_{cn}} \right) \overline{h}_{zn} \tag{4}$$

$$\nabla \times \boldsymbol{E} = \sum_{n=1}^{\infty} \left(\frac{\partial v_n}{\partial z} + k_{cn} \overline{e}_{zn} \right) \boldsymbol{u}_z \times \boldsymbol{e}_{tn} + \sum_{n=1}^{\infty} k_{cn} v_n \left(\frac{\nabla \times \boldsymbol{e}_{tn}}{k_{cn}} \right) \tag{5}$$

$$\nabla \times \boldsymbol{H} = \sum_{n=1}^{\infty} \left(-\frac{\partial i_n}{\partial z} + k_{cn} \overline{h}_{zn} \right) \boldsymbol{e}_{tn} + \boldsymbol{u}_z \sum_{n=1}^{\infty} k_{cn} i_n \left(\frac{\nabla \cdot \boldsymbol{e}_{tn}}{k_{cn}} \right) \tag{6}$$

where

$$v_n(z,t) = \int_\Omega \boldsymbol{E} \cdot \boldsymbol{e}_{tn} d\Omega, \quad i_n(z,t) = \int_\Omega \boldsymbol{H} \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn} d\Omega$$

$$\overline{h}_{zn}(z,t) = \int_\Omega \boldsymbol{H} \cdot \left(\frac{\nabla \times \boldsymbol{e}_{tn}}{k_{cn}} \right) d\Omega, \quad \overline{e}_{zn}(z,t) = \int_\Omega \boldsymbol{u}_z \cdot \boldsymbol{E} \left(\frac{\nabla \cdot \boldsymbol{e}_{tn}}{k_{cn}} \right) d\Omega$$

We will call v_n and i_n the time-domain modal voltage and time-domain modal current respectively. Substituting the above expansions into the time-domain Maxwell equations

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\mu \dfrac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t} - \boldsymbol{J}_m(\boldsymbol{r},t) \\ \nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \varepsilon \dfrac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{J}(\boldsymbol{r},t) \end{cases}$$

and comparing the transverse and longitudinal components, we obtain

$$-\frac{\partial i_n}{\partial z} + k_{cn}\overline{h}_{zn} = \varepsilon \frac{\partial v_n}{\partial t} + \int_\Omega \boldsymbol{J} \cdot \boldsymbol{e}_{tn} d\Omega \tag{7}$$

$$k_{cn} i_n = \varepsilon \frac{\partial \overline{e}_{zn}}{\partial t} + \int_\Omega \boldsymbol{u}_z \cdot \boldsymbol{J} \left(\frac{\nabla \cdot \boldsymbol{e}_{tn}}{k_{cn}} \right) d\Omega, \text{ for TM modes} \tag{8}$$

$$\frac{\partial v_n}{\partial z} + \overline{e}_{zn} k_{cn} = -\mu \frac{\partial i_n}{\partial t} - \int_\Omega \boldsymbol{J}_m \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn} d\Omega \tag{9}$$

$$k_{cn} v_n = -\mu \frac{\partial \overline{h}_{zn}}{\partial t} - \int_\Omega (\boldsymbol{u}_z \cdot \boldsymbol{J}_m) \left(\frac{\boldsymbol{u}_z \cdot \nabla \times \boldsymbol{e}_{tn}}{k_{cn}} \right) d\Omega, \text{ for TE modes} \tag{10}$$

$$-\mu \frac{\partial}{\partial t} \int_\Omega \frac{\boldsymbol{H} \cdot \boldsymbol{u}_z}{\sqrt{\Omega}} d\Omega = \int_\Omega \frac{\boldsymbol{u}_z \cdot \boldsymbol{J}_m}{\sqrt{\Omega}} d\Omega, \text{ for TE modes} \tag{11}$$

It is easy to show that the modal voltage and modal current for TEM mode satisfy the one-dimensional wave equation

$$\begin{cases} \dfrac{\partial^2 v_n^{TEM}}{\partial z^2} - \dfrac{1}{c^2} \dfrac{\partial^2 v_n^{TEM}}{\partial t^2} = \mu \dfrac{\partial}{\partial t} \displaystyle\int_\Omega \boldsymbol{J} \cdot \boldsymbol{e}_{tn} d\Omega - \dfrac{\partial}{\partial z} \displaystyle\int_\Omega \boldsymbol{J}_m \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn} d\Omega \\ \dfrac{\partial^2 i_n^{TEM}}{\partial z^2} - \dfrac{1}{c^2} \dfrac{\partial^2 i_n^{TEM}}{\partial t^2} = -\dfrac{\partial}{\partial z} \displaystyle\int_\Omega \boldsymbol{J} \cdot \boldsymbol{e}_{tn} d\Omega + \varepsilon \dfrac{\partial}{\partial t} \displaystyle\int_\Omega \boldsymbol{J}_m \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn} d\Omega \end{cases} \tag{12}$$

from (7) and (8). For TE modes the modal voltage v_n^{TE} satisfies the following one-dimensional Klein-Gordon equation, i.e.,

$$\frac{\partial^2 v_n^{TE}}{\partial^2 z} - \frac{1}{c^2} \frac{\partial^2 v_n^{TE}}{\partial t^2} - k_{cn}^2 v_n^{TE} = \mu \frac{\partial}{\partial t} \int_\Omega \boldsymbol{J} \cdot \boldsymbol{e}_{tn} d\Omega - \frac{\partial}{\partial z} \int_\Omega \boldsymbol{J}_m \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn} d\Omega$$

$$+k_{cn}\int_{\Omega}(\boldsymbol{u}_z\cdot\boldsymbol{J}_m)\left(\frac{\boldsymbol{u}_z\cdot\nabla\times\boldsymbol{e}_{tn}}{k_{cn}}\right)d\Omega \quad (13)$$

from (7), (9) and (10). The modal current i_n^{TE} can be determined by a time integration of $\partial v_n^{TE}/\partial z$. For TM modes, it can be shown that the modal current i_n^{TM} also satisfies the Klein-Gordon equation

$$\frac{\partial^2 i_n^{TM}}{\partial^2 z}-\frac{1}{c^2}\frac{\partial^2 i_n^{TM}}{\partial t^2}-k_{cn}^2 v_n^{TM}=-\frac{\partial}{\partial z}\int_{\Omega}\boldsymbol{J}\cdot\boldsymbol{e}_{tn}d\Omega+\varepsilon\frac{\partial}{\partial t}\int_{\Omega}\boldsymbol{J}_m\cdot\boldsymbol{u}_z \times\boldsymbol{e}_{tn}d\Omega-k_{cn}\int_{\Omega}\boldsymbol{u}_z\cdot\boldsymbol{J}\left(\frac{\nabla\cdot\boldsymbol{e}_{tn}}{k_{cn}}\right)d\Omega \quad (14)$$

from (7), (8) and (9). The modal voltage v_n^{TM} can be determined by a time integration of $\partial i_n^{TM}/\partial z$. Now we can see that the excitation problem in a waveguide is reduced to the solution of a series of inhomogeneous Klein-Gordon equations. Compared to the treatment in previous publications, where only homogeneous Klein-Gordon equations are involved but very complicated time-domain vector mode functions have to be introduced to solve an excitation problem [e.g., 21], our approach is much simpler, and at the same time more general since it can be applied to a hollow waveguide as well as a multiple conductor transmission line.

3. SOLUTIONS OF INHOMOGENEOUS KLEIN-GORDON EQUATION

To get the complete solution of the transient fields in the waveguide, we need to solve the inhomogeneous Klein-Gordon equation. This can be done by means of the retarded Green's function.

3.1. Retarded Green's Function of Klein-Gordon Equation

The retarded Green's function for Klein-Gordon equation is defined by

$$\begin{cases}\left(\dfrac{\partial^2}{\partial z^2}-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}-k_{cn}^2\right)G_n(z,t;z',t')=-\delta(z-z')\delta(t-t') \\ G_n(z,t;z',t')\Big|_{t<t'}=0\end{cases} \quad (15)$$

where the second equation represents the causality condition. Taking the Fourier transform, the first equation reduces to

$$\tilde{G}_n(p,\omega;z',t')=-c^2e^{-jpz'-j\omega t'}/\left(\omega^2-p^2c^2-k_{cn}^2c^2\right)$$

The inverse Fourier transform may then be written as

$$G_n(z,t;z',t') = -\frac{c^2}{(2\pi)^2}\int_{-\infty}^{\infty} e^{jp(z-z')}dp \int_{-\infty}^{\infty} \frac{e^{j\omega(t-t')}}{(\omega^2 - p^2c^2 - k_{cn}^2c^2)}d\omega$$

To calculate the second integral we may extend ω to the complex plane and use the residue theorem in complex variable analysis. There are two simple poles $\omega_{1,2} = \pm\sqrt{p^2c^2 + k_{cn}^2c^2}$ in the integrand. To satisfy the causality condition, we only need to consider the integral of $e^{j\omega(t-t')}/(\omega^2 - p^2c^2 - k_{cn}^2c^2)$ along a closed contour consisting of the real axis from $-\infty$ to ∞ and an infinite semicircle in the upper half plane. Note that the poles on the real axis have been pushed up a little bit by changing them into $\omega_{1,2} = \pm\sqrt{p^2c^2 + k_{cn}^2c^2} + j\xi$ to satisfy the causality condition, where ξ will be eventually made to approach zero. The contour integral along the large semicircle will be zero for $t > t'$. Using the residue theorem and [27]

$$\int_0^{\infty} \left(\frac{\sin q\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)\cos bx dx = \frac{\pi}{2}J_0\left(a\sqrt{q^2-b^2}\right)H(q-b),$$
$$a > 0,\ q > 0,\ b > 0$$

we obtain

$$\begin{aligned} G_n(z,t;z',t') &= \frac{c}{\pi}\int_0^{\infty}\left\{\frac{\sin\left[c(t-t')\sqrt{p^2+k_{cn}^2}\right]}{\sqrt{p^2+k_{cn}^2}}\right\}\cos p(z-z')dp \\ &= \frac{c}{2}J_0\left[k_{cn}c\sqrt{(t-t')^2 - |z-z'|^2/c^2}\right]H(t-t'-|z-z'|/c) \end{aligned} \quad (16)$$

where $J_0(x)$ is the Bessel function of first kind and $H(x)$ the unit step function. Note that when $k_{cn} = 0$ this reduces to Green's function of wave equation.

3.2. Solution of Inhomogeneous Klein-Gordon Equation

Consider the inhomogeneous Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k_{cn}^2\right)u_n(z,t) = f(z,t) \quad (17)$$

and (15). Both equations can be transformed into the frequency domain by Fourier transform

$$\left(\frac{\partial^2}{\partial z^2} + \beta_{cn}^2\right)\tilde{u}_n(z,\omega) = \tilde{f}(z,\omega)$$

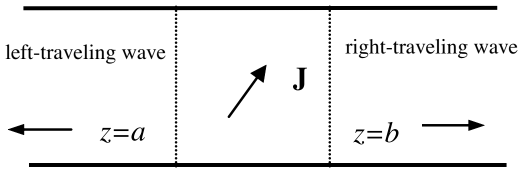

Figure 1. Left-traveling wave and right-traveling wave in waveguide.

$$\left(\frac{\partial^2}{\partial z^2}+\beta_{cn}^2\right)\tilde{G}_n(z,\omega;z',t') = -\delta(z-z')e^{-j\omega t'}$$

where $\beta_{cn}^2 = \omega^2/c^2 - k_{cn}^2$. Multiplying the first equation and the second equation by $\tilde{G}_n$ and $\tilde{u}$ respectively and then subtracting the resultant equations yield

$$\begin{aligned}&\tilde{u}_n(z,\omega)\frac{\partial^2\tilde{G}_n(z,\omega;z',t')}{\partial z^2}-\tilde{G}_n(z,\omega;z',t')\frac{\partial^2\tilde{u}_n(z,\omega)}{\partial z^2}\\&=-\delta(z-z')\tilde{u}_n\ (z,\omega)e^{-j\omega t'}-\tilde{f}(z,\omega)\tilde{G}_n(z,\omega;z',t')\end{aligned}\tag{18}$$

We assume that the source distribution $f(z,t)$ is limited in a finite interval (a,b) as shown in Figure 1. Taking the integration of (18) over the interval $[a,b]$ and then taking the inverse Fourier transform with respect to time, we obtain

$$\begin{aligned}u_n(z,t) \ = \ &\left[\int_{-\infty}^{\infty} G_n(z,t;z',t')\frac{\partial u_n(z',t')}{\partial z'}dt'\right.\\&\left.-\int_{-\infty}^{\infty} u_n(z',t')\frac{\partial G_n(z,t;z',t')}{\partial z'}dt'\right]_{z'=a}^{b}\\&-\int_a^b dz'\int_{-\infty}^{\infty} f(z',t')G_n(z,t;z',t')dt',\ \ z\in(a,b)\end{aligned}\tag{19}$$

where we have used the symmetry of Green's function about z and z'. Note that if $a\to-\infty,\ b\to\infty$, the above expression becomes

$$u_n(z,t) = -\int_{-\infty}^{\infty} dz'\int_{-\infty}^{\infty} f(z',t')G_n(z,t;z',t')dt',\ \ z\in(-\infty,\infty)\tag{20}$$

Next we will show that the solution in the region $(z_+,+\infty)(z_+\geq b)$ and $(-\infty,z_-)(z_-\leq a)$ may be expressed in terms of its boundary values at b and a. Without loss of generality, we assume that $a < 0$ and $b > 0$. Taking the integration of (18) over $[z_+, +\infty]$ with $z_+ \geq b$ and using integration by parts, we obtain

$$\tilde{G}_n(z_+, \omega; z', t') \frac{\partial \tilde{u}_n(z_+, \omega)}{\partial z} - \tilde{u}_n(z_+, \omega) \frac{\partial \tilde{G}_n(z_+, \omega; z', t')}{\partial z} = -\tilde{u}_n(z', \omega) e^{-j\omega t'}$$

where we have used the radiation condition at $z = +\infty$. Taking the inverse Fourier transform and letting $z' = z_+$ lead to

$$u_n(z, t) + cJ_0(k_{cn}ct)H(t) \otimes \frac{\partial u_n(z, t)}{\partial z} = 0, \ z \geq b > 0 \tag{21}$$

where we have replaced z_+ by z and $t - t'$ by t since z_+ and t' are arbitrary, and $\otimes$ denotes the convolution with respect to time. This equation is called right-traveling condition of the wave [21]. Similarly taking the integration of (18) over $(-\infty, z_-]$ with $z_- \leq a$, we may obtain left-traveling condition

$$u_n(z, t) - cJ_0(k_{cn}ct)H(t) \otimes \frac{\partial u_n(z, t)}{\partial z} = 0, \ z \leq a < 0 \tag{22}$$

Both (21) and (22) are integral-differential equations. If the source is turned on at $t = 0$, all the fields must be zero when $t < 0$. In this case, (21) and (22) can be solved by single-sided Laplace transform. Denoting the Laplace transform of u_n by $\tilde{u}_n$ we have

$$\begin{aligned} \tilde{u}_n(z, s) + \left[(s/c)^2 + k_{cn}^2\right]^{-1/2} \frac{\partial \tilde{u}_n(z, s)}{\partial z} &= 0, \ z \geq b > 0 \\ u_n(z, t) - \left[(s/c)^2 + k_{cn}^2\right]^{-1/2} \frac{\partial \tilde{u}_n(z, s)}{\partial z} &= 0, \ z \leq a < 0 \end{aligned}$$

The solutions of the above equations can be easily found. Making use of the inverse Laplace transforms listed in [26], the solutions of (21) and (22) can be expressed as

$$\begin{aligned} u_n(z, t) &= u_n\left(b, t - \frac{z-b}{c}\right) - ck_{cn}(z - b) \\ &\quad \cdot \int_0^{t-\frac{z-b}{c}} \frac{J_1\left[k_{cn}c\sqrt{(t-\tau)^2 - (z-b)^2/c^2}\right]}{\sqrt{(t-\tau)^2 - (z-b)^2/c^2}} u_n(b, \tau) d\tau \\ &\qquad (z \geq b > 0) \end{aligned}$$

$$u_n(z,t) = u_n\left(a, t+\frac{z-a}{c}\right) + ck_{cn}(z-a) \cdot \int_0^{t+\frac{z-a}{c}} \frac{J_1\left[k_{cn}c\sqrt{(t-\tau)^2-(z-a)^2/c^2}\right]}{\sqrt{(t-\tau)^2-(z-a)^2/c^2}} u_n(a,\tau)d\tau \quad (z \le a < 0)$$

The above two equations have also been derived by the method of impulse response function [16] and wave splitting technique [21]. Once the input signal is known the output signal after traveling a certain distance in the waveguide can be determined by a convolution operation. Some numerical results can be found in references [16, 21]. However there is a common mistake in studying the propagation of transients in a waveguide, which assumes a wideband input signal (such as a step function or a rectangular pulse) and a single-mode operation at the same time. The electric field at the output is then determined by the product of the response determined by one of the above two equations and the transverse mode function. This process totally ignores the contributions from higher order modes and is not accurate enough to describe the actual response in the waveguide. Indeed if the excitation pulse is wideband, the waveguide will be overmoded and the effects of higher order modes cannot be neglected in most situations. This will be demonstrated below.

4. TRANSIENT PROCESS IN WAVEGUIDES

Now the above general theory will be applied to study the propagation characteristics of electromagnetic pulse in waveguides. In general a wideband pulse in the waveguide will generate a number of higher order modes, and the field distributions inside the waveguide should be determined by (3) and (4), which will be approximated by a finite summation of m terms.

4.1. Transient Process in a Rectangular Waveguide

Let us first consider a rectangular waveguide of width a and height b as depicted in Figure 2. If the waveguide is excited by a line current extending across the waveguide located at $x_0 = a/2$, then current density is given by

$$\boldsymbol{J}(\boldsymbol{r}, t) = \boldsymbol{u}_y \delta(x - x_0)\delta(z - z_0) f(t)$$

Since the line current density is uniform in y direction, the field excited by the current will be independent of y. As a consequence, only TE_{n0}

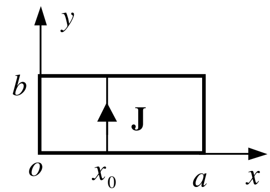

Figure 2. Cross-section of a rectangular waveguide.

mode will be excited. In this case we have

$$k_{cn} = \frac{n\pi}{a},\ \boldsymbol{e}_{tn}(x, y) = \boldsymbol{e}_{n0}^{TE}(x, y) = -\boldsymbol{u}_y \left(\frac{2}{ab}\right)^{1/2} \sin\frac{n\pi x}{a},\ n = 1, 2, 3\ldots$$

From (13), (20) and the above equation, we obtain

$$\begin{aligned} v_n^{TE}(z,t) &= \frac{b\eta}{2}\left(\frac{2}{ab}\right)^{1/2} \sin\frac{n\pi}{a}x_0 \\ &\quad \cdot \int_{-\infty}^{t-|z-z_0|/c} \frac{df(t')}{dt'} J_0\left[k_{cn}c\sqrt{(t-t')^2-|z-z_0|^2/c^2}\right] dt' \quad (23) \end{aligned}$$

where $\eta = \sqrt{\mu/\varepsilon}$. Thus the time-domain voltages v_n^{TE} for $\mathrm{TE}_{n0}(n = 2, 4, 6, \ldots)$ vanish. From (3) the total electric field in the waveguide may be approximated by a finite summation of m terms

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{u}_y E_y = -\boldsymbol{u}_y \left(\frac{2}{ab}\right)^{1/2} \sum_{n=1}^{m} v_n^{TE} \sin\frac{n\pi x}{a} \\ &= -\boldsymbol{u}_y \sum_{n=1}^{m} \frac{\eta}{a} \sin\frac{n\pi}{a}x_0 \sin\frac{n\pi x}{a} \\ &\quad \cdot \int_{-\infty}^{t-|z-z_0|/c} \frac{df(t')}{dt'} J_0\left[k_{cn}c\sqrt{(t-t')^2 - |z-z_0|^2/c^2}\right] dt' \quad (24) \end{aligned}$$

The above expression has been validated in Appendix B by using a totally different approach. The theory can also be validated by considering the time-domain response to a continuous sinusoidal wave turned on at $t = 0$. We should expect that the time-domain response tends to the well-known steady state response as time goes to infinity. Hereafter we assume that $x_0 = a/2$, and $z_0 = 0$ for all numerical examples. To validate the theory let $f(t) = H(t)\sin\omega t$. Then (23) may be expressed as

$$v_n^{TE}(z,t) = v_n^{TE}(z,t)\Big|_{\text{steady}} + v_n^{TE}(z,t)\Big|_{\text{transient}}, \quad t > |z|/c$$

where $k = \omega/c$ and $v_n^{TE}(z,t)\Big|_{\text{steady}}$ represents the steady-state part of the response and $v_n^{TE}(z,t)\Big|_{\text{transient}}$ the transient part of the response

$$\begin{aligned} v_n^{TE}(z,t)\Big|_{\text{steady}} &= \frac{b\eta}{2}\left(\frac{2}{ab}\right)^{1/2} ka\sin\frac{n\pi}{2} \\ &\quad \cdot \int_{|z|/a}^{\infty} \cos ka(ct/a-u) J_0\left[k_{cn}a\sqrt{u^2-|z|^2/a^2}\right] du \\ v_n^{TE}(z,t)\Big|_{\text{transient}} &= -\frac{b\eta}{2}\left(\frac{2}{ab}\right)^{1/2} ka\sin\frac{n\pi}{2} \\ &\quad \cdot \int_{ct/a}^{\infty} \cos ka(ct/a-u) J_0\left[k_{cn}a\sqrt{u^2-|z|^2/a^2}\right] du \end{aligned}$$

Note that the transient part of the response approaches to zero as $t \to \infty$. To investigate the steady-state part of the response the following calculations are needed [27]

$$\int_a^{\infty} J_0\left(b\sqrt{x^2-a^2}\right)\sin cx dx = \begin{cases} 0, & 0<c<b \\ \cos\left(a\sqrt{c^2-b^2}\right)/\sqrt{c^2-b^2}, & 0<b<c \end{cases}$$

$$\int_a^{\infty} J_0\left(b\sqrt{x^2-a^2}\right)\cos cx dx = \begin{cases} \exp\left(-a\sqrt{b^2-c^2}\right)/\sqrt{b^2-c^2}, & 0<c<b \\ -\sin\left(a\sqrt{c^2-b^2}\right)/\sqrt{c^2-b^2}, & 0<b<c \end{cases}$$

Thus we have

$$\begin{aligned} v_n^{TE}(z,t)\Big|_{\text{steady}} &= \frac{b\eta}{2}\left(\frac{2}{ab}\right)^{1/2} \frac{ka\sin\frac{n\pi}{2}}{\sqrt{(ka)^2-(k_{cn}a)^2}} \\ &\quad \cdot \begin{cases} \sin\left(ka\frac{ct}{a} - \frac{|z|}{a}\sqrt{(ka)^2-(k_{cn}a)^2}\right), & k>k_{cn} \\ \cos\left(ka\frac{ct}{a}\right)\exp\left[-|z/a|\sqrt{(k_{cn}a)^2-(ka)^2}\right], & k<k_{cn} \end{cases} \end{aligned}$$

Therefore only those modes satisfying $k > k_{cn}$ will propagate in the steady state. When $v_n^{TE}\Big|_{\text{steady}}$ are inserted into (24) it will be found

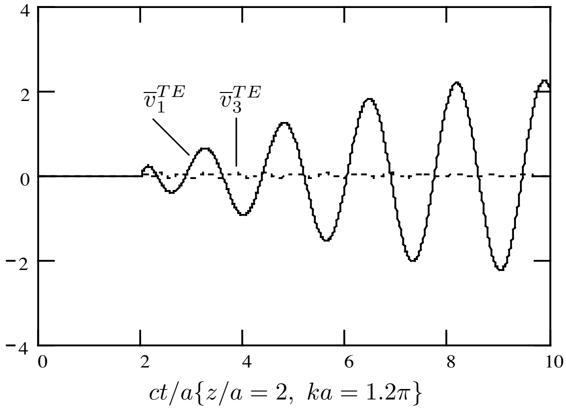

Figure 3. Time-domain modal voltages excited by a sinusoidal wave.

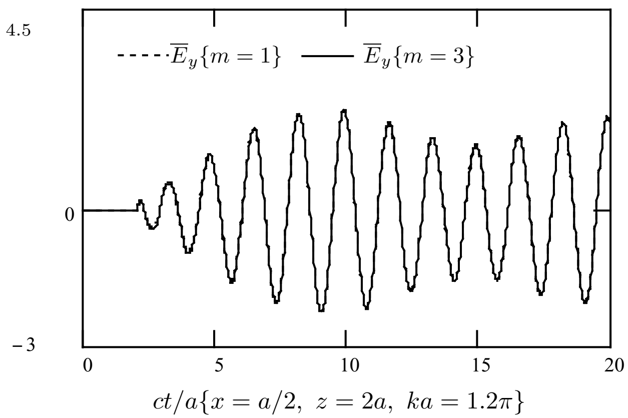

Figure 4. Electric fields excited by a sinusoidal wave.

that the steady state response of the electric field agrees with the traditional time harmonic theory of waveguides (see Eqn. (74), Chapter 5 of [20]). The variation of $\overline{v}_n^{TE} = \sqrt{2}v_n^{TE}/\eta\sqrt{b/a}$ and $\overline{E}_y = -aE_y/\eta$ with the time at $z = 2a$ for different modes have been shown in Figure 3 and Figure 4 respectively, where m stands for the number of terms chosen in (24). The operating frequency ω has been chosen in between the first cut-off frequency and the second one. It can be seen that the contribution from the higher order modes are negligible. However this is not true for other wideband excitation pulse.

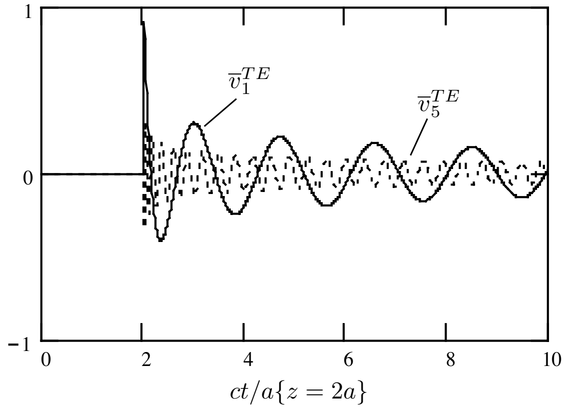

Figure 5. Time-domain modal voltages excited by a unit step waveform.

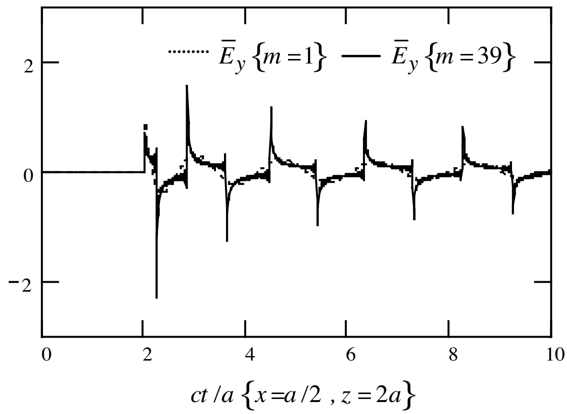

Figure 6. Electric fields excited by a unit step waveform.

Let us consider a unit step pulse, i.e., $f(t) = H(t)$. The time-domain voltages v_n^{TE} for the first and fifth mode are shown in Figure 5, which clearly indicates that the voltage for the higher order modes cannot be ignored in this case. As indicated by Figure 6, the electric fields at $z = 2a$, obtained by assuming $m = 1$ (only the dominant mode is used) and $m = 39$ (the first 39 modes are used), are quite different due to the significant contributions from the higher order modes. It is seen that time response of the field is totally different from the original excitation pulse (i.e., a unit step function) due to the fact that a hollow waveguide is essentially a high pass filter, which blocks all the low frequency components below the first cut-off frequency. In addition the waveguide exhibits severe dispersion. It should be noted that the singularities in the electric field distributions come from the time derivatives of the excitation waveform in (23), and the periodic-like performance of the field distribution results from the behavior of Bessel functions.

Thus it is clear that a hollow metal waveguide is not an ideal medium to carry a wideband signal that contains significant low frequency components below the first cut-off frequency. In this case one should use a multi-conductor transmission line supporting a TEM mode whose cut-off frequency is zero.

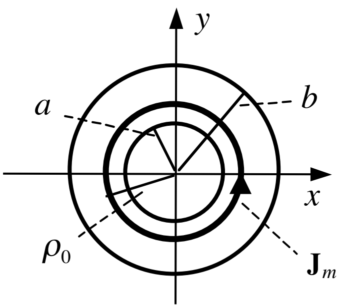

Figure 7. Cross-section of a coaxial waveguide.

4.2. Transient Process in a Coaxial Cable

To see how a pulse propagates in a TEM transmission line as well as the effects of the higher order modes we may consider a coaxial line consisting of an inner conductor of radius a and an outer conductor of radius b, as shown in Figure 7. Let the coaxial line be excited by a magnetic ring current located at $z = 0$, i.e.,

$$\boldsymbol{J}_m(\boldsymbol{r}, t) = \boldsymbol{u}_\theta f(t)\delta(z)\delta(\rho - \rho_0), \quad a < \rho_0 < b$$

where (ρ, φ, z) are the polar coordinates and $\boldsymbol{u}_\theta$ is the unit vector in θ direction. According to the symmetry, only TEM mode and those TM_{0q} modes that are independent of φ will be excited. The orthonormal vector mode functions for these modes are given by [28]

$$k_{c1} = 0, \quad \boldsymbol{e}_{t1}(\rho, \varphi) = \boldsymbol{u}_r \frac{1}{\rho\sqrt{2\pi \ln c_1}}$$

$$k_{cn} = \frac{\chi_n}{a},$$

$$\boldsymbol{e}_{tn}(\rho,\varphi) = \boldsymbol{u}_\rho \frac{\sqrt{\pi}}{2}\frac{\chi_n}{a}\frac{J_1(\chi_n\rho/a)N_0(\chi_n) - N_1(\chi_n\rho/a)J_0(\chi_n)}{\sqrt{J_0^2(\chi_n)/J_0^2(c_1\chi_n) - 1}},\ n \geq 2$$

where $c_1 = b/a$, $\boldsymbol{u}_\rho$ is the unit vector in ρ direction, and χ_n is the nth nonvanishing root of the eqauation $J_0(\chi_n c_1)N_0(\chi_n) - N_0(\chi_n c_1)J_0(\chi_n) = 0$. It follows from (12), (14) and (20) that

$$\begin{aligned} i_1^{TEM} &= -\frac{\pi}{\eta\sqrt{2\pi\ln c_1}} f(t - |z - z_0|/c) \\ i_n^{TM} &= -\frac{\pi\chi_n\rho_0\sqrt{\pi}}{2a\eta}\frac{J_1(\chi_n\rho_0/a)N_0(\chi_n) - N_1(\chi_n\rho_0/a)J_0(\chi_n)}{\sqrt{J_0^2(\chi_n)/J_0^2(c_1\chi_n) - 1}} \\ &\quad \cdot \int_{-\infty}^{t-|z-z_0|/c} \frac{df(t')}{dt'} J_0\left[k_{cn}c\sqrt{(t-t')^2 - |z-z_0|^2/c^2}\right] dt' \end{aligned}$$

Therefore if the highest frequency component of the excitation waveform is below the cut-off frequency of the first higher mode, the coaxial line will be an ideal medium for a distortion-free transmission of signals. From (4) the magnetic field in the coaxial cable is given by

$$\begin{aligned} \boldsymbol{H} &= \boldsymbol{u}_\varphi H_\varphi = \boldsymbol{u}_\varphi i_n^{TEM}\frac{1}{\rho\sqrt{2\pi\ln c_1}} \\ &\quad + \boldsymbol{u}_\varphi \sum_{n=2}^{\infty} i_n^{TM}\frac{\sqrt{\pi}\chi_n}{2a}\frac{J_1(\chi_n\rho/a)N_0(\chi_n) - N_1(\chi_n\rho/a)J_0(\chi_n)}{\sqrt{J_0^2(\chi_n)/J_0^2(c_1\chi_n) - 1}} \\ &= \frac{-\boldsymbol{u}_\varphi}{2\eta\rho\ln c_1} f(t - |z - z_0|/c) \\ &\quad - \boldsymbol{u}_\varphi \sum_{n=2}^{\infty} \frac{\pi^2\chi_n^2\rho_0}{4a^2\eta}\frac{J_1(\chi_n\rho_0/a)N_0(\chi_n) - N_1(\chi_n\rho_0/a)J_0(\chi_n)}{\sqrt{J_0^2(\chi_n)/J_0^2(c_1\chi_n) - 1}} \\ &\quad \cdot \frac{J_1(\chi_n\rho/a)N_0(\chi_n) - N_1(\chi_n\rho/a)J_0(\chi_n)}{\sqrt{J_0^2(\chi_n)/J_0^2(c_1\chi_n) - 1}} \\ &\quad \cdot \int_{-\infty}^{t-|z-z_0|/c} \frac{df(t')}{dt'} J_0\left[k_{cn}c\sqrt{(t-t')^2 - |z-z_0|^2/c^2}\right] dt' \end{aligned} \tag{25}$$

In the following all numerical examples are based on the assumptions that $\rho_0 = (a+b)/2$, and $z_0 = 0$. The time-domain currents for the first three modes excited by a unit step waveform $f(t) = H(t)$ are depicted in Figure 8, where $\bar{i}_1^{TEM} = -\eta i_1^{TEM}$ and $\bar{i}_n^{TM} = -\eta i_n^{TM}$. Figure 9 gives

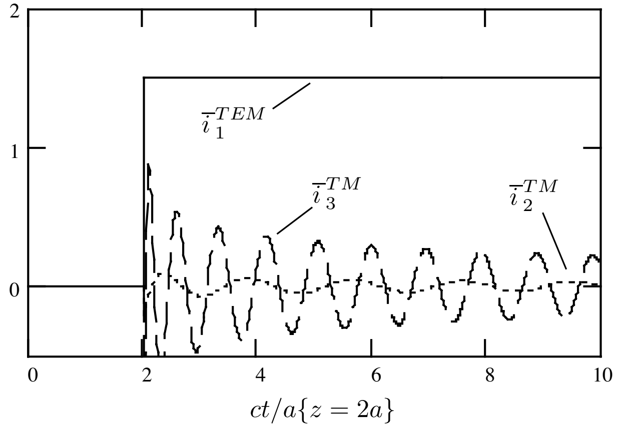

Figure 8. Time-domain modal currents excited by unit step waveform.

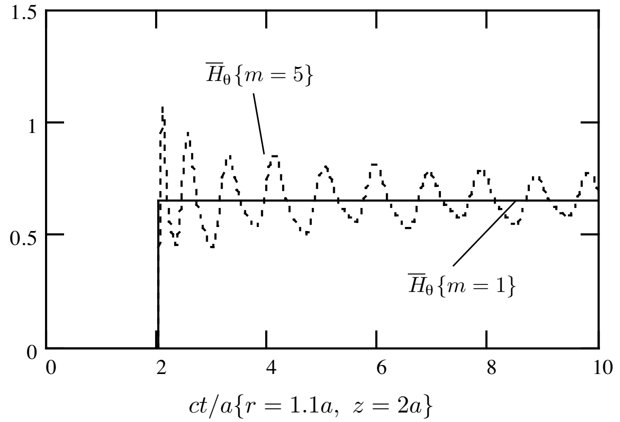

Figure 9. Magnetic fields excited by unit step waveform.

the magnetic fields $\overline{H}_\theta = -\eta H_\theta$ at $r = 1.1a,\ z = 2a$, where m denotes the number of terms chosen in (25). If we neglect the contribution of the higher modes by letting $m = 1$, the magnetic field is a perfect step function. When higher modes are included ($m = 5$, the first five modes are used), some ripples will occur in the response of the magnetic field, which stands for the contribution of higher order modes excited by the high frequency components of the unit step waveform. In this situation the signal has been distorted when it propagates in the coaxial line.

Fig. 10 and Fig. 11 give the time-domain currents and the

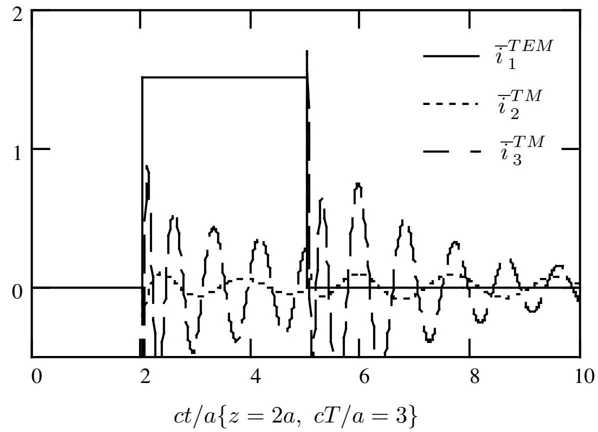

Figure 10. Time-domain modal currents excited by a rectangular pulse.

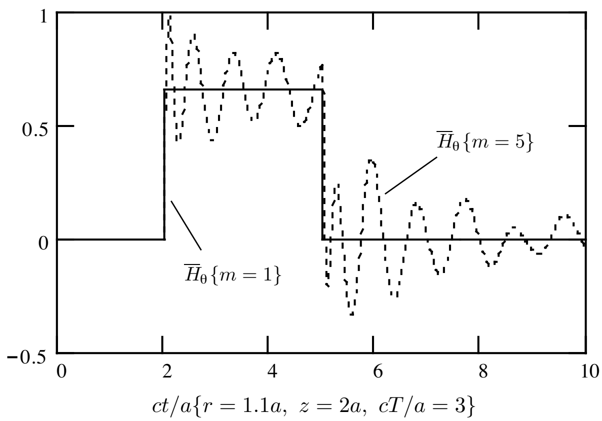

Figure 11. Magnetic fields excited by rectangular pulse.

magnetic fields in the coaxial line excited by a rectangular pulse $f(t) = H(t) - H(t - T)$ respectively. It can be seen that the ripples occur not only inside the pulse but also outside the pulse. Therefore when a rectangular pulse train passes through a transmission line the pulse will spread in time, which causes the pulse to smear into the time intervals of succeeding pulses. This phenomenon is very similar to the situation of a pulse train passing through a bandlimited channel, where the pulses will also spread in time, introducing intersymbol interference. Note that the time spread in the transmission line is caused by the higher order modes while the time spread in a bandlimited channel is due to the shortage of bandwidth. To reduce the effects of time spread in both cases, the pulse shaping techniques can be used to restrain the high frequency components.

5. CONCLUSIONS

In this paper a new and concise approach for the time-domain theory of waveguides has been presented. The theory has been applied to the time-domain analysis of excitation problems in typical waveguides. Numerical results for various typical excitation pulses have been expounded, which give a real physical picture of the transient process in a waveguide. In a special case where the excitation pulse is a sinusoidal wave turned on at $t = 0$, the steady state response is shown to approach to the well-known result from time-harmonic theory, which validates our theory. More validations can be found in the Appendices.

Our analysis results have also indicated that the contributions from the higher order modes excited by a wideband waveform are significant. Therefore the input signal at a given reference plane cannot simply be written as a single term of a separable function of space and time when studying the propagation of wideband signals in a waveguide or a feeding line of an antenna. Instead the expression of the input signal should consist of a number of such terms given by (3) and (4), each term having a different time variation than the others. The number of terms to be selected depends on the accuracy required as well as the bandwidth of excitation waveform. The wider the bandwidth of the excitation waveform, the more terms of the higher order modes must be included.

APPENDIX A. PROOF OF THE COMPLETENESS OF THE TRANSVERSE EIGENFUNCTIONS

To prove the completeness of the transverse eigenfunctions we need the following theorem [29, pp. 284].

Theorem: Let H be a real separable Hilbert space with $\dim H = \infty$. Let $\boldsymbol{B} : D(\boldsymbol{B}) \subset H \to H$ be a linear, symmetric operator. We further assume that $\boldsymbol{B}$ is strongly monotone, i.e., there exists a constant c_1 such that $(\boldsymbol{B}u, u) > c_1\|u\|^2$ for all $u \in D(\boldsymbol{B})$. Let $\boldsymbol{B}_F$ be the Friedrichs extension of $\boldsymbol{B}$ and $H_{\boldsymbol{B}}$ be the energy space of operator $\boldsymbol{B}$. We further assume that the embedding $H_{\boldsymbol{B}} \subset H$ is compact. Then the following eigenvalue problem

$$\boldsymbol{B}_F u = \lambda u, \quad u \in D(\boldsymbol{B}_F)$$

has a countable eigenfunctions $\{u_n\}$, which form a complete orthonormal system in the Hilbert space H, with $u_n \in H_{\boldsymbol{B}}$. Each eigenvalue corresponding to u_n has finite multiplicity. Furthermore we have $\lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_n \underset{n\to\infty}{\longrightarrow} \infty$.

Now consider the following eigenvalue problem for the transverse electric field in the waveguide

$$\begin{cases} \boldsymbol{B}(\boldsymbol{e}) = \nabla\times\nabla\times\boldsymbol{e} - \nabla(\nabla\cdot\boldsymbol{e}) = k_c^2\boldsymbol{e}, & \boldsymbol{r}\in\Omega \\ \boldsymbol{u}_n\times\boldsymbol{e} = \nabla\cdot\boldsymbol{e} = 0, & \boldsymbol{r}\in\Gamma \end{cases} \tag{A1}$$

where $k_c^2 = \omega^2\mu\varepsilon - \beta^2$, and $\boldsymbol{B} = \nabla\times\nabla\times -\nabla\nabla$. The domain of definition of operator $\boldsymbol{B}$ is defined as follows

$$D(\boldsymbol{B}) = \left\{\boldsymbol{e}\middle|\boldsymbol{e}\in(C^\infty(\Omega))^2, \boldsymbol{u}_n\times\boldsymbol{e} = \nabla\cdot\boldsymbol{e} = 0 \text{ on } \Gamma\right\}$$

where $C^\infty(\Omega)$ stands for the set of functions that have continuous partial derivatives of any order. Let $L^2(\Omega)$ stand for the space of square-integrable functions defined in Ω and $H = (L^2(\Omega))^2 = L^2(\Omega)\times L^2(\Omega)$. For two transverse vector fields $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in $(L^2(\Omega))^2$, the usual inner product is defined by

$$(\boldsymbol{e}_1, \boldsymbol{e}_2) = \int_\Omega \boldsymbol{e}_1\cdot\overline{\boldsymbol{e}}_2 d\Omega.$$

and the corresponding norm is denoted by $\|\cdot\| = (\cdot,\cdot)^{1/2}$. In the above a bar over a letter is used to represent the complex conjugate. Now we modify (A1) as equivalent form by adding a term $\xi\boldsymbol{e}$ on both sides

$$\begin{cases} \boldsymbol{A}(\boldsymbol{e}) = \nabla\times\nabla\times\boldsymbol{e} - \nabla(\nabla\cdot\boldsymbol{e}) + \xi\boldsymbol{e} = \left(k_c^2+\xi\right)\boldsymbol{e}, & \boldsymbol{r}=\Omega \\ \boldsymbol{u}_n\times\boldsymbol{e} = \nabla\cdot\boldsymbol{e} = 0, & \boldsymbol{r}\in\Gamma \end{cases}$$

where ξ is an arbitrary positive constant. In order to apply the previous theorem we need to prove that the operator $\boldsymbol{A}$ is symmetric, strongly monotone and the embedding $H_{\boldsymbol{A}}\subset H$ is compact. The proof will be carried out in three steps.

Step 1: We first show that $\boldsymbol{A}$ is symmetric, strongly monotone. For all $\boldsymbol{e}_1, \boldsymbol{e}_2 \in D(\boldsymbol{A}) = D(\boldsymbol{B})$, we have

$$\begin{aligned}(\boldsymbol{e}_1,\boldsymbol{e}_2)_{\boldsymbol{A}} &= (\boldsymbol{A}(\boldsymbol{e}_1),\boldsymbol{e}_2) = \int_\Omega[\nabla\times\nabla\times\boldsymbol{e}_1 - \nabla(\nabla\cdot\boldsymbol{e}_1)+\xi\boldsymbol{e}_1]\cdot\overline{\boldsymbol{e}}_2 d\Omega \\ &= \int_\Omega[\nabla\times\boldsymbol{e}_1\cdot\nabla\times\overline{\boldsymbol{e}}_2 + (\nabla\cdot\boldsymbol{e}_1)(\nabla\cdot\overline{\boldsymbol{e}}_2)+\xi\boldsymbol{e}_1\cdot\overline{\boldsymbol{e}}_2]d\Omega \quad \text{(A2)}\end{aligned}$$

Therefore the new operator $\boldsymbol{A}$ is symmetric. Thus we can assume that $\boldsymbol{e}$ is real. $\boldsymbol{A}$ is also strongly monotone since

$$(\boldsymbol{A}(\boldsymbol{e}), \boldsymbol{e}) = \int_\Omega [\nabla \times \boldsymbol{e} \cdot \nabla \times \boldsymbol{e} + (\nabla \cdot \boldsymbol{e})(\nabla \cdot \boldsymbol{e}) + \xi \boldsymbol{e} \cdot \boldsymbol{e}] d\Omega \geq \xi \|\boldsymbol{e}\|^2$$

Step 2: We then demonstrate some important properties of energy space $H_{\boldsymbol{A}}$. The energy space $H_{\boldsymbol{A}}$ is the completion of $D(\boldsymbol{A})$ with respect to the norm $\|\cdot\|_{\boldsymbol{A}} = (\cdot,\cdot)_{\boldsymbol{A}}^{1/2}$. Now let $\boldsymbol{e} \in H_{\boldsymbol{A}}$, and by definition, there exists admissible sequence $\{\boldsymbol{e}_n \in D(\boldsymbol{A})\}$ for $\boldsymbol{e}$ such that

$$\|\boldsymbol{e}_n - \boldsymbol{e}\| \underset{n\to\infty}{\longrightarrow} 0$$

and $\{\boldsymbol{e}_n\}$ is a Cauchy sequence in $H_{\boldsymbol{A}}$. From (A2) we obtain

$$\|\boldsymbol{e}_n - \boldsymbol{e}_m\|_{\boldsymbol{A}}^2 = \|\nabla \times \boldsymbol{e}_n - \nabla \times \boldsymbol{e}_m\|^2 + \|\nabla \cdot \boldsymbol{e}_n - \nabla \cdot \boldsymbol{e}_m\|^2 + \xi \|\boldsymbol{e}_n - \boldsymbol{e}_m\|^2$$

Consequently $\{\nabla \times \boldsymbol{e}_n\}$ and $\{\nabla \cdot \boldsymbol{e}_n\}$ are Cauchy sequences in H. As a result, there exist $\boldsymbol{h} \in H$, and $\rho \in L^2(\Omega)$ such that

$$\nabla \times \boldsymbol{e}_n \underset{n\to\infty}{\longrightarrow} \boldsymbol{h}, \quad \nabla \cdot \boldsymbol{e}_n \underset{n\to\infty}{\longrightarrow} \rho$$

From integration by parts

$$\int_\Omega \nabla \times \boldsymbol{e}_n \cdot \boldsymbol{\varphi} d\Omega = \int_\Omega \boldsymbol{e}_n \cdot \nabla \times \boldsymbol{\varphi} d\Omega, \quad \forall \boldsymbol{\varphi} \in (C_0^\infty(\Omega))^2$$

$$\int_\Omega (\nabla \cdot \boldsymbol{e}_n) \varphi d\Omega = -\int_\Omega \boldsymbol{e}_n \cdot \nabla \varphi d\Omega, \quad \forall \varphi \in C_0^\infty(\Omega)$$

we obtain

$$\int_\Omega \boldsymbol{h} \cdot \boldsymbol{\varphi} d\Omega = \int_\Omega \boldsymbol{e} \cdot \nabla \times \boldsymbol{\varphi} d\Omega, \quad \forall \boldsymbol{\varphi} \in (C_0^\infty(\Omega))^2$$

$$\int_\Omega \rho \varphi d\Omega = -\int_\Omega \boldsymbol{e} \cdot \nabla \varphi d\Omega, \quad \forall \varphi \in C_0^\infty(\Omega).$$

In the above $C_0^\infty(\Omega)$ is the set of all functions in $C^\infty(\Omega)$ that vanish outside a compact set of Ω. Therefore $\nabla \times \boldsymbol{e} = \boldsymbol{h}$ and $\nabla \cdot \boldsymbol{e} = \rho$ in the generalized sense. For arbitrary $\boldsymbol{e}_1, \boldsymbol{e}_2 \in H_{\boldsymbol{A}}$, there are two admissible functions $\{\boldsymbol{e}_{1n}\}$ and $\{\boldsymbol{e}_{2n}\}$ such that $\|\boldsymbol{e}_{1n} - \boldsymbol{e}_1\| \underset{n\to\infty}{\longrightarrow} 0$ and $\|\boldsymbol{e}_{2n} - \boldsymbol{e}_2\| \underset{n\to\infty}{\longrightarrow} 0$. We define

$$\begin{aligned}(\boldsymbol{e}_1, \boldsymbol{e}_2)_{\boldsymbol{A}} &= \lim_{n\to\infty} (\boldsymbol{e}_{1n}, \boldsymbol{e}_{2n})_{\boldsymbol{A}} \\ &= \int_\Omega [\nabla \times \boldsymbol{e}_1 \cdot \nabla \times \overline{\boldsymbol{e}}_2 + (\nabla \cdot \boldsymbol{e}_1)(\nabla \cdot \overline{\boldsymbol{e}}_2) + \xi \boldsymbol{e}_1 \cdot \overline{\boldsymbol{e}}_2] \, d\Omega\end{aligned}$$

where the derivatives must be understood in the generalized sense.

Step 3: Finally we prove that the embedding $H_{\boldsymbol{A}} \subset H$ is compact. Let

$$\boldsymbol{J}(\boldsymbol{e}) = \boldsymbol{e}, \quad \boldsymbol{e} \in H_{\boldsymbol{A}}$$

Then the linear operator $\boldsymbol{J} : H_{\boldsymbol{A}} \to H$ is continuous since

$$\|\boldsymbol{J}(\boldsymbol{e})\|^2 = \|\boldsymbol{e}\|^2 \leq \xi^{-1}\left(\xi\|\boldsymbol{e}\|^2 + \|\nabla\times\boldsymbol{e}\|^2 + \|\nabla\cdot\boldsymbol{e}\|^2\right) = \xi^{-1}\|\boldsymbol{e}\|_{H_{\boldsymbol{A}}}.$$

A bounded sequence $\{\boldsymbol{e}_n\} \subset H_{\boldsymbol{A}}$ implies

$$\begin{aligned}\|\boldsymbol{e}_n\|^2_{H_{\boldsymbol{A}}} &= \xi\|\boldsymbol{e}_n\|^2 + \|\nabla\times\boldsymbol{e}_n\|^2 + \|\nabla\cdot\boldsymbol{e}_n\|^2 \\ &= \int_\Omega \left[\xi(e_{nx})^2 + \xi(e_{ny})^2 + (\nabla e_{nx})^2 + (\nabla e_{ny})^2\right] d\Omega \leq c'\end{aligned}$$

where c' is a constant. So the compactness of the operator $\boldsymbol{J}$ follows from the above inequality and the following Rellich's theorem.

Rellich's theorem [25]: Any sequence $\{f_n\}$, which satisfies $\|f_n\|^2 = \int_\Omega f_n^2 d\Omega \leq c_1$ and $\|\nabla f_n\|^2 = \int_\Omega (\nabla f_n)^2 d\Omega \leq c_2$, where c_1 and c_2 are constants, has a subsequence still denoted by $\{f_n\}$ such that $\lim\limits_{m,n\to\infty} \int_\Omega w(f_m - f_n)^2 d\Omega = 0$, where w is the weight function.

From the previous theorem, the set of eigenfunctions $\{\boldsymbol{e}_n\}$ constitutes a complete system in $(L^2(\Omega))^2$. The proof is completed.

APPENDIX B. A DIFFERENT APPROACH TO THE TRANSIENT RESPONSE OF RECTANGULAR WAVEGUIDE

To validate (24), let us take a totally different approach to the excitation problem considered in Section 4.1. The electromagnetic fields in the waveguide satisfy the wave equations

$$\nabla\times\nabla\times\boldsymbol{E} + \mu\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2} = -\mu\frac{\partial\boldsymbol{J}}{\partial t}, \quad \nabla\times\nabla\times\boldsymbol{H} + \mu\varepsilon\frac{\partial^2\boldsymbol{H}}{\partial t^2} = \nabla\times\boldsymbol{J}$$

From the wave equations and the property of exciting source only the y component of the electric field $\boldsymbol{E}$ is excited. The y component of the electric field E_y satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right) E_y = \mu f'(t)\delta(x - x_0)\delta(z - z_0)$$

and the boundary condition $E_y(x,z,t)\Big|_{x=0,a} = 0$. Making use of the Fourier transform pair with respect to z and t,

$$\tilde{E}_y(x,p,\omega) = \boldsymbol{F}(E_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(x,z,t)e^{-jpz-j\omega t}dzdt$$

$$E_y(x,z,t) = \boldsymbol{F}^{-1}(\tilde{E}_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}_y(x,p,\omega)e^{jpz+j\omega t}dpd\omega$$

we obtain

$$\begin{cases} \left(\dfrac{\partial^2}{\partial x^2} - p^2 + \dfrac{\omega^2}{c^2}\right)\tilde{E}_y = j\omega\mu\tilde{f}(\omega)e^{-jpz_0}\delta(x-x_0) \\ \tilde{E}_y(x,p,\omega)\Big|_{x=0,a} = 0 \end{cases} \tag{B1}$$

where $c = 1/\sqrt{\mu\varepsilon}$ and $\tilde{f}(\omega)$ is the Fourier transform of $f(t)$. According to the boundary condition we may use the well-known complete orthonormal set $\{\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x|n = 1,2,\ldots\}$ to expand the electric field [30]. So we have

$$\tilde{E}_y = \sum_{n=1}^{\infty}\tilde{e}_n(z,t)\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x \tag{B2}$$

Inserting this into (B1) we get

$$\sum_{n=1}^{\infty}\tilde{e}_n(z,t)\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x\left[-\left(\frac{n\pi}{a}\right)^2 - p^2 + \frac{\omega^2}{c^2}\right] = j\omega\mu\tilde{f}(\omega)e^{-jpz_0}\delta(x-x_0)$$

Multiplying both sides by $\sqrt{2/a}\sin(n\pi x/a)$ and taking the integration over $[0,a]$ gives

$$\tilde{e}_n(z,t) = \frac{j\omega\mu c^2\tilde{f}(\omega)\sqrt{2/a}\sin(n\pi x_0/a)e^{-jpz_0}}{\omega^2 - c^2(n\pi/a)^2 - p^2c^2}$$

The electric field can be obtained by taking the inverse Fourier transform

$$E_y = \sum_{n=1}^{\infty}\boldsymbol{F}^{-1}\left[\tilde{e}_n(z,t)\right]\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x$$

where

$$\boldsymbol{F}^{-1}\left[\tilde{e}_n(z,t)\right] = \mu c^2\sqrt{\frac{2}{a}}\sin\frac{n\pi x_0}{a}\boldsymbol{F}^{-1}\left[\frac{j\omega\tilde{f}(\omega)e^{-jpz_0}}{\omega^2-c^2(n\pi/a)^2-p^2c^2}\right]$$

$$= \frac{\mu c^2}{(2\pi)^2}\sqrt{\frac{2}{a}}\sin\frac{n\pi x_0}{a}f'(t)\otimes\int_{-\infty}^{\infty}e^{jpz-jpz_0}dp\int_{-\infty}^{\infty}\frac{e^{j\omega t}}{\omega^2-p^2c^2-c^2(n\pi/a)^2}d\omega$$

The integral with respect to ω can be evaluated by residue theorem and result is

$$\int_{-\infty}^{\infty}\frac{e^{j\omega t}d\omega}{\omega^2-p^2c^2-c^2(n\pi/a)^2} = -\frac{2\pi\sin t\sqrt{p^2c^2+c^2(n\pi/a)^2}}{\sqrt{p^2c^2+c^2(n\pi/a)^2}}$$

Thus we have

$$\begin{aligned}\boldsymbol{F}^{-1}[\tilde{e}_n(z,t)] &= -\frac{\mu c^2}{\pi}\sqrt{\frac{2}{a}}\sin\frac{n\pi x_0}{a}f'(t)\\ &\quad\otimes\int_0^{\infty}\cos p(z-z_0)\frac{\sin tc\sqrt{p^2+(n\pi/a)^2}}{\sqrt{p^2+(n\pi/a)^2}}dp\\ &= -\frac{\eta}{2}\sqrt{\frac{2}{a}}\sin\frac{n\pi x_0}{a}f'(t)\\ &\quad\otimes J_0\left[\mu_n c\sqrt{t^2-|z-z_0|/c^2}\right]H(t-|z-z_0|/c)\end{aligned}$$

Finally the electric field is given by

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{u}_y E_y\\ &= -\boldsymbol{u}_y\sum_{n=1}^{\infty}\frac{\eta}{a}\sin\frac{n\pi x_0}{a}\sin\frac{n\pi}{a}x\\ &\quad\cdot\int_{-\infty}^{t-|z-z_0|/c}\frac{df(t')}{dt'}J_0\left[k_{cn}c\sqrt{(t-t')^2-|z-z_0|^2/c^2}\right]dt' \qquad \text{(B3)}\end{aligned}$$

which is exactly the same as (24).

APPENDIX C. A DIFFERENT APPROACH TO THE TRANSIENT RESPONSE OF COAXIAL WAVEGUIDE

To validate (25), the excitation problem discussed in Section 4.2 will be studied by directly solving Maxwell equations. The electromagnetic fields in the coaxial waveguide satisfy

$$\nabla\times\nabla\times\boldsymbol{H}+\mu\varepsilon\frac{\partial^2\boldsymbol{H}}{\partial t^2}=-\varepsilon\frac{\partial\boldsymbol{J}_m}{\partial t},\quad \nabla\times\nabla\times\boldsymbol{E}+\mu\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2}=-\nabla\times\boldsymbol{J}_m \tag{C1}$$

From the above equations and the symmetry of the source only the field components E_ρ, E_z, and H_φ are excited. Thus we have

$$\nabla\times\boldsymbol{H}=-\frac{\partial H_\varphi}{\partial z}\boldsymbol{u}_\rho+\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)\boldsymbol{u}_z$$

The boundary condition $\boldsymbol{u}_n\times\boldsymbol{E}=0$ on the conductor requires $\frac{\partial}{\partial\rho}(\rho H_\varphi)\Big|_{\rho=a,b}=0$. Equation (C1) reduces to

$$\frac{\partial^2 H_\varphi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial H_\varphi}{\partial\rho}-\frac{H_\varphi}{\rho^2}+\frac{\partial^2 H_\varphi}{\partial z^2}-\mu\varepsilon\frac{\partial^2 H_\varphi}{\partial t^2}=\varepsilon f'(t)\delta(z-z_0)\delta(\rho-\rho_0)$$

Making use of the Fourier transform pair with respect to z and t,

$$\tilde{H}_\varphi(\rho,\varphi,p,\omega)\;=\;\boldsymbol{F}(H_\varphi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_\varphi(\rho,\varphi,z,t)e^{-jpz-j\omega t}dzdt$$

$$H_\varphi(\rho,\varphi,z,t)\;=\;\boldsymbol{F}^{-1}(\tilde{H}_\varphi)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{H}_\varphi(\rho,\varphi,p,\omega)e^{jpz+j\omega t}dpd\omega$$

we get

$$\begin{cases}\dfrac{\partial^2\tilde{H}_\varphi}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial\tilde{H}_\varphi}{\partial\rho}+\left(\mu^2-\dfrac{1}{\rho^2}\right)\tilde{H}_\varphi=j\omega\varepsilon\tilde{f}(\omega)e^{-jpz_0}\delta(\rho-\rho_0)\\ \dfrac{\partial(\rho\tilde{H}_\varphi)}{\partial\rho}\Big|_{\rho=a,b}=\rho\dfrac{\partial\tilde{H}_\varphi}{\partial\rho}+\tilde{H}_\varphi\Big|_{\rho=a,b}=0\end{cases} \tag{C2}$$

where $\mu^2=\omega^2\mu\varepsilon-p^2$. To solve this equation, we may use the method of eigenfunction expansion [30]. We first consider the following eigenvalue problem

$$\begin{cases}\dfrac{\partial^2 h_\varphi}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial h_\varphi}{\partial\rho}+\left(\mu^2-\dfrac{1}{\rho^2}\right)h_\varphi=0\\ \rho\dfrac{\partial h_\varphi}{\partial\rho}+h_\varphi\Big|_{\rho=a,b}=0\end{cases} \tag{C3}$$

This is a typical eigenvalue problem of Sturm-Liouville type. The general solution of the above equation is

$$h_\varphi=a_1J_1(\mu\rho)+a_2N_1(\mu\rho)$$

Applying the boundary conditions yields

$$a_1/a_2 = -N_0(\mu a)/J_0(\mu a) = -N_0(\mu b)/J_0(\mu b)$$

The above equation determines the eigenvalues μ_n, $n = 1, 2, \ldots$, which satisfy

$$J_0(c'\chi_n)N_0(\chi_n) - J_0(\chi_n)N_0(c'\chi_n) = 0$$

where $\chi_n = \mu_n a$ is the nth root of the above equation, and $c' = b/a$. The eigenfunctions corresponding to μ_n are given by

$$h_{\varphi n}(\rho) = b_n[J_1(\chi_n\rho/a)N_0(\chi_n) - N_1(\chi_n\rho/a)J_0(\chi_n)]$$

The constants b_n can be determined by using the normalized condition $\int_a^b \rho h_{\varphi n}^2 d\rho = 1$, which gives $b_n = \frac{\pi\chi_n}{\sqrt{2}a}\left[\frac{J_0^2(\chi_n)}{J_0^2(c'\chi_n)} - 1\right]^{-1/2}$. The normalized eigenfunctions may be expressed as

$$h_{\varphi n}(\rho) = \frac{\pi\chi_n}{\sqrt{2}a}\frac{[J_1(\chi_n\rho/a)N_0(\chi_n) - N_1(\chi_n\rho/a)J_0(\chi_n)]}{[J_0^2(\chi_n)/J_0^2(c'\chi_n) - 1]^{1/2}}, \quad n = 1, 2, \ldots$$

In solving (C3) we have assumed that $\mu \neq 0$. If $\mu = 0$, (C3) reduces to

$$\begin{cases} \dfrac{\partial^2 h_\varphi}{\partial\rho^2} + \dfrac{1}{\rho}\dfrac{\partial h_\varphi}{\partial\rho} - \dfrac{h_\varphi}{\rho^2} = 0 \\ \dfrac{\rho\partial h_\varphi}{\partial\rho} + h_\varphi\Big|_{\rho=a,b} = 0 \end{cases}$$

The general solution of the above equation is $h_\varphi = c_1\rho + c_2/\rho$, where c_1 and c_2 are two constants. The boundary condition requires that $c_1 = 0$. Therefore $h_\varphi = c_2/\rho$. Applying the normalized condition we obtain eigenfunction corresponding to $\mu = 0$

$$h_{\varphi 0} = \frac{1}{\rho\sqrt{\ln c'}}$$

Thus the set of eigenfunctions $\{h_{\varphi n},\ n = 0, 1, 2, \ldots\}$ is complete by Sturm-Liouville theory [30], and can be used to expand the solution of (C2)

$$\tilde{H}_\varphi = \sum_{n=0}^{\infty}\tilde{g}_n h_{\varphi n} \tag{C4}$$

where $\tilde{g}_n$ are the expansion coefficients to be determined. Substituting this into (C2) we obtain

$$\sum_{n=0}^{\infty}\tilde{g}_n\left[\frac{\partial^2 h_{\varphi n}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial h_{\varphi n}}{\partial\rho} + \left(\mu^2 - \frac{1}{\rho^2}\right)h_{\varphi n}\right] = j\omega\varepsilon\tilde{f}(\omega)e^{-jpz_0}\delta(\rho - \rho_0)$$

i.e.,

$$\sum_{n=0}^{\infty} \tilde{g}_n \left(\mu^2 - \mu_n^2\right) h_{\varphi n} = j\omega\varepsilon \tilde{f}(\omega) e^{-jpz_0} \delta(\rho - \rho_0)$$

Multiplying both sides by $\rho h_{\varphi n}$ and taking the integration over $[a, b]$, we get

$$\tilde{g}_n = \frac{j\omega\varepsilon c^2 \rho_0 \tilde{f}(\omega) e^{-jpz_0} h_{\varphi n}(\rho_0)}{\omega^2 - p^2 c^2 - \mu_n^2 c^2}$$

The magnetic field can then be obtained by taking the inverse Fourier transform of (C4)

$$H_\varphi = \sum_{n=0}^{\infty} \boldsymbol{F}^{-1}(\tilde{g}_n) h_{\varphi n}$$

where

$$\boldsymbol{F}^{-1}(\tilde{g}_n) = \varepsilon c^2 \rho_0 h_{\varphi n}(\rho_0) \boldsymbol{F}^{-1} \left[\frac{j\omega \tilde{f}(\omega) e^{-jpz_0}}{\omega^2 - p^2 c^2 - \mu_n^2 c^2}\right]$$

Similar to the discussion in Appendix B, we may obtain

$$\boldsymbol{F}^{-1}(\tilde{g}_n) = -\frac{\rho_0 h_{\varphi n}(\rho_0)}{2\eta} \cdot \int_{-\infty}^{t-|z-z_0|/c} \frac{df(t')}{dt'} J_0 \left[k_{cn} c \sqrt{(t-t')^2 - |z-z_0|^2/c^2}\right] dt'$$

The magnetic field is given by

$$\begin{aligned} \boldsymbol{H} \ = \ \boldsymbol{u}_\varphi H_\varphi &= \frac{-\boldsymbol{u}_\varphi}{2\eta\rho \ln c'} f(t - |z-z_0|/c) \\ &-\boldsymbol{u}_\varphi \sum_{n=1}^{\infty} \frac{\pi^2 \chi_n^2 \rho_0}{4a^2\eta} \frac{[J_1(\chi_n\rho_0/a) N_0(\chi_n) - N_1(\chi_n\rho_0/a) J_0(\chi_n)]}{\left[J_0^2(\chi_n)/J_0^2(c'\chi_n) - 1\right]^{1/2}} \\ &\cdot \frac{[J_1(\chi_n\rho/a) N_0(\chi_n) - N_1(\chi_n\rho/a) J_0(\chi_n)]}{\left[J_0^2(\chi_n)/J_0^2(c'\chi_n) - 1\right]^{1/2}} \\ &\cdot \int_{-\infty}^{t-|z-z_0|/c} \frac{df(t')}{dt'} J_0 \left[k_{cn} c \sqrt{(t-t')^2 - |z-z_0|^2/c^2}\right] dt' \end{aligned}$$

which is exactly the same as (25).

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