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ON UNIQUENESS THEOREM OF A VECTOR FUNCTION

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Abstract—Based on a generalized Helmholtz's identity, definitions of an irrotational vector and a solenoidal vector are reviewed, and new definitions are presented. It is pointed out that the well-known uniqueness theorem of a vector function is incomplete. Although the divergence and curl are specified, for problems with finite boundary surfaces, normal components are not sufficient for uniquely determining a vector function. A complete uniqueness theorem and its two corollaries are then presented. It is proven that a vector function can be uniquely determined by specifying its divergence and curl in the problem region, its value (both normal and tangential components) on the boundary.

1. INTRODUCTION

Existence, uniqueness and stability of solutions are of primary importance for differential equations. In reality, the existence is assured by physical considerations of the equations governing physical processes and the physical sources. Uniqueness is the fundamental for developing various solving techniques. It is well-known that the existing uniqueness theorem of a vector function is given as: A vector function $\mathbf{F}(\mathbf{r})$ in V enclosed by surface S, which is continuously differentiable, and whose normal components are given over the boundary S, is uniquely determined by its divergence and curl. In equations, the solution to the system

$$\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{s}(\mathbf{r}) \tag{1a}$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = c(\mathbf{r}) \tag{1b}$$

$$\mathbf{n} \cdot \mathbf{F}(\mathbf{r})|_{\mathbf{r}\ on\ S} = \mathbf{n} \cdot \mathbf{F}_0(\mathbf{r})|_{\mathbf{r}\ on\ S} \tag{1c}$$

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is unique. The terms on the right-hand sides are given [1, p. 314], [2, Sect. 1.16].

This is a critical theorem of vector analysis, mathematical physics and its applications to boundary problems of various disciplines such as gravity theory and electromagnetism [3]. Intuitively, we question the above uniqueness theorem as follows. It is hard to imagine that if both divergence and curl are given explicitly, both parts of a vector function can be determined just by its normal components on the boundary surface that correspond to the divergence part. If the tangential components over the boundary are undetermined, the vector may not be unique since the boundary is just special part of the solution space. The popular proof introduces the scalar potential directly from the divergence equation of the difference vector equation [2, pp. 92–93]. But it is not difficult to notice that any difference vector function always satisfies both solenoidal and irrotational equations. There is no reason to introduce scalar potential only. In fact, although (1) is consistent with the existing uniqueness theorem in electrostatics that requires normal components on S, it is inconsistent with that in magnetostatics that requires tangential components [4]. However, both the electrostatics and magnetostatics are special cases of (1) with $\mathbf{s} = 0$ or $c = 0$. Therefore, the above existing theorem is questionable.

In order to resolve this theoretical difficulty, the well-known Helmholtz's theorem is extended to piecewise continuously differential vector functions first, then the definitions of irrotational and solenoidal vectors are revisited. It is found that the existing definitions are incomplete. The incompleteness is the root cause of the above questionable uniqueness theorem. Complete definitions are given as an important theoretical application of the generalized Helmholtz's theorem. Then, a complete uniqueness theorem is proposed in Section 3. The new conditions of the introduction of potentials are discussed and some comments are made on the proofs of exiting uniqueness theorem in the last section to complete the paper.

2. DEFINITIONS OF IRROTATIONAL AND SOLENOIDAL FIELDS

In vector analysis, Helmholtz's decomposition theorem is a very important decomposition according to the divergence and curl characteristics of a vector function. There are many statements of Helmholtz's theorem [5–8] etc. Recently, in [9], it is extended to a more general case for piecewise continuously differential vectors. It states,

Theorem 1. *Any finite, integrable and piecewise continuously differentiable vector function* $\mathbf{F}(\mathbf{r})$ *given in a space* V *enclosed by* S *can be completely and uniquely decomposed into a sum of an irrotational part and a solenoidal part. The two parts are independent. Mathematically, it is the identity.*

$$\begin{aligned}\mathbf{F}(\mathbf{r}) &= \mathbf{F}_i + \mathbf{F}_s \\ &= -\nabla\left[\int_V \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' + \sum_l\left(\oint_{S_l}\frac{\mathbf{F}_l(\mathbf{r}')\cdot\mathbf{n}}{4\pi|\mathbf{r}-\mathbf{r}'|}d^2\mathbf{r}'\right)\right] \\ &\quad +\nabla\times\left[\int_V \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' + \sum_l\left(\oint_{S_l}\frac{\mathbf{F}_l(\mathbf{r}')\times\mathbf{n}}{4\pi|\mathbf{r}-\mathbf{r}'|}d^2\mathbf{r}'\right)\right] \quad (2)\end{aligned}$$

where a partial volume V_l *is bounded by surface* S_l. *In each partial volume,* $\mathbf{F}$ *is continuously differentiable.*

Of course, all vectors must be second order differentiable. In electromagnetic theory, (2) could be explained as follows. The terms in the first bracket represent scalar potentials due to volume charge sources and surface (layer) charge distributions on all discontinuous surfaces. Similarly, the terms in the second bracket are vector potentials due to volume current sources and surface current distributions. Many authors have claimed that Helmholtz's theorem is equivalent to the uniqueness theorem of a vector function [3, 2]. This opinion is questioned in three aspects in [9]. First, if it is true, there is no need to prove the uniqueness theorem by introducing scalar potentials; secondly, it is imperative to notice that all the distributions include free and induced sources [4, Chap. III and IV]. From this point of view, the above theorem does not imply the uniqueness theorem since it is well-known that the induced sources should not be specified; thirdly, there are other vector boundary value problems in which divergence and curl are not given explicitly. Of course, the correct uniqueness theorem must be compatible with (2) since the solutions to any problems should satisfy the decomposition identity as vectors. As an important theoretical application of (2), let's revisit the definitions of irrotational and solenoidal vectors.

Concepts in vector analysis were mostly introduced with the needs of the development of electromagnetic theory in the 18th century. In electromagnetic theory, divergence and curl of a vector are incorporated into the well-known Maxwell's equations, the basic physical laws governing electromagnetic behaviors. Especially, the concepts of irrotational and solenoidal fields are introduced and used widely in electrostatics and magnetostatics. Traditionally, in all references $[1, 10, 8]$, an irrotational vector is defined by the equation

$$\nabla \times \mathbf{F} = 0 \tag{3}$$

and a solenoidal vector is defined by

$$\nabla \cdot \mathbf{F} = 0 \tag{4}$$

(3) and (4) are considered as the necessary and sufficient conditions for introducing scalar and vector potentials [6]. However, as mathematical concepts the definitions are somewhat intuitive, they are not reviewed according to the above decomposition theorem. The first glance shows us more conditions are needed. Then, we propose the following precise definitions.

Definition 1. *A finite, integrable and piecewise continuously differentiable vector* $\mathbf{F}$ *defined in a finite region* V *bounded by a surface* S*, is irrotational if and only if*

$$\nabla \times \mathbf{F} = 0 \tag{5}$$

in each uniform partial volume V_l *of* V*, in which the vector is continuously differentiable, and*

$$\mathbf{n} \times \mathbf{F} = 0 \tag{6}$$

on the surface S*.*

In order to be irrotational, the volume integral and the surface integrals of the second term of (2) must vanish , that is $\mathbf{F}_s \equiv 0$. (5) assures the volume integral be zero. It seems that it requires that all tangential components on all discontinuous surfaces V_l must be zero. However, if no free surface sources on S_l (5) implies the boundary condition between two interior adjacent volumes V_p and V_q [4],

$$\mathbf{n}_p \times \mathbf{F}_p + \mathbf{n}_q \times \mathbf{F}_q = 0 \tag{7}$$

Notice that $\mathbf{n}_p = -\mathbf{n}_q$. So only the tangential component on S has to be specified. The tangential component of $\mathbf{F}$ on S must vanish everywhere in order to be applicable to any shapes in general cases, which leads to the requirement (6). (6) is not derivable from (5). According to (2), (5) is even impossible at any point in V without (6)! The physical meaning is also very clear. There must be no surface current sources on S. Similarly, we have

Definition 2. *A finite, integrable and piecewise continuously differentiable vector* $\mathbf{F}$ *defined in a region* V *bounded by a finite surface* S*, is solenoidal if and only if*

$$\nabla \cdot \mathbf{F} = 0 \tag{8}$$

in each uniform partial volume V_l *of* V*, in which the vector is continuously differentiable, and*

$$\mathbf{n} \cdot \mathbf{F} = 0 \tag{9}$$

on the surface S*.*

If S goes to infinity, the requirement $|\mathbf{r}|^2|\mathbf{F}|$ being bounded is equivalent to (6) and (9).

An immediate conclusion of the above definitions and the identity (2) is the following theorem about a null vector,

Theorem 2. *If a vector function is both irrotational and solenoidal, it must be a null vector function.*

The above theorem was recognized in free space by many authors [10]. However, the conditions (6) and (9) were not noticed by those authors. In fact, the conditions are implied in the infinite boundary requirements. Note that a nonzero vectorial constants can not be uniquely decomposed into irrotational and solenoidal parts everywhere. Although the divergence and curl of nonzero vectorial constants are zero, these nonzero vectorial constants are excluded by the requirements (6) and (9) on S. Fortunately, in real physical problems, constant vectors do not exist except some approximations in certain areas.

3. A COMPLETE UNIQUENESS THEOREM OF A VECTOR FUNCTION

Based on the above Theorem 2, we propose a complete uniqueness theorem of a vector function.

Theorem 3. *A vector function* $\mathbf{F}(\mathbf{r})$ *in* V *bounded by the surface* S *can be uniquely determined by its divergence, curl and boundary values (both normal and tangential components) over the boundary* S*, i.e., the solution to the system*

$$\nabla \times \mathbf{F}(\mathbf{r}) \;\; = \;\; \mathbf{s}(\mathbf{r}) \tag{10a}$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = c(\mathbf{r}) \tag{10b}$$

$$\mathbf{F}(\mathbf{r})|_{\mathbf{r}\ on\ S} = \mathbf{F}_0(\mathbf{r})|_{\mathbf{r}\ on\ S} \tag{10c}$$

is unique.

In physics, all real sources are at least first order differentiable, therefore the vector fields are second order differentiable. Helmholtz's theorem can be employed.

Proof. Assume there are two different solutions $\mathbf{F}_1(\mathbf{r})$ and $\mathbf{F}_2(\mathbf{r})$. According to (10), the difference vector function $\mathbf{F}_d(\mathbf{r}) = \mathbf{F}_1(\mathbf{r}) - \mathbf{F}_2(\mathbf{r})$ satisfies the homogeneous equation system

$$\nabla \times \mathbf{F}_d(\mathbf{r}) = 0 \tag{11a}$$

$$\nabla \cdot \mathbf{F}_d(\mathbf{r}) = 0 \tag{11b}$$

$$\mathbf{F}_d(\mathbf{r})|_{\mathbf{r}\ on\ S} = 0 \tag{11c}$$

Therefore, $\mathbf{F}_d$ is both irrotational and solenoidal according to the new definitions and must be a zero vector function according to Theorem 2. The solution is then unique.

It is very interesting to point out that D. A. Woodside wrote '... whose tangential and normal components on the closed surface S are given...' [11, Theorem $H2$]. Unfortunately, his Theorem U that requires normal components is obviously contradictory to his Theorem $H2$. Although he mentioned the authors of [6, 10], those authors never made such a statement in their versions of uniqueness theorems. He did not provide good reasons although he repeated it in his Theorem V for four-vector fields in Minkowski space. It is noticeable that the above proof is completed without the concepts of either scalar or vector potentials. The introduction of potentials will be discussed in the last section.

Two corollaries can be immediately deduced.

Corollary 1. *An irrotational vector function $\mathbf{F}(\mathbf{r})$ can be uniquely determined by its divergence in V and normal components (since the tangential components are zero) over the boundary S. That is, the solution to the mathematical problem*

$$\nabla \times \mathbf{F}(\mathbf{r}) = 0 \tag{12a}$$

$$\mathbf{n} \times \mathbf{F}(\mathbf{r})|_{\mathbf{r}\ on\ S} = 0 \tag{12b}$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = c(\mathbf{r}) \tag{12c}$$

$$\mathbf{n} \cdot \mathbf{F}(\mathbf{r})|_{\mathbf{r}\ on\ S} = \mathbf{n} \cdot \mathbf{F}_0(\mathbf{r})|_{\mathbf{r}\ on\ S} \tag{12d}$$

is unique.

Corollary 2. *A solenoidal vector function* $\mathbf{F}(\mathbf{r})$ *can be uniquely determined by its curl and tangential components over the boundary* S *(since the normal components are zero). That is, the solution to the mathematical problem*

$$\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{s}(\mathbf{r}) \tag{13a}$$
$$\mathbf{n} \times \mathbf{F}(\mathbf{r})|_{\mathbf{r}\ on\ S} = \mathbf{n} \times \mathbf{F}_0(\mathbf{r})|_{\mathbf{r}\ on\ S} \tag{13b}$$
$$\nabla \cdot \mathbf{F}(\mathbf{r}) = 0 \tag{13c}$$
$$\mathbf{n} \cdot \mathbf{F}(\mathbf{r})|_{\mathbf{r}\ on\ S} = 0 \tag{13d}$$

is unique.

The first corollary will have applications in electrostatics; and the second in magnetostatics.

4. COMMENTS ON THE EXISTING UNIQUENESS THEOREM AND ITS PROOFS

As seen, the above complete uniqueness theorem requires both tangential and normal values as boundary conditions. The proof does not need the concept of potentials. However, the proof of the existing uniqueness that requires normal components relies on the introduction of scalar potentials. All the existing proofs introduce a scalar potential from the zero divergence of the difference vector of two possible solutions [1, 2, 11]. Now the proof is questionable if the new theorem is correct. What is the problem? Obviously the difference vector has both zero divergence and zero curl. Which potential, scalar ϕ or vector $\mathbf{A}$, should be introduced? Different potentials require different boundary conditions although traditionally only scalar potential is used [2]. According to the new definitions, the scalar potential in [2, p. 93] cannot be introduced without zero tangential boundary condition. This is the mistake that has been used in all literatures. Let's review the introduction of potentials first.

Literally, we can still use scalar potential ϕ and vector potential $\mathbf{A}$ because of the zero identities,

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A} \tag{14}$$

In the existing proof [2],

$$\mathbf{F} = -\nabla\phi \tag{15}$$

is used for the difference vector. (15) indicates that the second term in (14) must be zero. That is the solenoidal part in the Helmholtz's decomposition theorem (2) must vanish. We now know that this is the case in which the conditions (5) and (6) described in the definition

(1) must be satisfied. The difference vector satisfies (5). (6) indicates the tangential components are specified as well. Therefore the new uniqueness theorem is proven in the traditional way. Similarly, if we want to introduce the vector potential **A** only, (8) and (9) must be satisfied. Although it is not necessary, the proof process is summarized as follows to complete the discussions.

Proof. The difference vector $\mathbf{F}_d$ satisfies,

$$\nabla \times \mathbf{F}_d = 0 \tag{16a}$$

$$n \times \mathbf{F}_{d|on\ S} = 0 \tag{16b}$$

$$\nabla \cdot \mathbf{F}_d = 0 \tag{16c}$$

$$\mathbf{n} \cdot \mathbf{F}_{d|on\ S} = 0 \tag{16d}$$

Based on both (16a) and (16b), the scalar potential can be used for the difference vector,

$$\mathbf{F}_d(\mathbf{r}) = \mathbf{F}_1(\mathbf{r}) - \mathbf{F}_2(\mathbf{r}) = -\nabla\phi_d \not\equiv 0 \tag{17}$$

Substituting (17) into (16c) and (16d) yields,

$$\nabla^2\phi_d(\mathbf{r}) = 0 \tag{18a}$$

$$\mathbf{n}\frac{\partial\phi_d(\mathbf{r})}{\partial n}|_{\mathbf{r}\ on\ S} = 0 \tag{18b}$$

Using the following Greens theorem [12, p. 488],

$$\int_V \left[\phi_d\nabla^2\phi_d + |\nabla\phi_d|^2\right] dV = \int_S \phi_d\frac{\partial\phi_d(\mathbf{r})}{\partial n}dS \tag{19}$$

one has

$$\int_V |\nabla\phi_d|^2\, dV = 0 \tag{20}$$

Notice that the Green's theorem does not require any information about interior surfaces. Discontinuities are allowed on the boundaries between two interior regions [12, p. 488]. Although the surface integral of (19) includes both ϕ_d and its normal derivative, only the normal derivative associated with the vector is proper. Since $|\nabla\phi_d|^2 \geq 0$, (20) yields

$$\mathbf{F}_d(\mathbf{r}) = -\nabla\phi_d \equiv 0 \tag{21}$$

(21) contradicts with the assumption (17), thus the theorem is proved.

□

Similarly, the theorem can be proven by introducing vector potential of the difference vector or both scalar and vector potentials of the original vector. All proofs will conclude the same theorem consistently. The inconsistency in the existing theorems and its proofs is resolved. It is worthy of investigating the impacts of the newly proven uniqueness theorem on applications, especially in electromagnetics. In fact, the new theorem is used in [13] to complete the uniqueness theorem in dynamic theory of electromagnetics. It is also noticed that the above classical vector field problem is described as a special case of Hodge decomposition in the modern theory of differential forms on manifolds [14, 15]. It will be interesting to extend the present discussions to abstract spaces.

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