

**ADVANCEMENT OF ALGEBRAIC FUNCTION
APPROXIMATION IN EIGENVALUE PROBLEMS OF
LOSSLESS METALLIC WAVEGUIDES TO INFINITE
DIMENSIONS, PART II: TRANSFER OF RESULTS IN
FINITE DIMENSIONS TO INFINITE DIMENSIONS**

N. Yener

Technical Education Faculty
Kocaeli University
Umuttepe Campus 41380 Izmit, Kocaeli, Turkey

Abstract—In this phase of the attempt to advance finite dimensional algebraic function approximation technique in eigenvalue problems of lossless metallic guides filled with anisotropic and/or inhomogeneous media, to exact analysis in infinite dimensions, it is seen that the problem in infinite dimensions, can be reduced to finite dimensions, by virtue of a result in perturbation theory. Furthermore, it is found that analysis results of algebraic function approximation, can be adapted to infinite dimensions too, at worst by introduction of some additional arguments.

1. INTRODUCTION

Having completed examination of properties of the operator T in Part I, in Section 2 of Part II we shall discuss reduction of the infinite dimensional problem to finite dimensions. In Section 3 we shall transfer the results on eigenvector inner products in finite dimensions obtained in [1–4], to infinite dimensions. We need to consider arguments of Section 3, in generalizing, in Section 4, results of algebraic function approximation in eigenvalue problems of lossless closed guides to exact solutions obtained in infinite dimensions.

In Sections 2 and 3, $T(p)$ will be assumed to be bounded and holomorphic. Only in one part of Section 4, existence of poles in $\hat{Z}(p)\hat{Y}(p)$ will be considered while adapting Appendix B of [2], to infinite dimensions.

If it were possible we would like to set up an approach to treat the infinite dimensional physical problem in a finite dimensional setting.

This would not only simplify the mathematics involved, but might allow us to inherit the results already obtained by algebraic function approximation for the eigenvalue problem of the original guiding structure in [1, 2].

In the sequel, we shall find that as a nicety such a scheme is possible, by virtue of a standard result in perturbation theory that the study of a finite system of eigenvalues of an infinite dimensional operator can be reduced to a problem in a finite dimensional space [5, p. 181].

2. REDUCTION OF THE INFINITE DIMENSIONAL PROBLEM TO FINITE DIMENSIONS

In this section, by Facts 1 through 9, we shall arrive at the result that a finite part of the discrete spectrum of $T(p)$ can be contained in a finite radius circle, centered at origin of the eigenvalue plane.

Fact 1. *In Sections 2 and 3 of Part II, we shall exclusively assume matrix elements of $\hat{Z}(p)\hat{Y}(p)$ are holomorphic, evaluated away from poles $\hat{Z}(p)\hat{Y}(p)$ might have, due to poles that constituent parameter matrices $\underline{\varepsilon}$ and $\underline{\mu}$ of the guide filling media can have.*

By Fact 11 of Part I, this assumption makes T in l^2 bounded in Sections 2 and 3, since none of its eigenvalues can have a pole making the eigenvalue infinite. Furthermore T is holomorphic in Sections 2 and 3. Holomorphicity argument for T follows the same lines as for T^{-1} given in the proof of Fact 4 of Part I.

Fact 2. *By Fact 11 of Part I, no eigenvalue of T can have a pole if one also considers holomorphicity of $\hat{Z}(p)\hat{Y}(p)$ given by Fact 1.*

Fact 3. *Therefore T in l^2 is bounded with only finite eigenvalues in Sections 2 and 3. This implies its spectrum which we will denote by $\Sigma(T)$ henceforth, is nonempty [5, p. 176].*

Fact 4. *By Facts 13, 14 of Part I and Facts 1 and 2 of Part II, the spectrum of T we shall take up in Sections 2 and 3 is discrete, containing only isolated points of finite eigenvalues.*

Fact 5. *Further to Fact 4, we shall assume that the algebraic multiplicity of each member of the spectrum of T is finite.*

Facts 4 and 5 permit us to consider hereafter the set of eigenvectors of T as complete whenever algebraic multiplicities of the eigenvalues are

the same as their geometric multiplicities. Otherwise the generalized eigenvectors are added to the set to yield a complete set.

Fact 6. *Non-emptiness of the spectrum of T due to Fact 3 means, we can always choose a finite radius circle in the complex eigenvalue plane, centered at the origin and in which there exists at least one eigenvalue of T . Call this radius r_0 .*

Fact 7. *(Bolzano-Weierstrass theorem). Every bounded infinite (i.e., consisting of an infinite number of points) point set has at least one limit point [6, p. 7].*

Fact 8. *An operator with discrete spectrum has no limit points in its spectrum. Therefore considering Fact 4, we conclude in the circle of radius r_0 of Fact 6, there are no limit points for the members of the spectrum of our operator T .*

Fact 9. *Fact 8 coupled with Fact 7, implies T has only a finite system of eigenvalues in the circle of radius r_0 . By Fact 6, we repeat, r_0 can be chosen so that the circle contains at least one eigenvalue.*

We shall focus on the finite system of eigenvalues with finite algebraic multiplicities of the infinite dimensional operator T , and which system is prescribed by Facts 5 and 9. Because, as we shall see below, investigation of properties of this subset of the spectrum of T which we shall denote by $\sum(T)$ can be made considering it simply as a set of eigenvalues of a finite dimensional operator.

We now proceed to construct the finite dimensional Jordan canonical form matrix $J_{1\text{ fin}}$ for this finite system of eigenvalues inside circle of radius r_0 . This will involve determination of eigenprojection and eigennilpotent operators discussed in [5, pp. 180–181]. This matrix will bear information on the finite system of eigenvalues of T , contained in the circle of radius r_0 . Furthermore, its characteristic equation will be an algebraic equation, an equation of finite degree. This is the essential feature that resolves the problem of bridging between finite and infinite dimensional formulations. For, we shall now have exact eigenvalues expressed as roots of a finite dimensional polynomial equation. Formulation of the problem as the solution of an algebraic equation was in fact the basis of the algebraic function approximation in [1, 2]. But now in this way we have the advantage of containing the exact eigenvalues in an algebraic equation, for which there exists a well developed theory along with already obtained results of [1, 2] for the assumed type of lossless closed guiding systems prescribed in Part I.

Fact 10. *$J_{1\text{ fin}}$ is made up of Jordan blocks for each of the eigenvalues in the finite system, all of which have finite algebraic multiplicities [7,*

p. 368].

Suppose the sum of the algebraic multiplicities of the eigenvalues in the system is m .

The characteristic equation for $J_{1\text{ fin}}$ will then yield an m th degree polynomial equation in $\hat{\gamma}^2(p)$ and p , which we will represent as,

$$G_m(\hat{\gamma}^2, p) = 0. \quad (1)$$

Since T is holomorphic in p near p_0 , by Fact 1, the finite system of eigenvalues $\hat{\gamma}^2(p)$ of $T(p)$, consists of branches of one or several analytic functions which have at most algebraic singularities near p_0 , by Fact 5 of Part I.

Fact 11. *On the other hand by definition functions with only algebraic singularities are algebroidal functions. Also because by another definition algebroidal functions are the roots of an algebraic equation, we conclude (1) is an algebraic equation in $\hat{\gamma}^2(p)$ and p .*

An additional remark is that $J_{1\text{ fin}}$ informs us about the geometric multiplicities of the eigenvalues as well as their algebraic multiplicities.

By analogy with $J_{1\text{ fin}}$, obtained for the system of eigenvalues in the circle of radius r_0 from $\hat{Z}(p)\hat{Y}(p)$, we assume $J_{2\text{ fin}}$ is the Jordan form for the same system constructed from $\hat{Y}(p)\hat{Z}(p)$ of the operator \bar{T} . (See Section 2 of Part I, for the definition of \bar{T} .)

Fact 12. *Note the two matrices $\hat{Z}(p)\hat{Y}(p)$ and $\hat{Y}(p)\hat{Z}(p)$ share the same eigenvalues even though their proper and generalized eigenvectors, hence their Jordan forms may be different. Both Jordan forms, $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$, have the same characteristic equation (1).*

Next we find the finite dimensional proper and generalized eigenvectors of $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$. Denote them by v_{fi} and i_{fi} . Here the subscript i indicates which one of the vectors (or modes) is considered.

In order to bridge to infinite dimensions from $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$, observe that the corresponding Jordan forms $J_{1\text{ inf}}$ and $J_{2\text{ inf}}$ of the infinite dimensional problem are obtainable as either $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$ is augmented by additional Jordan blocks until all the infinite system of discrete eigenvalues with finite algebraic multiplicities are included. (Recall that by Fact 2, T has only finite eigenvalues. Actually the same applies to \bar{T} also, since T and \bar{T} share the same eigenvalues.)

The proper and generalized eigenvectors of the infinite system represented by $J_{1\text{ inf}}$ and $J_{2\text{ inf}}$ are derived from those of Jordan blocks for an eigenvalue and inserting zeroes in those rows which complement the blocks to $J_{1\text{ inf}}$ and $J_{2\text{ inf}}$. Call the resulting eigenvector \hat{v}_{Ji} or \hat{i}_{Ji}

according to whether it belongs to $J_{1\text{inf}}$ or $J_{2\text{inf}}$. Here subscript i identifies the particular vector in either set.

Since now we have matrix definition $J_{1\text{inf}}$ of $T(p)$, with respect to the basis consisting of proper and generalized eigenvectors of $T(p)$ that can be computed from $\hat{Z}(p)\hat{Y}(p)$, we can determine the basis with respect to which $\hat{Z}(p)\hat{Y}(p)$ represents $T(p)$, using change of basis matrix which also relates $\hat{Z}(p)\hat{Y}(p)$ and $J_{1\text{inf}}$. A similar derivation holds for $\hat{Y}(p)\hat{Z}(p)$.

Now we have two infinite dimensional bases in X and Y which we shall denote by $\{\hat{v}_{J_i}\}$ and $\{\hat{i}_{J_i}\}$ in respective order. Note that the sets of generalized and proper eigenvectors of $\hat{Z}(p)\hat{Y}(p)$ and $\hat{Y}(p)\hat{Z}(p)$, which we denoted by $\{\hat{v}_i\}$ and $\{\hat{i}_i\}$ respectively in Section 2 of Part I, are also bases in X and Y, to which $\{\hat{v}_{J_i}\}$ and $\{\hat{i}_{J_i}\}$ are alternatives.

Fact 13. Assume $\sum(T) = \sigma_1 \cup \sigma_2$ where σ_1 (which is bounded) and σ_2 are two sets such that a simply closed piecewise smooth positively oriented path denoted by Γ , belonging to the resolvent set of T , separates σ_1 from σ_2 . If,

$$P = \frac{1}{2\pi j} \int_{\Gamma} (\lambda - T)^{-1} d\lambda \quad (2)$$

then the decomposition $X = M_1 \oplus M_2$ where $M_1 = PX$ and $M_2 = (1 - P)X$, yields a decomposition of T into parts T_{M_1} and T_{M_2} where $\sum(T_{M_1}) = \sigma_1$ and $\sum(T_{M_2}) = \sigma_2$. Moreover T_{M_1} is bounded, but T_{M_2} can be unbounded. Here the projection P is equal to the sum of the eigenprojections for all the eigenvalues of T , lying inside Γ [7, p. 361].

In our problem Γ may be considered as the circle of radius r_0 of Fact 6. $J_{1\text{fin}}$ will be the $m \times m$ matrix definition of part of T in M_1 , namely T_{M_1} , with respect to the finite dimensional basis consisting of proper and generalized eigenvectors of T_{M_1} .

$J_{1\text{fin}}$ will have the mate $Z(p)Y(p)$ where $Z(p)$ and $Y(p)$ are truncated matrices obtained from $\hat{Z}(p)$ and $\hat{Y}(p)$ of (1) of Part I, and were used in algebraic function approximation of the exact eigenvalues of the physical problem in [1, 2].

Similarly $J_{2\text{fin}}$ will mate with $Y(p)Z(p)$ of again [1, 2].

Hence we have established a correspondence between a finite system of exact eigenvalues and their finite dimensional eigenvectors corresponding to $J_{1\text{fin}}$ (or $J_{2\text{fin}}$) and a finite system of approximating eigenvalues and their finite dimensional eigenvectors corresponding to $Z(p)Y(p)$ (or $Y(p)Z(p)$).

Since as noted above for both $J_{1\text{fin}}$ and $Z(p)Y(p)$, the characteristic equations are algebraic, without hesitation we can carry

over the results obtained in the investigation of the eigenvalues of $Z(p)Y(p)$ that were reported in [1,2], to the case of matrix $J_{1\text{fin}}$. However it is to be noted that the roots of characteristic equation of $J_{1\text{fin}}$, namely (1), are now exact eigenvalues of the infinite system. Therefore all results thus carried over from approximate analysis of [1, 2] to infinite dimensions, will bear the significant property that they will now be stated for exact eigenvalues.

This applies to all results presented in [1,2], and that were obtained directly from the characteristic equation of $Z(p)Y(p)$.

In this way we shall be able to generalize results of the finite dimensional ‘approximate’ analysis on eigenvalues, as results of the infinite dimensional ‘exact’ analysis.

Section 4 outlines how results of [1–4], apply in the infinite dimensional ‘exact’ case.

3. TRANSFER OF RESULTS ON EIGENVECTOR INNER PRODUCTS IN FINITE DIMENSIONS TO INFINITE DIMENSIONS

We need one more supplement to complete generalization to the infinite dimensional case. This consists in showing how some properties of proper and generalized eigenvectors of $Z(p)Y(p)$, $Y(p)Z(p)$, $J_{1\text{fin}}$, $J_{2\text{fin}}$, $J_{1\text{inf}}$, $J_{2\text{inf}}$, $\hat{Z}(p)\hat{Y}(p)$ and $\hat{Y}(p)\hat{Z}(p)$ relate.

Fact 14. *In this connection, the first feature we observe is the equality of the inner products of the proper and generalized eigenvectors v_{fk} and i_{fl} of $J_{1\text{fin}}$ and $J_{2\text{fin}}$, to the inner product of the corresponding elements \hat{v}_{Jk} and \hat{i}_{Jl} of $\{\hat{v}_{Ji}\}$ and $\{\hat{i}_{Ji}\}$:*

$$v_{fk}^+ i_{fl} = \hat{v}_{Jk}^+ \hat{i}_{Jl}. \quad (3)$$

Here $(\cdot)^+$ shows the adjoint and k and l indicate which eigenvalue, the eigenvectors are associated with. (3) is trivial, since the right side infinite dimensional vector components are obtained by inserting zeroes into those rows which complement the left side finite dimensional vectors to infinite dimensions. For instance, \hat{v}_{Jk} is obtained from v_{fk} , by inserting zeroes in all rows that correspond to Jordan blocks in $J_{1\text{inf}}$ other than those corresponding to eigenvalue $\hat{\gamma}_k^2$.

Next we search a relation similar to (3) between elements of $\{\hat{v}_{Ji}\}$ and $\{\hat{i}_{Ji}\}$ and those of $\{\hat{v}_i\}$ and $\{\hat{i}_i\}$ of Part I. To this end denote the change of basis matrix between bases for matrix representations $\hat{Z}(p)\hat{Y}(p)$ and $J_{1\text{inf}}$ of T , by K . In other words, K relates $\{\hat{v}_i\}$

and $\{\hat{v}_{J_i}\}$ through the relation $\hat{v}_i = K^T \hat{v}_{J_i}$. Here $(\cdot)^T$ denotes the transpose.

Similarly denote the change of basis matrix between the bases for matrix definitions of $\hat{Y}(p)\hat{Z}(p)$ and $J_{2\text{inf}}$ of \overline{T} , by L . In other words, L relates $\{\hat{i}_i\}$ and $\{\hat{i}_{J_i}\}$ through the relation $\hat{i}_i = L^T \hat{i}_{J_i}$.

Restricting ourselves to the $j\omega$ imaginary axis of the p plane and using Foster matrix properties of $\hat{Z}(p)$ and $\hat{Y}(p)$, outlined in the Introduction of Part I, we find $K^{-1}|_{p=j\omega} = L^+|_{p=j\omega}$ when the matrices $\hat{Z}(p)\hat{Y}(p)$ and $\hat{Y}(p)\hat{Z}(p)$ are non-defective.

Fact 15. *Using the relation $K^{-1}|_{p=j\omega} = L^+|_{p=j\omega}$ we find for the proper eigenvectors \hat{v}_k and \hat{i}_l of the k th and l th eigenvalues of $\hat{Z}(j\omega)\hat{Y}(j\omega)$ and $\hat{Y}(j\omega)\hat{Z}(j\omega)$ given by (1) of Part I and those of $J_{1\text{inf}}$ and $J_{2\text{inf}}$ for the same eigenvalues,*

$$\hat{v}_k^+(j\omega)\hat{i}_l(j\omega) = \hat{v}_{J_k}^+(j\omega)\hat{i}_{J_l}(j\omega) \quad (4)$$

where $\hat{Z}(j\omega)\hat{Y}(j\omega)$ and $\hat{Y}(j\omega)\hat{Z}(j\omega)$ are non-defective.

(3) and (4) are relations on the inner products of proper and generalized eigenvectors of operator $T(j\omega)$ in different bases. Based on them we can ascertain some statements that were made during the algebraic function approximation, now in infinite dimensions.

The first such feature we find, regards continuity of the voltage and current eigenvector inner products appearing in (3) and also in (4).

We know that in infinite dimensions, when $T(p)$ is holomorphic, the eigenvalues belonging to a finite system, i.e., those of $J_{1\text{fin}}$ or $J_{2\text{fin}}$, may have only algebraic singularities by Fact 5 of Part I.

Fact 16. *Since poles or pole branch points are ruled out by assumption of a holomorphic T , the only singularities permitted for the eigenvalues are algebraic branch points where the eigenvalue is finite and continuous.*

Proof. This follows from Fact 5 of Part I and the finiteness of the eigenvalues at the singularity.

Because proper and generalized eigenvectors of infinite Jordan forms can be chosen as vectors proportional to the columns of an infinite dimensional identity matrix, and since each such vector can be chosen as continuous then, $\hat{v}_{J_i}^+(j\omega)\hat{i}_{J_i}(j\omega)$ will be continuous at $p = j\omega_0$.

This brings, together with (3), Fact 17.

Fact 17. $v_{f_i}^+(j\omega)\hat{i}_{f_i}(j\omega)$ is continuous at an algebraic branch point where the eigenvalue is finite, when $\hat{Z}(j\omega)\hat{Y}(j\omega)$ and $\hat{Y}(j\omega)\hat{Z}(j\omega)$ matrices are defective or non-defective at the singularity.

This property will be needed in generalizing results of Appendix A1 of [2] to infinite dimensions. This involves identification of an eigenvalue, at an algebraic branch point where it is finite, as a defective multiple eigenvalue.

Fact 18. Another result obtained from the finite dimensional system and we need to transfer to infinite dimensions, is that when a propagation constant (square root of an eigenvalue) is complex, the corresponding proper eigenvectors of $\hat{Z}(j\omega)\hat{Y}(j\omega)$ and $\hat{Y}(j\omega)\hat{Z}(j\omega)$ satisfy,

$$\hat{v}_i^+(j\omega)\hat{i}_i(j\omega) = 0. \quad (5)$$

Proof. Suppose in infinite dimensions $\hat{\gamma}_i^2(j\omega)$ is the non-real eigenvalue which makes $\hat{\gamma}_i(j\omega)$ complex. Furthermore let $\hat{v}_i(j\omega)$ be the eigenvector in eigenvalue equation

$$\hat{Z}(j\omega)\hat{Y}(j\omega)\hat{v}_i(j\omega) = \hat{\gamma}_i^2(j\omega)\hat{v}_i(j\omega). \quad (6)$$

Next pick a sequence of $m \times m$ truncations of $\hat{Z}(j\omega)$ and $\hat{Y}(j\omega)$. Denote by $Z_m(j\omega)$ and $Y_m(j\omega)$ the infinite dimensional matrices that are the same as $\hat{Z}(j\omega)$ and $\hat{Y}(j\omega)$ except for entries in rows and columns that are removed in the truncation process and for which zeroes are substituted. Then consider the sequence of vectors $\{v_i^m\}$ such that v_i^m is the infinite dimensional vector whose first m components are the first m of $\hat{v}_i(j\omega)$, and whose remaining components are zeroes. v_i^m and its dual i_i^m , obtained in like manner from $\hat{i}_i(j\omega)$, do not satisfy the transmission line equations (1) of Part I for $\hat{\gamma}_i^2$ exactly, but rather up to vectors negligible in length as $m \rightarrow \infty$ [8, p.180]. This modified form of the transmission line equations is as follows:

$$\hat{\gamma}_i(j\omega)v_i^m(j\omega) = Z_m(j\omega)i_i^m(j\omega) + \varepsilon_{V_i}^m, \quad (7)$$

$$\hat{\gamma}_i(j\omega)i_i^m(j\omega) = Y_m(j\omega)v_i^m(j\omega) + \varepsilon_{I_i}^m. \quad (8)$$

From this we obtain the modified version of eigenvalue equations as below:

$$\hat{\gamma}_i^2(j\omega)v_i^m(j\omega) = Z_m Y_m(j\omega)v_i^m(j\omega) + \gamma_i(j\omega)\varepsilon_{V_i}^m + Z_m \varepsilon_{I_i}^m, \quad (9a)$$

$$\hat{\gamma}_i^2(j\omega)i_i^m(j\omega) = Y_m Z_m(j\omega)i_i^m(j\omega) + \gamma_i(j\omega)\varepsilon_{I_i}^m + Y_m \varepsilon_{V_i}^m. \quad (9b)$$

Using (9) and Foster matrix properties of Z_m and Y_m (see Part I), we find

$$v_i^{m+}(j\omega)i_i^m(j\omega) = \frac{1}{\hat{\gamma}_i^{2*}(j\omega)} \left[v_i^{m+}(j\omega)Y_m(j\omega)Z_m(j\omega)i_i^m(j\omega) + \hat{\gamma}_i^*(j\omega)\varepsilon_{V_i}^{m+}i_i^m(j\omega) - \varepsilon_{I_i}^{m+}Z_m(j\omega)i_i^m(j\omega) \right], \quad (10)$$

or dropping dependence on $j\omega$,

$$v_i^{m+}i_i^m = \frac{1}{\hat{\gamma}_i^{2*}} \left[\hat{\gamma}_i^2 v_i^{m+}i_i^m - \gamma_i v_i^{m+}\varepsilon_{I_i}^m - v_i^{m+}Y_m\varepsilon_{V_i}^m + \gamma_i^* \varepsilon_{V_i}^{m+}i_i^m - \varepsilon_{I_i}^{m+}Z_m i_i^m \right]. \quad (11)$$

From this, follow:

$$\left(\hat{\gamma}_i^{2*} - \hat{\gamma}_i^2 \right) v_i^{m+}i_i^m = \left[-\hat{\gamma}_i v_i^{m+}\varepsilon_{I_i}^m - v_i^{m+}Y_m\varepsilon_{V_i}^m + \gamma_i^* \varepsilon_{V_i}^{m+}i_i^m - \varepsilon_{I_i}^{m+}Z_m i_i^m \right], \quad (12)$$

and,

$$\left| \hat{\gamma}_i^{2*} - \hat{\gamma}_i^2 \|v_i^{m+}i_i^m\| \right| = \left[-\hat{\gamma}_i v_i^{m+}\varepsilon_{I_i}^m - v_i^{m+}Y_m\varepsilon_{V_i}^m + \gamma_i^* \varepsilon_{V_i}^{m+}i_i^m - \varepsilon_{I_i}^{m+}Z_m i_i^m \right]. \quad (13)$$

We rewrite (13) as:

$$\left| \hat{\gamma}_i^{2*} - \hat{\gamma}_i^2 \|v_i^{m+}i_i^m\| \right| = \left| \hat{\gamma}_i v_i^{m+}\varepsilon_{I_i}^m - \left(\hat{\gamma}_i i_i^{m+} - \varepsilon_{I_i}^{m+} \right) \varepsilon_{V_i}^m - \gamma_i^* \varepsilon_{V_i}^{m+}i_i^m + \varepsilon_{I_i}^m \left(\hat{\gamma}_i v_i^{m+} - \varepsilon_{V_i}^{m+} \right) \right|. \quad (14)$$

We thus find an upper bound for the left side of (13) as

$$\left| \hat{\gamma}_i^{2*} - \hat{\gamma}_i^2 \|v_i^{m+}i_i^m\| \right| \leq |\hat{\gamma}_i| \|v_i^{m+}\| \|\varepsilon_{I_i}^m\| + |\hat{\gamma}_i| \|i_i^{m+}\| \|\varepsilon_{V_i}^m\| + |\hat{\gamma}_i| \|\varepsilon_{V_i}^{m+}\| \|i_i^m\| + |\hat{\gamma}_i| \|\varepsilon_{I_i}^{m+}\| \|v_i^m\|. \quad (15)$$

Since $\|\varepsilon_{V_i}^m\|$ and $\|\varepsilon_{I_i}^m\|$ tend to zero as $m \rightarrow \infty$, and $\|v_i^m\|$ and $\|i_i^m\|$ are finite because \hat{v}_i , \hat{i}_i , are in l^2 , we infer $\lim_{m \rightarrow \infty} v_i^{m+}(j\omega)i_i^m(j\omega) = \hat{v}_i^+(j\omega)\hat{i}_i(j\omega) = 0$, whenever $\hat{\gamma}_i^2(j\omega) \neq \hat{\gamma}_i^{2*}(j\omega)$.

Note that the proof does not cover generalized eigenvectors which do not satisfy an equation system such as (9) that holds for proper eigenvectors.

Fact 19. *Still another result we wish to transfer to infinite dimensions is the orthogonality of eigenvectors of $\hat{Z}(j\omega)\hat{Y}(j\omega)$ and $\hat{Y}(j\omega)\hat{Z}(j\omega)$ for different modes, i.e.,*

$$\hat{v}_k^+(j\omega)\hat{i}_l(j\omega) = 0, \quad (16)$$

when $k \neq l$ provided the eigenvectors of the system are complete.

Proof. Consider the sequence of vectors v_i^m and i_i^m defined in the proof of Fact 18.

Now find an upper bound for the absolute value of the difference $v_k^{m+}(j\omega)i_l^m(j\omega) - \hat{v}_k^+(j\omega)\hat{i}_l(j\omega)$, where \hat{v}_k and \hat{i}_l are the exact infinite dimensional eigenvectors.

$$\begin{aligned} \left| v_k^{m+}i_l^m - \hat{v}_k^+\hat{i}_l \right| &= \left| (v_k^{m+} - \hat{v}_k^+)i_l^m + \hat{v}_k^+i_l^m - \hat{v}_k^+(\hat{i}_l - i_l^m) - \hat{v}_k^+i_l^m \right| \\ &= \left| (v_k^{m+} - \hat{v}_k^+)i_l^m - \hat{v}_k^+(\hat{i}_l - i_l^m) \right| \\ &\leq \left| (v_k^{m+} - \hat{v}_k^+)i_l^m \right| + \left| \hat{v}_k^+(\hat{i}_l - i_l^m) \right| \\ &\leq \|v_k^{m+} - \hat{v}_k^+\| \|i_l^m\| + \|\hat{v}_k^+\| \|\hat{i}_l - i_l^m\|. \end{aligned} \quad (17)$$

Since v_k^m approaches \hat{v}_k and i_l^m approaches \hat{i}_l as $m \rightarrow \infty$, we conclude the rightmost side of the inequality approaches zero as $m \rightarrow \infty$. Hence, $v_k^{m+}(j\omega)i_l^m(j\omega) \rightarrow \hat{v}_k^+(j\omega)\hat{i}_l(j\omega)$ as $m \rightarrow \infty$.

Now utilizing equations (9), we obtain

$$v_k^{m+}i_l^m = \frac{1}{\hat{\gamma}_k^{2*}} \left[v_k^{m+}Y_mZ_m + \gamma_k^*\varepsilon_{V_k}^{m+} - \varepsilon_{I_k}^{m+}Z_m \right] i_l^m, \quad (18)$$

$$\begin{aligned} v_k^{m+}i_l^m &= \frac{1}{\hat{\gamma}_k^{2*}} \left[\hat{\gamma}_l^2 v_k^{m+}i_l^m - \hat{\gamma}_l v_k^{m+}\varepsilon_{I_l}^m + \gamma_k^*\varepsilon_{V_k}^{m+}i_l^m \right. \\ &\quad \left. - \varepsilon_{I_k}^{m+}(\gamma_l v_l^m - \varepsilon_{V_l}^m) - (\gamma_k^*i_k^{m+} - \varepsilon_{I_k}^{m+})\varepsilon_{V_l}^m \right], \end{aligned} \quad (19)$$

$$\begin{aligned} (\hat{\gamma}_k^{2*} - \hat{\gamma}_l^2) v_k^{m+}i_l^m &= -\hat{\gamma}_l v_k^{m+}\varepsilon_{I_l}^m + \gamma_k^*\varepsilon_{V_k}^{m+}i_l^m \\ &\quad - \varepsilon_{I_k}^{m+}(\gamma_l v_l^m - \varepsilon_{V_l}^m) - (\Gamma_k^*i_k^{m+} - \varepsilon_{I_k}^{m+})\varepsilon_{V_l}^m. \end{aligned} \quad (20)$$

Because the right side of (20) tends to zero for reasons similar to those in proof of Fact 18, we get

$$\lim_{m \rightarrow \infty} \left[\hat{\gamma}_k^{2*}(j\omega) - \hat{\gamma}_l^2(j\omega) \right] v_k^{m+}(j\omega)i_l^m(j\omega) = 0. \quad (21)$$

Hence if $\hat{\gamma}_k^{2*}(j\omega) \neq \hat{\gamma}_l^2(j\omega)$,

$$\lim_{m \rightarrow \infty} v_k^{m+}(j\omega)i_l^m(j\omega) = \hat{v}_k^+(j\omega)\hat{i}_l(j\omega) = 0, \quad (22)$$

which is the basic orthogonality relation in infinite dimensions when $\hat{\gamma}_k^{2*}(j\omega) \neq \hat{\gamma}_l^2(j\omega)$.

Suppose $\hat{\gamma}_k^{2*}(j\omega) = \hat{\gamma}_l^2(j\omega)$ with $k \neq l$. This implies $\hat{\gamma}_k^2(j\omega) = \hat{\gamma}_l^{2*}(j\omega)$. Then at this point we have a degenerate (multiple) eigenvalue $\hat{\gamma}_k^2(j\omega)$. We further suppose the multiple eigenvalue is non-defective, i.e., the operator T possesses a complete set of eigenvectors at the singularity. Orthogonality of $\hat{v}_{k\alpha}(j\omega)$ and $\hat{v}_{k\beta}(j\omega) = \hat{\gamma}_k(j\omega)\hat{Z}^{-1}(j\omega)\hat{v}_{k\beta}(j\omega)$, when $\hat{v}_{k\alpha}(j\omega)$ and $\hat{v}_{k\beta}(j\omega)$ are linearly independent eigenvectors associated with $\hat{\gamma}_k^2(j\omega)$, can be argued by referring to [9, p. 232], where a result of [10, p. 827] has been freely cited in support of a more general orthogonality statement, i.e., one in infinite dimensions even though the result is given for a finite ‘polarization space’ in [10]. Following the same logic, without going into a rigorous proof, we shall carry over the remark in [10, p. 65], concerning orthogonality of eigenvectors of a non-defective multiple eigenvalue in finite dimensions, into infinite dimensions. This will give us the requested result.

Hence when $\hat{\gamma}_k^{2*}(j\omega) = \hat{\gamma}_l^2(j\omega)$, but $\hat{v}_{k\alpha}(j\omega)$ and $\hat{v}_{k\beta}(j\omega)$ are linearly independent,

$$\hat{\gamma}_k(j\omega)\hat{v}_{k\alpha}^+(j\omega)Z^{-1}(j\omega)\hat{v}_{k\beta}(j\omega) = \hat{v}_{k\alpha}^+(j\omega)\hat{i}_{k\beta}(j\omega) = 0, \quad \text{where } \alpha, \beta = 1, 2, \dots, n. \quad (23)$$

In (23), n is the algebraic multiplicity, which is the same as geometric multiplicity, since the eigenvalue is non-defective.

Fact 20. *Having shown that when $\{\hat{v}_i\}$ comprises of a complete set of eigenvectors, $\hat{\gamma}_l(j\omega)\hat{v}_k^+(j\omega)Z^{-1}(j\omega)\hat{v}_l(j\omega) = \hat{v}_k^+(j\omega)\hat{i}_l(j\omega) = 0$ if $k \neq l$, we go onto proving that this inner product can be normalized under the assumption of completeness of eigenvectors and when the mode is not a complex wave.*

Proof. Consider an eigenvector $\hat{v}_k(j\omega)$. Construct $c = Z^{-1}(j\omega)\hat{v}_k(j\omega)$ assuming non-singularity of $Z^{-1}(j\omega)$. Under the hypothesis of completeness of the eigenvectors $c = \sum_k a_k \hat{v}_k(j\omega)$ holds. Consider the terms in the form c^+c which is nonzero because $Z^{-1}(j\omega)$ is non-singular. $a_k^*(\hat{v}_k^+)c = a_k^*(\hat{v}_k^+)Z^{-1}\hat{v}_k$ and $a_l^*(\hat{v}_l^+)c = a_l^*(\hat{v}_l^+)Z^{-1}\hat{v}_k$ are the two types of terms in c^+c . By Fact 19 the second type vanishes since $k \neq l$. Of the first type, there is only one term and it must be nonzero because c^+c is nonzero. So we have

$$\hat{\gamma}_k(j\omega)\hat{v}_k^+(j\omega)Z^{-1}(j\omega)\hat{v}_k(j\omega) = \hat{v}_k^+(j\omega)\hat{i}_k(j\omega) \neq 0. \quad (24)$$

By Facts 16 and 17 we already proved at a singularity where the eigenvalue is continuous, $v_{fi}^+ i_{fi}$ is continuous too.

Now we wish to carry also the results of Facts 18, 19, and 20, which were on the product $\hat{v}_k^+(j\omega)\hat{i}_l(j\omega)$ in infinite dimensions onto the product $\hat{v}_{fk}^+(j\omega)\hat{i}_{fl}(j\omega)$, that appeared in (3) as finite dimensional.

This is done easily since the relation (4) indicates results found in infinite dimensions for $\hat{v}_k^+(j\omega)\hat{i}_l(j\omega)$ are also implied for $\hat{v}_{fk}^+(j\omega)\hat{i}_{fl}(j\omega)$ whenever the matrices $\hat{Z}(j\omega)\hat{Y}(j\omega)$ and $\hat{Y}(j\omega)\hat{Z}(j\omega)$ are non-defective. Through relation (3) we find, as desired, the equivalence in finite dimensions too.

Let us summarize these results for the matrix $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$ with exact eigenvalues $\hat{\gamma}_k^2$ and $\hat{\gamma}_l^2$ of the physical problem.

Fact 21. *Whenever on the $j\omega$ axis, a propagation constant (square root of an eigenvalue) of the exact system is complex, and if $\hat{Z}(j\omega)\hat{Y}(j\omega)$ and $\hat{Y}(j\omega)\hat{Z}(j\omega)$ are non-defective, the corresponding proper eigenvectors of $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$ satisfy,*

$$v_{fi}^+(j\omega)i_{fi}(j\omega) = 0,$$

Fact 22. *Whenever the operator T has a complete set of eigenvectors, so will $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$, and*

$$v_{fk}^+(j\omega)i_{fl}(j\omega) = \delta_{kl}$$

will hold, where δ_{kl} is Kronecker's delta. If $k = l$ and $\hat{\gamma}_k^2$ is non-real, we have Fact 21 again.

4. TRANSFER OF RESULTS ON APPROXIMATE EIGENVALUES TO EXACT EIGENVALUES

In this section we list how the results of approximate analysis in [1, 2] transform into results on exact eigenvalues by the discussion in Parts I and II.

Now $\hat{\gamma}^2(p)$ is a root of (1), and in place of (2) in [2], which is the algebraic equation obtained from the characteristic equation of the truncated matrix product $Z(p)Y(p)$, we have (1). (1) is again an algebraic equation, but with algebraic functions as roots, as opposed to (2) in [2] whose roots are algebraic functions.

Algebraic functions have only algebraic singularities like the algebraic functions we used in the approximation in the eigenvalue problem. Therefore all that concern us vis-à-vis singularities are the

same in infinite dimensions as in finite dimensions. Many of the results already obtained for approximate eigenvalues transfer to exact eigenvalues directly. The remaining results transfer with additional arguments.

Let us see how each of these results, previously obtained in [1, 2, 8], can be formulated for the exact propagation constant.

In Section 2.1 of [2], the result that ‘transition frequency $j\omega_B$ from complex to non-complex wave mode can not take place at a point where $\gamma_1^2(p)$, for the subject mode, is regular,’ applies in infinite dimensions. In fact this result is general and independent of the determination method of propagation constant, as indicated in [2].

But that at the transition frequency $j\omega_B$, from complex to non-complex wave mode in infinite dimensions, $\hat{\gamma}_1^2(j\omega_B)$ is finite and multiple, i.e., that it is a root of the discriminant of (1), is a new result.

The points needed to transfer the results of Appendix A.1 of [2], that a finite multiple eigenvalue must be defective, to the infinite dimensional exact eigenvalue, are given as follows.

Facts 1 and 2 of Appendix A.1 of [2], carry over to infinite dimensions and then to matrices $J_{1\text{ fin}}$ and $J_{2\text{ fin}}$, by virtue of Facts 17, 21 and 22 of this paper. Similar to their counterparts for approximate eigenvalues in [2], for finite dimensions, these three facts imply for $J_{1\text{ fin}}$ that the complex wave mode frequency interval can not end at a singularity where the exact eigenvalue is finite, multiple but non-defective. The exact eigenvalue at such a singularity is defective.

Considering Section 2.2 of [2], by virtue of a non-zero defective multiple eigenvalue at $j\omega_B$, in infinite dimensions too at the upper end frequency point of the complex wave interval, bifurcation property exists and forces the complex propagation constant to split into two γ^2 , which are real for $\omega > \omega_B$. Results of Section 2 of [2] are fully transferable to exact eigenvalues. It is however noted that bifurcation property reported for exact propagation constants in [9] was assumed to hold without a justification, also for roots of the algebraic equation in [2] which yields only approximate propagation constants, and numerous results were derived based on it. In Appendix A we give a proof of bifurcation for a propagation constant obtained from the algebraic equation of a truncated system such that if this function is accurate enough in approximating the exact bifurcating propagation constant, is non-zero at the frequency where it has a singular derivative, then it bifurcates at this singularity. Existence of such an approximating propagation constant is also shown. Therefore we must assume that in the development in [2] always such a root of the algebraic equation has been considered.

Results of Appendix A.2 of the same reference are also fully transferable to exact eigenvalues. Similar case holds for Sections 3.1 and 3.2 too.

Adaptation of Appendix B of [2], to infinite dimensions, can be made under the light of Fact 11 of Part I, which states that if at $p = p_0 = j\omega_0$, $\hat{Z}(p)\hat{Y}(p)$ has a pole, then at least one eigenvalue $\hat{\gamma}_1^2(p)$, has a pole at $p = p_0 = j\omega_0$.

At this point we remark that in Appendix B of [2], the preclusion of a pole branch point was proved considering the singularity to be on the $j\omega$ axis, without explaining the reason for it. We now complete this proof of [2], by pointing out that the only type of singularity $Z(p)Y(p)$, and thus $\gamma^2(p)$, can have is poles on the $j\omega$ axis anyway, because $Z(p)$ and $Y(p)$ are Foster matrices.

The same applies in infinite dimensions, when again the only type of pole or pole branch point $\hat{\gamma}^2(p)$ can have is on the $j\omega$ axis, for the same reason. Therefore we proceed with adaptation of Appendix B of [2], considering a pole at $p = p_0 = j\omega_0$, of $\hat{Z}(p)\hat{Y}(p)$.

In the case $\hat{Z}(p)\hat{Y}(p)$ has a pole at $p = p_0 = j\omega_0$, let us consider T^{-1} , when $(\hat{Z}\hat{Y})^{-1}$ exists. Then at $p = p_0 = j\omega_0$, one eigenvalue $\hat{\gamma}_1^2(p)$ of T^{-1} will either be regular with zero value, or have a singularity where $\hat{\gamma}_1^2(p)|_{p=p_0} = \frac{1}{\hat{\gamma}_1^2(p)}|_{p=p_0} = 0$, is finite. In the latter case, $\hat{\gamma}_1^2(p)$ will have an algebraic branch point with no negative powers in the Puiseux-Laurent expansion about it. But from Section 3 of [2], we have regularity of $\hat{\gamma}_1^2(p)$ at a cutoff frequency. This makes $\hat{\gamma}_1^2(p)$ single valued in the neighborhood of p_0 , and forces $\hat{\gamma}_1^2(p)$ to have the same property. Recall p_0 is a pole of $\hat{\gamma}_1^2(p)$, and now due to this single valuedness, we have impossibility of $p = p_0 = j\omega_0$ being a pole branch point for $\hat{\gamma}_1^2(p)$. In this way, for exact eigenvalues as well, we rule out the case of a pole branch point for the type of guiding systems prescribed.

Coexistence of complex and backward wave modes in [4] can be argued in a similar way, for infinite dimensions.

[1] completely carries over to infinite dimensions except Section 3 paragraph ii) which became inapplicable due to Appendix B of [2].

5. CONCLUSION

We have reduced the exact infinite dimensional problem to finite dimensions. Furthermore we observed that results of the algebraic function approximation method which is not an exact technique, can be carried to infinite dimensions, by introduction of additional arguments.

APPENDIX A. PROOF THAT IF γ OBTAINED FROM A ROOT OF THE ALGEBRAIC EQUATION SUFFICIENTLY APPROXIMATES A BIFURCATING EXACT PROPAGATION CONSTANT, IF γ IS NONZERO WHERE $d\gamma/d\omega$ IS SINGULAR, THEN IT BIFURCATES AT THIS SINGULARITY

Consider first the following theorem in [8, p. 176].

Fact A.1. *Except possibly at frequencies where the eigenvectors of an infinite number of truncations $Z(p)Y(p)$ do not form complete sets, i.e., these truncations are defective, the sets of eigenvalues of the truncated $Z(p)Y(p)$ converge to the eigenvalues of the infinite dimensional $\hat{Z}(p)\hat{Y}(p)$ as the order of $Z(p)$ and $Y(p)$ tends to infinity [8, p. 176].*

Exclusion of frequencies where an infinite number of truncations $Z(p)Y(p)$ are defective, is not very prohibitive for application of Fact A.1. This is true especially as we shall see below, if these frequencies are discrete frequency points.

This discreteness is not a very demanding constraint for a physical guiding structure and it is sufficient for applying Fact A.1.

Indeed since an excluded frequency p_0 is assumed to be discrete, i.e., isolated, it has a neighborhood in which no other excluded frequency exists. For each p in the punctured neighborhood where Fact A.1 is applicable, the sequence $\{\gamma_{in}^2(p)\}$ of truncated system eigenvalues converges to $\hat{\gamma}_i^2(p)$ of $\hat{Z}(p)\hat{Y}(p)$, as the order of truncated $Z(p)$ and $Y(p)$ tends to infinity. Because each element of the sequence $\{\gamma_{in}^2(p)\}$ and $\hat{\gamma}_i^2(p)$ are continuous functions of p at p_0 whenever these functions are finite, convergence of $\{\gamma_{in}^2(p_0)\}$ to $\hat{\gamma}_i^2(p_0)$ will be insured as well. I.e., eigenvalues of a sequence of truncated $Z(p_0)Y(p_0)$ at p_0 also converge to the eigenvalue $\hat{\gamma}_i^2(p_0)$ of $\hat{Z}(p_0)\hat{Y}(p_0)$, even though p_0 is an excluded frequency.

Fact A.2. *There exists a truncation $Z(j\omega)Y(j\omega)$ and hence an approximate propagation constant γ close enough to the exact propagation constant bifurcating at $j\omega_0$, such that if additionally at $j\omega_1$ which is in a small enough neighborhood of $j\omega_0$, γ is nonzero, finite and $d\gamma/d\omega$ is singular, then the associated dispersion characteristic of γ bifurcates at $j\omega_1$*

Proof. The proof may be given in two parts. The first part will show that the approximating propagation constant splits at $j\omega_1$. The second part will show the existence of an approximating propagation constant that bifurcates.

If $\gamma^2(j\omega)$ is a finite root of the algebraic equation obtained from the characteristic equation associated with $Z(j\omega)Y(j\omega)$, wherein $Z(j\omega)$ and $Y(j\omega)$ are truncated matrices, and if $d\gamma/d\omega$ tends to infinity and $\gamma(j\omega) \neq 0$, then $d\gamma^2/d\omega$ will tend to infinity. I.e., the root of the algebraic equation will be finite, multiple and hence singular. The singularity is an exceptional point and the eigenvalue must split at this point as per [5].

Therefore we have splitting for eigenvalues of the truncated, finite dimensional matrix product $Z(p)Y(p)$ as well. But our claim is not simply splitting of the propagation constant, but its bifurcation, which is splitting into two. This is also true as follows.

Since eigenvalues of both the truncated systems and the infinite system are continuous functions of the complex frequency p whenever these eigenvalues are finite, the convergence on any compact set Q of the complex plane is uniform for some subsequence of truncated systems. Similarly, for any compact set Q and finite set of eigenvalues $(\hat{\gamma}^2(p))$ of the infinite system, we have due to Fact A.1 that, a subsequence of truncated systems exists with eigenvalues that converge uniformly on Q to corresponding eigenvalues $(\hat{\gamma}^2(p))$ [8, p. 183].

Even if an excluded point p_0 in Q is singular for eigenvalues of an infinite number of truncations $Z(p)Y(p)$ causing these matrices to be defective in the way discussed above for Fact A.1, the uniform convergence of a finite set of eigenvalues over Q is still true, so long as these excluded points are discrete. Because we saw above that even in this case, at p_0 , $\{\gamma_{in}^2(p_0)\}$ converges to $\gamma_i^2(p_0)$ for some subsequence of truncations, preserving continuity of the sequence elements and the limit which are all continuous functions of p , since they are all finite, that is since they can at most have algebraic branch points where the eigenvalues are finite.

Consider the finite set $(\hat{\gamma}^2(p))$ of exact eigenvalues within which bifurcation occurs for one element [9] at point $p_0 = j\omega_0$.

There exists a subsequence of truncated systems whose eigenvalues converge uniformly to this exact set on a region Q including p_0 even if p_0 is an isolated singularity for eigenvalues of an infinite number of truncations $Z(p)Y(p)$ in the way discussed above.

Assume Q is a neighborhood of the point p_0 . Convergence is uniform over Q , and therefore there will exist truncated systems with eigenvalues arbitrarily close to the set $(\hat{\gamma}^2(p))$ over Q . Not only will there exist truncations with eigenvalue sequences $\{\gamma_{in}^2(p_0)\}$ that converge to $\hat{\gamma}_i^2(p_0)$ for each $\hat{\gamma}_i^2(p) \in (\hat{\gamma}_i^2(p))$, but due to uniform property of the convergence, eigenvalues $\{\gamma_{in}^2(p)\}$ in Q will be as close to $\hat{\gamma}_i^2(p)$ as desired for integer values n greater than a certain integer N , which is independent of the p in Q . Because N is independent of p ,

and because for a finite system of exact eigenvalues a subsequence of truncated systems exists with eigenvalues convergent uniformly on Q to this finite system, if there is bifurcation for a propagation constant of one element of this finite system of exact eigenvalues ($\hat{\gamma}^2(p)$) in Q , then there will exist a truncated system with eigenvalues that will approximate the set ($\hat{\gamma}^2(p)$) over Q . Therefore one approximating propagation constant will exist that will not simply split, but in particular bifurcate, because the limit exact propagation constant bifurcates.

To see the truth of the last sentence in more detail, suppose now as per [5], the approximating propagation constant characteristic splits, but that it splits into more than two branches at the singularity $j\omega_1$, in the neighborhood Q of $j\omega_0$, where the exact propagation constant bifurcates.

For each $\varepsilon > 0$ close to zero, there will exist by the foregoing proof, an integer $N(\varepsilon)$ to make

$$|\hat{\gamma}_i(j\omega) - \gamma_{in}(j\omega)| < \varepsilon, \quad \text{for } n > N(\varepsilon) \quad (\text{A1})$$

over Q . Recall that we have only one Puiseux series expansion for $\gamma_{in}(j\omega)$ on both sides of the singularity so that both the split and to split $\gamma_{in}(j\omega)$ satisfy (A1). By making ε smaller, $\gamma_{in}(j\omega)$ may be forced to split into two branches since choosing ε smaller will improve approximation of the exact eigenvalue $\hat{\gamma}_i(j\omega)$ by the $\gamma_{in}(j\omega)$. This will be achieved by forcing the second, third, ..., etc., branches of the splitting in the above supposition to overlap, by making ε smaller.

On the other hand if two or more branches overlap as result of choosing ε smaller, two or more solutions of the approximating algebraic equation will have overlapped. This will mean vanishing of the discriminant of the algebraic equation identically over a region, contradicting its irreducibility hypothesis. Therefore such a case is ruled out.

Hence γ obtained from the root of the algebraic equation which is a sufficiently accurate approximation of the bifurcating exact propagation constant, must bifurcate, if it has to split, since as shown above, it can not split into more than two branches.

REFERENCES

1. Yener, N., "Application of algebraic function theory to backward wave problems," *Jour. of Electromagnetic Waves and Applications*, Vol. 18, No. 10, 1399–1417, 2004.
2. Yener, N., "Algebraic function approximation in eigenvalue

- problems of lossless metallic waveguides (Revisited),” *Progress In Electromagnetics Research*, PIER 55, 147–174, 2005.
3. Yener, N., “Algebraic function approximation in eigenvalue problems of lossless metallic waveguides: Examples,” *Jour. of Electromagnetic Waves and Applications*, Vol. 20, No. 6, 731–745, 2006.
 4. Yener, N., “On the existence of backward waves in metallic waveguides,” *Jour. of Electromagnetic Waves and Applications*, Vol. 18, No. 6, 769–779, 2004.
 5. Kato, T., *Perturbation Theory for Linear Operators*, Springer Verlag, New York, 1980.
 6. Knopp, K., *Theory of Functions*, Part I, Dover Publications, New York, 1996.
 7. Baumgartel, H., *Analytic Perturbation Theory for Matrices and Operators*, Birkhauser Verlag, Basel, 1985.
 8. Noble, D. F., “Circuit properties of dispersive coupled transmission lines and waveguides,” Ph.D. dissertation, School of Electrical Engineering, Cornell University, Ithaca, N.Y., 1971.
 9. Mrozowski, M., *Guided Electromagnetic Waves*, Research Studies Press Ltd., UK, 1997.
 10. Felsen, L. and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice-Hall, Inc., Englewood Cliffs, 1973.