ANALYSIS OF LINEAR TAPERED WAVEGUIDE BY
TWO APPROACHES

S. Dwari, A. Chakraborty, and S. Sanyal

Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology
Kharagpur-721302, India

Abstract—This paper presents the analysis of linear tapered waveguide. Voltage-standing-wave-ratio (VSWR) is obtained from transmission matrix of the taper waveguide. Taper section is divided into number of section having uniform length. Transmission matrix of taper waveguide is found by multiplication of transmission matrix of each section. Transmission matrix of each section is obtained as the product of three matrices. One is of the initial length of transmission line, second one is due to discontinuity and third one is of the final length of transmission line. Transmission matrix of discontinuity is obtained by two methods. One is by equivalent circuit of step discontinuity and another is by moment method. The results are seen to be in good agreement with [1, 2] and [3].

1. INTRODUCTION

In a tapered section the dimension of waveguide varies smoothly and for this reason there are possibilities of providing a transition from one impedance level to another. Ref. [4] and then [5] analyzed taper line by dividing it into a number of sections of equal length into the direction of propagation. Ref. [1] found the transmission matrix of each section as the product of five matrices and multiplied the matrices of all sections.

This paper proposed that the transmission matrix of each section is obtained as the product of three matrices. They are contributed by initial length of transmission line, discontinuity and the final length of transmission line. Transmission matrix of discontinuity is obtained by two procedures. They are by equivalent circuit of step discontinuity and by moment method. The final transmission matrix of tapered waveguide is obtained by multiplication of matrices of all
sections. Voltage-standing-wave-ratio (VSWR) is obtained from final
transmission matrix.

2. TRANSMISSION MATRIX OF TAPER SECTION

The linear tapered waveguide is shown in Figure 1 and it can be
considered as if it is formed with huge number of step discontinuities.
Let the number of sections be \( N \) and the length of taper be \( L \).
Let \( 2a_0, 2b_0 \) are initial broad and narrow dimensions respectively and
\( 2a_1, 2b_1 \) are respective final broad and narrow dimensions.

Figure 1. (a) Tapered waveguide of length \( L \) connecting uniform
waveguides of dimensions \( 2a_0, 2b_0 \) and \( 2a_1, 2b_1 \). (b) Taper waveguide
of Figure 1(a) divided into sections. (c) Expanded view of \( n \)th section.

The \( n \)th section is blown up and shown in Figure 1(c). Each
section consists of an initial length of waveguide, a step discontinuity
and a final length of waveguide. Each of the constituents can be
expressed by a transmission matrix of order 2.

For \( n \)th section initial and final broad dimensions are

\[
2a_{0n} = 2a_0 + \frac{2(a_1 - a_0)}{N}(n - 1) \tag{1}
\]

\[
2a_{1n} = 2a_0 + \frac{2(a_1 - a_0)}{N}n \tag{2}
\]
Narrow wall dimensions are

\[ 2b_{0n} = 2b_0 + \frac{2(b_1 - b_0)}{N}(n - 1) \]  

(3)

\[ 2b_{1n} = 2b_0 + \frac{2(b_1 - b_0)}{N}n \]  

(4)

The matrix of each section is the product of three matrices. Let matrix \([T_1]\) is for initial length, \([T_3]\) is of final length and \([T_2]\) is for discontinuity. So transmission matrix of each section is given by

\[ [T] = [T_1][T_2][T_3] \]  

(5)

The transmission matrices of all the sections are multiplied to obtain the final transmission matrix of the taper.

Suppose \(d = \frac{L}{N}\) is the length of each section, \(k_{0n}\) is the propagation constant of initial length and \(k_{1n}\) is the propagation constant of final length of \(n\)th section. Then \([T_1]\) can be written as

\[ [T_1] = \begin{bmatrix} e^{jk_{0n}d} & 0 \\ 0 & e^{-jk_{0n}d} \end{bmatrix} \]  

(6)

\([T_3]\) can be written as

\[ [T_3] = \begin{bmatrix} e^{jk_{1n}d} & 0 \\ 0 & e^{-jk_{1n}d} \end{bmatrix} \]  

(7)

Let \(\Gamma_{1n}\) be the reflection co-efficient in the forward direction at discontinuity of \(n\)th section and \(\Gamma_{2n}\) be the reflection co-efficient in the backward direction at discontinuity of the \(n\)th section. Then \([T_2]\) can be written as

\[ [T_2] = \frac{1}{1 + \Gamma_{1n}} \begin{bmatrix} 1 & -\Gamma_{2n} \\ \Gamma_{1n} & 1 + \Gamma_{1n} + \Gamma_{2n} \end{bmatrix} \]  

(8)

To obtain \([T_2]\), reflection co-efficients should be determined. Reflection coefficient can be determined by two ways. One is using equivalent circuit representing step discontinuities shown in Figure 2 as described in [6]. Second is by using method of moment.

**3. DETERMINATION OF CIRCUIT PARAMETERS OF EQUIVALENT CIRCUIT FOR STEP DISCONTINUITIES**

Discontinuity of each section of linear taper waveguide is step discontinuity. This step discontinuity can be represented by equivalent
3.1. Evaluation of Circuit Parameters of Equivalent Circuit for Step Discontinuities in Narrow Dimension

Consider the waveguide junction as shown in Figure 3. Let junction is at \( z = 0 \). Waveguide of dimension \( 2a \times 2b \) extends up to \( z = -\infty \) and waveguide of dimension \( 2a \times 2(b + W) \) extends up to \( z = +\infty \). The dimensions are such that only dominant mode can propagate in each section. Let there be an incident wave from \( -z \) direction.

In the region \( z < 0 \) transverse fields are

\[
E_t^- = \left( e^{-j\beta z} + \Gamma e^{j\beta z} \right) \frac{V_0}{1 + \Gamma} e_0 + \sum_i V_i e^{\alpha_i z} e_i
\]

\[
H_t^- = Y_0^- \left( e^{-j\beta z} - \Gamma e^{j\beta z} \right) \frac{V_0}{1 + \Gamma} h_0 - \sum_i Y_i V_i e^{\alpha_i z} h_i
\]

where \( e_i, h_i \) are the mode vectors, \( \alpha_i \) are the cutoff mode-attenuation constants, \( Y_i \) are the characteristics admittances, and \( \Gamma \) is the reflection coefficient for the dominant mode. The subscript 0 signifies the
dominant mode. It is assumed that at \( z = \infty \) matched load is there. So in the region \( z > 0 \)

\[
E_t^+ = \hat{V}_0 e^{-j\beta z} e_0 + \sum_i \hat{V}_i e^{-\hat{\alpha}_i z} \hat{e}_i \\
H_t^+ = Y_0^+ \hat{V}_0 e^{-j\beta z} \hat{h}_0 + \sum_i \hat{Y}_i \hat{V}_i e^{-\hat{\alpha}_i z} \hat{h}_i
\]  

(10)

where the carets are used to differentiate the above parameters from there \( z < 0 \) counterparts. At junction region

\[
\int_0 E^+ \times H^+ \cdot ds = \int_0 E^- \times H^- \cdot ds
\]  

(11)

So

\[
Y_0^+ \hat{V}_0^2 + \sum_i \hat{Y}_i \hat{V}_i^2 = \frac{1 - \Gamma}{1 + \Gamma} Y_0^- V_0^2 - \sum_i Y_i V_i^2
\]  

(12)

Relative admittance observed from \( z < 0 \) is

\[
\frac{1 - \Gamma}{1 + \Gamma} = \frac{Y}{Y_0^-} = \frac{G}{Y_0^-} + j \frac{B}{Y_0^-}
\]  

(13)

\( Y_0 \) is real and \( Y_i \) are imaginary. For real \( V_i \) and \( \hat{V}_i \)

\[
\frac{jB}{Y_0^-} = \sum_i \frac{Y_i V_i^2 + \hat{Y}_i \hat{V}_i^2}{Y_0^- V_0^2}
\]  

(14)

\[
\frac{G}{Y_0^-} = \frac{Y_0^+ \hat{V}_0^2}{Y_0^- V_0^2}
\]  

(15)

For equivalent circuit of Figure 2, with matched condition at \( z = \infty \), it is evident that

\[
\frac{G}{Y_0^-} = n^2 \frac{Y_0^+}{Y_0^-}
\]  

(16)

From Eq. (15) and Eq. (16)

\[
n^2 = \frac{\hat{V}_0^2}{\hat{V}_0^2}
\]  

(17)

For the junction shown in Figure 3, the dominant mode vectors [6] are

\[
\hat{e}_0 = \hat{u}_y \sqrt{\frac{1}{2ab}} \sin \frac{\pi}{2a} (x + a)
\]  

(18)

\[
\hat{e}_0 = \hat{u}_y \sqrt{\frac{1}{2a(b + W)}} \sin \frac{\pi}{2a} (x + a)
\]  

(19)
Assumed tangential electric field in the aperture

\[ \overline{E}_t^a = \hat{a}_y \sin \frac{\pi}{2a} (x + a) \]  

Therefore following [6]:

\[ V_0 = \int_{x=-a}^{a} \int_{y=-b}^{b} \frac{1}{\sqrt{2ab}} \sin^2 \frac{\pi(x + a)}{2a} \, dxdy = \sqrt{2ab} \]  

\[ \hat{V}_0 = \int_{x=-a}^{a} \int_{y=-b}^{b} \frac{1}{\sqrt{2a(b + W)}} \sin^2 \frac{\pi(x + a)}{2a} \, dxdy = \sqrt{\frac{2a}{b + W} b} \]  

Therefore

\[ \frac{\hat{V}_0}{V_0} = \sqrt{\frac{b}{b + W}} \]  

\[ n^2 = \frac{b}{b + W} \]  

This is the transformation ratio of the transformer of equivalent circuit.

Now the first summation in the numerator of Equation (14) is zero and second summation is related to aperture susceptance \( B_a \) by

\[ \sum_i \hat{Y}_i \hat{V}_i^2 = j|V|^2 B_a = j4b^2 B_a \]  

Now

\[ V_0 = \int_{x=-a}^{a} \int_{y=-b}^{b} \sin \frac{\pi(x + a)}{2a} \frac{1}{\sqrt{2ab}} \sin \frac{\pi(x + a)}{2a} \, dxdy = \sqrt{2ab} \]

So

\[ B = \frac{2B_a b}{a} \]  

3.1.1 Determination of Aperture Susceptance \( B_a \)

Let \( E_x = 0 \) and \( E_y = f(x, y) \) be known over the cross section at \( z = 0 \). For TE to x mode at \( z > 0 \) scalar potential function is:

\[ \psi = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin \frac{m\pi}{2a} (x + a) \cos \frac{n\pi}{2(b + W)} (y + b + W) e^{-\gamma mn z} \]
where $A_{mn}$ are mode amplitudes and $\gamma_{mn}$ are the mode propagation constants. In particular $E_y$ at $z = 0$ is given by

$$E_y|_{z=0} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \gamma_{mn} A_{mn} \sin \frac{m\pi}{2a} (x + a) \cos \frac{n\pi}{2(b + W)} (y + b + W) \quad (29)$$

This is the form of double Fourier series: a sine series in $x$ and a cosine series in $y$. It is thus evident that $\gamma_{mn} A_{mn}$ are fourier coefficients of $E_y$, or

$$\gamma_{mn} A_{mn} = E_{mn} = \frac{\epsilon_n}{2a(b + W)} \int_{-a}^{a} \int_{-(b+W)}^{(b+W)} E_y|_{z=0} \sin \frac{m\pi}{2a} (x + a) \cos \frac{n\pi}{2(b + W)} (y + b + W) dx dy \quad (30)$$

where $\epsilon_n = 1$ for $n = 0$ and $\epsilon_n = 2$ for $n > 0$ (Numann’s number). The $A_{mn}$ and hence the field are now evaluated. The $z$ directed complex power at $z = 0$ is

$$P = \int \int_{z=0} \left( \vec{E} \times \vec{H}^* \right) \cdot \hat{u}_z ds$$

$$= - \int_{-a}^{a} \int_{-(b+W)}^{(b+W)} [E_y H_x^*]_{z=0} dx dy$$

$$= \int_{-a}^{a} \int_{-(b+W)}^{(b+W)} \left[ \sum_{m,n} E_{mn} \sin \frac{m\pi(x + a)}{2a} \cos \frac{n\pi(y + b + W)}{2(b + W)} \right]$$

$$\sum_{p,q} \frac{k^2 - \frac{p\pi}{2a}}{j\omega \mu \gamma_{pq}^*} E_{pq}^* \sin \frac{p\pi(x + a)}{2a} \cos \frac{q\pi(y + b + W)}{2(b + W)} dx dy \quad (31)$$

This reduces to

$$P = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (Y_0)_{mn}^* |E_{mn}|^2 \frac{2a(b + W)}{\epsilon_{mn}} \quad (32)$$
Assume
\[
E_y|_{z=0} = \begin{cases} 
\sin \frac{\pi(x+a)}{2a} & -b < y < b \\
0 & |y| > b
\end{cases} \tag{33}
\]

Only non-zero amplitudes are:
\[
E_{10} = \gamma_{10}A_{10} = \frac{b}{b+W} \tag{34}
\]
\[
E_{1n} = \gamma_{1n}A_{1n} = \frac{2}{n\pi} \left[ \sin \frac{n\pi b}{b+W} \left( b + \frac{W}{2} \right) - \sin \left( \frac{n\pi W}{2(b+W)} \right) \right] \tag{35}
\]

Therefore
\[
P = \frac{2ab^2}{b+W} \left[ (Y_0)_{10}^* + 8 \sum_{n=1}^{\infty} (Y_0)_{1n}^* \left( \frac{\sin \left( \frac{n\pi b}{b+W} \cos \left( \frac{n\pi}{2} \right) \right)}{n\pi b} \right)^2 \right] \tag{36}
\]

Now
\[
(Y_0)_{10} = \frac{k^2 - \left( \frac{\pi}{2a} \right)^2}{\omega \mu \beta} = \sqrt{1 - \left( \frac{f_c}{f} \right)^2} \tag{37}
\]
\[
(Y_0)_{1n} = \frac{k^2 - \left( \frac{\pi}{2a} \right)^2}{-j\omega \mu \alpha} = \frac{j4(b+W)(Y_0)_{10}}{\lambda_g \sqrt{n^2 - \left( \frac{4(b+W)}{\lambda_g} \right)^2}} \tag{38}
\]

Therefore
\[
\frac{P^*}{|V|^2} = (Y_0)_{10} \left[ \frac{a}{2(b+W)} + j \frac{16a}{\lambda_g} \sum_{n=1}^{\infty} \frac{\sin^2 \left( \frac{n\pi b}{2(b+W)} \cos \left( \frac{n\pi}{2} \right) \right)}{\left( \frac{n\pi b}{b+W} \right)^2 \sqrt{n^2 - \left( \frac{4(b+W)}{\lambda_g} \right)^2}} \right] \tag{39}
\]
where $V = 2b$. The imaginary part of this is the aperture susceptance $B_a$.

$$B_a = \frac{16a}{\lambda_g Z_0} \sum_{n=1}^{\infty} \frac{\sin^2 \left( \frac{n \pi b}{2(b + W)} \right) \cos^2 \left( \frac{n \pi}{2} \right)}{n \pi b \left( b + W \right)^2 \sqrt{n^2 - \left( \frac{4(b + W)}{\lambda_g} \right)^2}}$$  \hspace{1cm} (40)

### 3.2. Evaluation of Circuit Parameters of Equivalent Circuit for Step Discontinuities at Both Broad and Narrow Dimension

Consider the waveguide junction as shown in Figure 4. Let junction is at $z = 0$. Waveguide of dimension $2a \times 2b$ extends upto $z = -\infty$ and waveguide of dimension $2(a + W_1) \times 2(b + W)$ extends upto $z = +\infty$. The dimensions are such that only dominant mode can propagate in each section. Let there be an incident wave from $-z$ direction. Following similar procedure as described earlier it can be shown:

$$n^2 = \frac{16(a + W_1)^3 ab}{\pi^2 (b + W) W_1^2 (2a + W_1)^2} \cos^2 \frac{\pi a}{2(a + W_1)}$$  \hspace{1cm} (41)

$$B = \frac{2B_0 b}{a}$$  \hspace{1cm} (42)

$$B_a = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{4a^2 \lambda^2 \left[ k^2 - \left( \frac{m \pi}{2(a + W_1)} \right)^2 \right] (b + W)}{n^2 \pi^6 b^2 \eta (a + W_1)} \left[ \frac{1}{1 - \left( \frac{ma}{a + W_1} \right)^2} \right]^2$$
Figure 5. Cross section of waveguide junction for step discontinuity at narrow dimension.

\[
\frac{\sin^2 \left( \frac{m\pi}{2} \right) \cos^2 \left( \frac{n\pi}{2} \right) \cos^2 \left( \frac{m\pi a}{2(a + W_1)} \right) \sin^2 \left( \frac{n\pi b}{2(b + W)} \right)}{\sqrt{\left( \frac{m\lambda}{2(a + W_1)} \right)^2 + \left( \frac{n\lambda}{2(b + W)} \right)^2} - 1}
\]

(43)

4. REFLECTION COEFFICIENT FOR STEP DISCONTINUITIES BY METHOD OF MOMENT

Method of moment can be used for the analysis of waveguide discontinuities. Evaluation of reflection coefficient using moment method for step discontinuities (1) in narrow dimension only and (2) in both narrow & broad dimension are as follows:

4.1. Reflection Coefficient for Step Discontinuities in Narrow Dimension by Method of Moment

Consider the waveguide junction as shown in Figure 5. Let junction is at \( z = 0 \). Waveguide of dimension \( 2a \times 2b_1 \) extends up to \( z = -\infty \) and waveguide of dimension \( 2a \times 2b_2 \) extends upto \( z = +\infty \). It is assumed that \( b_2 > b_1 \). The dimensions are such that only dominant mode can propagate in each section. Let there be an incident wave from \(-z\) direction. In the present analysis, the following assumptions are made:

- the \( x \) component of the electric field at the plane of the aperture is ignored;
- only the \( x \) component of the magnetic field at the aperture plane is considered;
- the excitation in the feed waveguide is TE\(_{10}\) dominant wave with the incident electric field \( y \)-directed and uniform in the same direction, the assumption of uniform \( y \)-directed electric
field having no component in the \( x \)-direction follows as a natural consequence;

- the electric field \( E_y \) in the aperture is assumed to vary only in the \( x \)-direction and is constant in the \( y \)-direction.

At the region of window the tangential component of the magnetic field on both sides should be identical. In this analysis two sources producing these fields — the source in the waveguide exciting the TE\(_{10}\) mode and the magnetic current source at the discontinuity are considered. Using the principle of superposition, the \( x \) components of the magnetic field at the plane of discontinuity are derived.

When aperture is shortened (no magnetic current source) the \( x \) component of magnetic fields are denoted by \( H_{x}^{I/1} \) and \( H_{x}^{II/1} \) on guides with narrow dimensions \( 2b_1 \& 2b_2 \) respectively.

When the generator is shortened the \( x \) component of magnetic fields are denoted by \( H_{x}^{I/2} \) and \( H_{x}^{II/2} \) on guides with narrow dimensions \( 2b_1 \& 2b_2 \) respectively.

At the plane of discontinuity using principle of superposition it is possible to write

\[
H_{x}^{I/1} + H_{x}^{I/2} = H_{x}^{II/1} + H_{x}^{II/2}
\]  
(44)

Electric field at plane of discontinuity is considered as

\[
\vec{E}_{dis} = \hat{u}_y \sum_{p=1}^{M} E'_p e'_p, \quad p = 1, 2, \ldots, M.
\]  
(45)

Where basis function \( e'_p \) \( (p = 1, 2, \ldots, M) \) are defined by

\[
e'_p = \begin{cases} 
\sin \frac{p\pi(x + a)}{2a}, & -a < x < a \\
0, & \text{otherwise}
\end{cases}
\]  
(46)

As in [6] internally scattered electric field is given by

\[
\vec{E}_{\text{scat}} = \sum_n \sum_{p=1}^M V_{e} e'_p e_{mn} + V_{m} e'_p e_{mn}
\]  
(47)

Magnetic fields are related to the electric fields as follows:

\[
\begin{align*}
H_x^e &= Y_{mn1}^e E_y^e \\
H_x^m &= Y_{mn1}^m E_y^m
\end{align*}
\]  
\text{for wave propagating in } -z \text{ direction.}  
(48)

\[
\begin{align*}
H_x^e &= -Y_{mn2}^e E_y^e \\
H_x^m &= -Y_{mn2}^m E_y^m
\end{align*}
\]  
\text{for wave propagating in } +z \text{ direction.}  
(49)
Where \( Y_{mn} \) are the characteristic admittances of TE \(_{mn}\) \& TM \(_{mn}\) modes for the waveguide of \(-z\) direction and \( Y'_{mn} \) \& \( Y''_{mn} \) are the characteristic admittances of TE \(_{mn}\) \& TM \(_{mn}\) modes for the waveguide of \(+z\) direction.

For waveguide of \(-z\) direction modal vectors are given by

\[
\vec{e}_{mn}^e = \frac{1}{\pi} \sqrt{\frac{ab_1 \epsilon_m \epsilon_n}{(mb_1)^2 + (na)^2}} \left[ u_x \left( \frac{n\pi}{2b_1} \right) \cos \left( \frac{m\pi}{2a} (x + a) \right) \right] \\
\times \sin \left( \frac{n\pi}{2b_1} (y + b_1) \right) - \hat{u}_y \left( \frac{m\pi}{2a} \right) \sin \left( \frac{m\pi}{2a} (x + a) \right) \\
\times \cos \left( \frac{n\pi}{2b_1} (y + b_1) \right)
\]

(50)

\[
\vec{e}_{mn}^m = -\frac{2}{\pi} \sqrt{\frac{ab_1}{(mb_1)^2 + (na)^2}} \left[ u_x \left( \frac{m\pi}{2a} \right) \cos \left( \frac{m\pi}{2a} (x + a) \right) \right] \\
\times \sin \left( \frac{n\pi}{2b_1} (y + b_1) \right) + \hat{u}_y \left( \frac{n\pi}{2b_1} \right) \sin \left( \frac{m\pi}{2a} (x + a) \right) \\
\times \cos \left( \frac{n\pi}{2b_1} (y + b_1) \right)
\]

(51)

Similarly for waveguide of \(+z\) direction modal vectors are given by

\[
\vec{e}_{mn}^e = \frac{1}{\pi} \sqrt{\frac{ab_2 \epsilon_m \epsilon_n}{(mb_2)^2 + (na)^2}} \left[ u_x \left( \frac{n\pi}{2b_2} \right) \cos \left( \frac{m\pi}{2a} (x + a) \right) \right] \\
\times \sin \left( \frac{n\pi}{2b_2} (y + b_2) \right) - \hat{u}_y \left( \frac{m\pi}{2a} \right) \sin \left( \frac{m\pi}{2a} (x + a) \right) \\
\times \cos \left( \frac{n\pi}{2b_2} (y + b_2) \right)
\]

(52)

\[
\vec{e}_{mn}^m = -\frac{2}{\pi} \sqrt{\frac{ab_2}{(mb_2)^2 + (na)^2}} \left[ u_x \left( \frac{m\pi}{2a} \right) \cos \left( \frac{m\pi}{2a} (x + a) \right) \right] \\
\times \sin \left( \frac{n\pi}{2b_2} (y + b_2) \right) + \hat{u}_y \left( \frac{n\pi}{2b_2} \right) \sin \left( \frac{m\pi}{2a} (x + a) \right) \\
\times \cos \left( \frac{n\pi}{2b_2} (y + b_2) \right)
\]

(53)

where \( \epsilon_m, \epsilon_n \) are the Neumann’s number satisfying

\[
\epsilon_i = \begin{cases} 
1 & \text{for } i = 0 \\
2 & \text{otherwise} 
\end{cases}
\]

(54)
Following [6] modal voltages of waveguide of \(-z\) direction are given by

\[
V_{mn}^e = \int_{\text{aperture}} \bar{E}_{\text{dis}} \cdot \bar{e}_{mn}^e \, ds
\]

\[
= \frac{mb_1 \sin(n\pi)}{n\pi} \sqrt{\frac{ab_1 \epsilon_m \epsilon_n}{(mb_1)^2 + (na)^2}} \sum_{p=1}^{M} E_p' \left[ \frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right]
\]

\[
(55)
\]

\[
V_{mn}^m = \int_{\text{aperture}} \bar{E}_{\text{dis}} \cdot \bar{e}_{mn}^m \, ds
\]

\[
= \frac{2a \sin(n\pi)}{\pi} \sqrt{\frac{ab_1}{(mb_1)^2 + (na)^2}} \sum_{p=1}^{M} E_p' \left[ \frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right]
\]

\[
(56)
\]

Similarly modal voltages of waveguide of \(+z\) direction are given by

\[
V_{mn}^e = \frac{mb_2}{n\pi} \sqrt{\frac{ab_2 \epsilon_m \epsilon_n}{(mb_2)^2 + (na)^2}} \sum_{p=1}^{M} E_p' \left[ \frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right]
\]

\[
\times \left[ \sin \left( \frac{n\pi (b_2 + b_1)}{2b_2} \right) - \sin \left( \frac{n\pi (b_2 - b_1)}{2b_2} \right) \right]
\]

\[
(57)
\]

\[
V_{mn}^m = \frac{2a}{\pi} \sqrt{\frac{ab_2}{(mb_2)^2 + (na)^2}} \sum_{p=1}^{M} E_p' \left[ \frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right]
\]

\[
\times \left[ \sin \left( \frac{n\pi (b_2 + b_1)}{2b_2} \right) - \sin \left( \frac{n\pi (b_2 - b_1)}{2b_2} \right) \right]
\]

\[
(58)
\]

Now

\[
H_{x}^{1/1} = 2H_{x}^{\text{inc}} = -2Y_0 \cos \frac{\pi x}{2a}
\]

\[
(59)
\]

Where \(H_{x}^{\text{inc}}\) is the \(x\) component of magnetic field due to incident TE\(_{10}\) mode and \(Y_0\) is the characteristic admittance of the line due to TE\(_{10}\) mode.

Now

\[
H_{x}^{1/1} = 0
\]

\[
(60)
\]

\[
H_{x}^{1/2} = -\sum_{m} \sum_{n} \sum_{p=1}^{M} E_p' \left[ \sin c(p+m)\pi - \sin c(p-m)\pi \right] \left[ \frac{ab_1}{(mb_1)^2 + (na)^2} \right]
\]

\[
\times \left[ \sin c(n\pi) \left( \frac{m^2 b_1 \epsilon_m \epsilon_n V_{mn1}^e}{2a} + \frac{2n^2 a V_{mn1} m}{b_1} \right) \right]
\]
\[ H_{x}^{1/2} = \sum_{m} \sum_{n} \sum_{p=1}^{M} E_p' \left[ \sin c(p+m)\pi - \sin c(p-m)\pi \right] \left[ \frac{ab_2}{(mb_2)^2 + (na)^2} \right] \times \left( \frac{1}{n\pi} \right) \left[ \sin \frac{n\pi (b_2 + b_1)}{2b_2} - \sin \frac{n\pi (b_2 - b_1)}{2b_2} \right] \times \sin \left\{ \frac{m\pi}{2a} (x + a) \right\} \cos \left\{ \frac{n\pi}{2b_2} (y + b_2) \right\} \] (61)

Therefore continuity equation will be

\[ H_{x}^{1/1} + H_{x}^{1/2} = H_{x}^{11/2} \] (63)

The weighting function \( W_q \) is selected to be the same form as the basis function.

\[ W_q = \begin{cases} \sin \frac{q\pi(x + a)}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases} \] (64)

where \( q = 1, 2, \ldots, M \).

Inner multiplication is defined as

\[ \langle H, W_q \rangle = \iint_{\text{Aperture at discontinuity}} H \cdot W_q \, dxdy \] (65)

So

\[ \langle H_{x}^{1/1}, W_q \rangle + \langle H_{x}^{1/2}, W_q \rangle = \langle H_{x}^{11/2}, W_q \rangle \] (66)

or

\[ L_1 + L_2 = L_3 \] (67)

Therefore

\[ L_1 = \frac{4Y_0ab_1}{\pi} \left[ \frac{1}{q+1} \cos \frac{\pi}{2} (2q+1) + \frac{1}{q-1} \cos \frac{\pi}{2} (2q-1) \right] \] (68)

\[ L_2 = 2 \sum_{p=1}^{M} E_p' \left[ \sum_{m} \sum_{n} \left[ \frac{m^2b_2\epsilon_m\epsilon_nY_m^{e}Y_{mn1}}{2a} + \frac{2n^2aY_m^{m}b_1}{b_1} \right] \right] \]

\[ \times \left[ \frac{a^2b_1^2}{(mb_1)^2 + (na)^2} \right] (\sin(n\pi))^2 [\sin c(p + m)\pi - \sin c(p - m)\pi] \]
\[ L_3 = -2 \sum_{p=1}^{M} E'_p \left[ \sum_m \sum_n \left[ \frac{m^2 b_2 \epsilon_m \epsilon_n Y_{mn}^m}{2a} + \frac{2n^2 a Y_{mn}^m}{b_2} \right] \right] \]
\[ \times \left[ \frac{a^2 b_2^2}{(mb_2)^2 + (na)^2} \right] \left( \frac{1}{n\pi} \right)^2 \left[ \sin \frac{n\pi(b_2+b_1)}{2b_2} - \sin \frac{n\pi(b_2-b_1)}{2b_2} \right]^2 \]
\[ \times \left[ \sin c(p+m)\pi - \sin c(p-m)\pi \right] [\sin c(q+m)\pi - \sin c(q-m)\pi] \]
\[
\times \left[ \sin c(q + m)\pi - \sin c(q - m)\pi \right] - \sin c(q - m)\pi \right] \]
\[
\times \left[ \sin c(p + m)\pi - \sin c(p - m)\pi \right] [\sin c(q + m)\pi - \sin c(q - m)\pi] \]
\[
(70)
\]

To solve the unknown coefficients \( E'_p (p = 1, 2, \ldots, M) \), Equation (67) can be written in the matrix form for all \( p \) and \( q \) as below:
\[
[A_1] + [A_2] E'_p = [A_3] E'_p \]
\[
(71)
\]
or,
\[
\begin{align*}
E'_p &= \{[A_3] - [A_2]\}^{-1} [A_1] \\
E_i &= \text{Incident electric field} = \cos \frac{\pi x}{2a} \\
E_R &= \text{Reflected electric field} \\
&= -\cos \frac{\pi x}{2a} - \sum_{p=1}^{M} E'_p \left[ \sin c(p+1)\pi - \sin c(p-1)\pi \right] \sin \frac{\pi(x+a)}{2a}
\end{align*}
\]
\[
(72)
\]

Reflection coefficient is given by
\[
\Gamma_1 = -1 - \sum_{p=1}^{M} E'_p \left[ \sin c(p+1)\pi - \sin c(p-1)\pi \right]
\]
\[
(75)
\]

Similarly if there be an incident wave from \( +z \) direction then
\[
E_i = \text{Incident electric field} = \cos \frac{\pi x}{2a}
\]
\[
E_R = \text{Reflected electric field} \\
= -\cos \frac{\pi x}{2a} - \frac{b_1}{b_2} \sum_{p=1}^{M} E'_p \left[ \sin c(p+1)\pi - \sin c(p-1)\pi \right] \sin \frac{\pi(x+a)}{2a}
\]
\[
(77)
\]
Figure 6. Cross section of waveguide junction for step discontinuity at both broad and narrow dimension.

Reflection coefficient is given by

\[ \Gamma_2 = -1 - \frac{b_1}{b_2} \sum_{p=1}^{M} E'_p \left[ \sin c(p+1)\pi - \sin c(p-1)\pi \right] \]  

4.2. Reflection Coefficient for Step Discontinuities at Both Broad and Narrow Dimension by Method of Moment

Consider the waveguide junction as shown in Figure 6. Let junction is at \( z = 0 \). Waveguide of dimension \( 2a_1 \times 2b_1 \) extends up to \( z = -\infty \) and waveguide of dimension \( 2a_2 \times 2b_2 \) extends up to \( z = +\infty \). It is considered that \( a_1 > a_2 \) and \( b_2 > b_1 \). The dimensions are such that only dominant mode can propagate in each section. Following similar procedure as described in previous section it can be shown:

If there be an incident wave from \(-z\) direction then reflection coefficient:

\[ \Gamma_1 = -1 - \frac{a_2}{a_1} \sum_{p=1}^{M} E'_p \left[ \cos \frac{\pi(p+1)}{2} \sin c \frac{\pi(pa_1 + a_2)}{2a_1} 
- \cos \frac{\pi(p-1)}{2} \sin c \frac{\pi(pa_1 - a_2)}{2a_1} \right] \]  

If there be an incident wave from \(+z\) direction then reflection coefficient:

\[ \Gamma_2 = -1 - \frac{b_1}{b_2} \sum_{p=1}^{M} E'_p \left[ \sin c(p+1)\pi - \sin c(p-1)\pi \right] \]
5. RESULTS

From final transmission matrix of taper waveguide reflection coefficient and the VSWR are calculated. For tapering in narrow dimension considering \(2a_0 = 5.81 \text{ cm}, 2b_0 = 1.20 \text{ cm}, 2a_1 = 5.81 \text{ cm}, 2b_1 = 2.91 \text{ cm}, f = 3.96 \text{ GHz}, \text{ and } L = 4 \text{ cm}, \) VSWR is plotted against number of sections \(N\) in Figure 7 using equivalent circuit representation of step discontinuities. For large number of sections VSWR is converged. In convergence region the plot of VSWR with taper length is shown in Figure 8 along with previously published results.

Calculating reflection coefficient of step discontinuities by moment method, for tapering in narrow dimension with same dimensional parameter as above, the plot of VSWR against length is shown in Figure 9 along with previously published results. Similarly for tapering in both dimension with \(2a_0 = 0.90 \text{ inch}, 2b_0 = 0.40 \text{ inch}, 2a_1 = 0.75 \text{ inch}, 2b_1 = 0.60 \text{ inch}, \) \(f = 8.7 \text{ GHz} \text{ and } L = 2.85 \text{ inch}, \) VSWR is plotted against number of sections \(N\) in Figure 10 using equivalent circuit representation of step discontinuities. In convergence region the plot of VSWR with frequency is shown in Figure 11 along with previously published results.

Calculating reflection coefficient of step discontinuities by moment method, for double taper section with same dimensional parameter as above, the plot of VSWR against frequency is shown in Figure 12 along with previously published results.

Figure 10. VSWR versus number of sections for double taper section.


6. CONCLUSION

The results obtained in the presented methods are to be in good agreement with results of [1, 2] and [3]. Here it is assumed that only TE_{10} mode is propagating along the tapered line. Even if other modes are generated they are not supported by the structure.

REFERENCES