Progress In Electromagnetics Research, PIER 66, 191–198, 2006

PECULIARITIES IN THE DIELECTRIC RESPONSE OF NEGATIVE-PERMITTIVITY SCATTERERS

A. Sihvola

Electromagnetics Laboratory
Helsinki University of Technology
P. O. Box 3000, FIC02015 HUT, Finland

Abstract—This study analyzes polarizability properties of spherically layered small inclusions that possess negative permittivity. Conditions for invisibility to external electric fields are derived. The complementary principle for two-dimensional scatterers is used to derive special properties of self-complementary inclusions. A singular behavior between the limits of invisibility and infinite response is underlined for a hollow circular shell. A similar, although not as drastic, phenomenon is shown to take place for the three-dimensional hollow sphere.

1. INTRODUCTION

The essence of the electromagnetic response of a small scatterer is contained in its field-induced dipole moment, determined by the polarizability which is a function of the geometry and permittivity of the particle. In applications like modeling of materials, the dipole polarizability is the most important of the multipolar parameters. All higher-order multipole fields decay more strongly with distance than those of the dipole.

The normalized polarizability of a small, homogeneous dielectric sphere is

$$\alpha = \frac{\alpha_{\mathrm{abs}}}{\epsilon_0 V} = 3\frac{\epsilon - 1}{\epsilon + 2} \tag{1}$$

where ϵ is the relative permittivity of the sphere (the absolute polarizability is made dimensionless through division by the free-space permittivity ϵ_0 and the volume of the sphere V). Obviously the dielectric response of a small sphere behaves very strangely if the permittivity is allowed to be negative. The polarizability grows without limit if the permittivity approaches the value -2. This is the so-called Fröhlich mode [1, 2]. One could also call it the electrostatic resonance [3].

Negative-valued permittivities are no anomalies. Several metals display such values for optical or infrared frequencies. And indeed, the present interest in so-called metamaterials which are being studied for their possible potential in microwave and higher-frequency imaging applications [4, 5] are often based on the simultaneously negative permittivity and permeability parameters [6]. Such artificial materials can be fabricated by embedding various resonating "molecules" into a neutral matrix. And this area of research on negatively refracting materials is presently getting very much attention in the electromagnetics community [7].

This study points out some interesting observations about the peculiarities of the dipole response of spherical scatterers of negative permittivity. In particular, the focus is on how closely a very high and very low observability of a small particle are connected when its permittivity approaches certain negative values. The invisibility or at least low observability of scatterers with negative material parameter values has been studied also in the full-wave regime [8], and recent studies have appeared dealing with even a complete cloaking of arbitrary objects by tracing and dragging wave rays past it [9–12]. In contrast, in this paper the focus is on small scatterers and on a very singular behavior between the full transparency and infinite visibility.

2. COMPLEMENTARITY PRINCIPLE FOR TWO-DIMENSIONAL SCATTERERS

A generalization of (1) into other spatial dimensions is obvious. Depending on the dimension D of the problem, the "sphere" polarizability is

$$\alpha = D\frac{\epsilon - 1}{\epsilon + D - 1} \tag{2}$$

and therefore, the polarizability of a circle (a two-dimensional sphere) is

$$\alpha = 2\frac{\epsilon - 1}{\epsilon + 1} \tag{3}$$

which has a singularity when the relative permittivity attains the value $\epsilon = -1$.

For a two-layer sphere, Figure 1, with core of ϵ_2 and shell of ϵ_1, the polarizability, generalization of (1), reads [13]

$$\alpha = 3\frac{(\epsilon_1 - 1)(\epsilon_2 + 2\epsilon_1) + \beta(2\epsilon_1 + 1)(\epsilon_2 - \epsilon_1)}{(\epsilon_1 + 2)(\epsilon_2 + 2\epsilon_1) + 2\beta(\epsilon_1 - 1)(\epsilon_2 - \epsilon_1)} \tag{4}$$

where $\beta = a_2^3/a_1^3$ is the volume fraction of the core of the total volume.

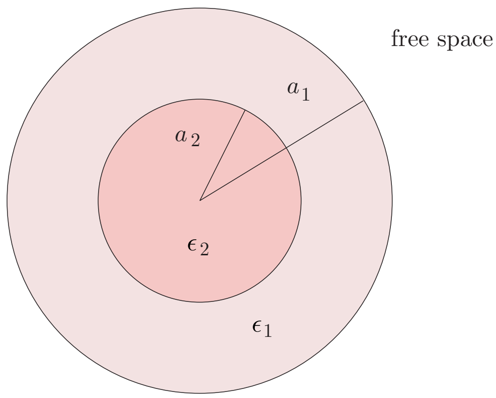

Figure 1. The core-and-shell sphere.

The corresponding polarizability value for the two-dimensional case (concentric circles, i. e., cross-cuts of cylinders) obeys the formula

$$\alpha = 2\frac{(\epsilon_1 - 1)(\epsilon_2 + \epsilon_1) + \beta(\epsilon_1 + 1)(\epsilon_2 - \epsilon_1)}{(\epsilon_1 + 1)(\epsilon_2 + \epsilon_1) + \beta(\epsilon_1 - 1)(\epsilon_2 - \epsilon_1)} \tag{5}$$

where now β refers to the fractional area a_2^2/a_1^2.

A look at the relation (3) gives rise to an interesting observation concerning two-dimensional scatterer responses. The polarizabilities of "complementary circles," meaning circles with inverse permittivities (ϵ and $1/\epsilon$) are opposite numbers:

$$\alpha_{1/\epsilon} = 2\frac{1/\epsilon - 1}{1/\epsilon + 1} = -2\frac{\epsilon - 1}{\epsilon + 1} = -\alpha_\epsilon \tag{6}$$

The theorems by Keller and Mendelson [14, 15], even if derived for effective conductivities of composites and not permittivities nor single inclusions corroborate that this result is valid for any isotropic two-dimensional scatterer.

For anisotropic scatterers, the corresponding relation is only little more complicated. For example, the polarizabilities of an ellipse with relative permittivity ϵ and semiaxes a and b in the x and y directions, respectively, are determined by the depolarization factors $N_x = b/(a+b)$ and $N_y = a/(a+b)$ [13]:

$$\alpha_\epsilon^x = \frac{\epsilon_r - 1}{1 + N_x(\epsilon_r - 1)} = \frac{\epsilon_r - 1}{1 + \frac{b}{a+b}(\epsilon_r - 1)} \tag{7}$$

$$\alpha_\epsilon^y = \frac{\epsilon_r - 1}{1 + N_y(\epsilon_r - 1)} = \frac{\epsilon_r - 1}{1 + \frac{a}{a+b}(\epsilon_r - 1)} \tag{8}$$

From this we can derive the relations corresponding to Equation (6) and generalizing it, between the polarizability components of complementary scatterers (inclusions of inverse permittivities):

$$\alpha_{1/\epsilon}^x = -\alpha_\epsilon^y, \quad \alpha_{1/\epsilon}^y = -\alpha_\epsilon^x \tag{9}$$

The two orthogonal polarizability components are pairwise opposite numbers when the permittivity is inverted. Again, this applies to any two-dimensional shape, not only to the ellipse, and can be exploited effectively to speed up the accuracy of the numerical evaluation of the polarizabilities of arbitrary scatterers [16].

And for a more general case of an inhomogeneous scatterer where the permittivity is not uniform, the corresponding complementarity relation reads

$$\alpha(\epsilon(\mathbf{r})) = -\alpha(1/\epsilon(\mathbf{r})) \tag{10}$$

for isotropic scatterers. One can easily check the validity of this in the special case of a two-layer circle (Equation (5)): there, replacing ϵ_1 by $1/\epsilon_1$ and ϵ_2 by $1/\epsilon_2$ changes the sign of α.

3. INVISIBILITY AND EMERGENCE OF INFINITY

Next, let us search for the conditions when the dipole response of a scatterer vanishes, of course other than the trivial case of homogeneous $\epsilon = 1$. From the previos results we can solve the case of a invisible hollow spherical shell. In other words the question is to find the permittivity of the layer in the situation $\epsilon_2 = 1$ and $\alpha = 0$. Equation (4), for the 3D situation, tells that the shell permittivity has then to be $\epsilon_1 = -1/2$.

In Figure 2 the field solution over a cut through the center of the sphere is depicted for an invisible three-dimensional hollow shell. As can be seen, the potential lines everywhere outside the sphere are equidistant and hence no dipole field is created. But also, as expected, the internal behavior is more turbulent.

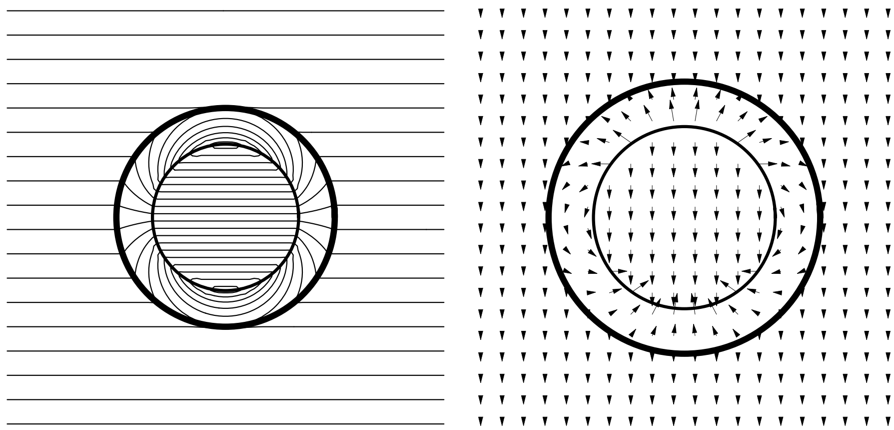

Figure 2. The equipotential lines (left) and the electric field (right) for an invisible ($\epsilon_1 = -1/2$) spherical shell in a uniform external electric field. Note that the solutions outside the scatterer are not affected by the presence of the sphere. The potential and the field patterns have been calculated from the exact solution of the Laplace equation for the twolayer sphere [17].

Similarly in the two-dimensional case, Equation (5), the invisibility condition requires that the shell permittivity has to be $\epsilon_1 = -1$.

In fact, the complementarity principle can be used to understand the invisibility of the two-dimensional case. Since the complementary operation reverses the sign of the polarizability and in the case of a shell with permittivity $\epsilon_1 = -1$ the complementarity operation keeps the problem the same (because then the inverse of the permittivity remains the same; $1/\epsilon_1 = -1$), one has to conclude that the polarizability has to vanish because $\alpha = -\alpha$. Therefore a 2D shell with permittivity of -1 has to be transparent.

However, there is another possibility to reconcile a change of sign with unchanged amplitude: infinite polarizability. And such a thing happens in the case when the core shrinks to zero. Equation (3) tells that a homogeneous self-complementary ($\epsilon = -1$) homogeneous sphere has infinite polarizability.

At first sight, this is counter-intuitive. At least such an inclusion shows extremely singular behavior: a homogeneous particle displaying a very strong Fröhlich resonance (a homogeneous circle of $\epsilon = -1$) suddenly not only loses the mode but even becomes totally invisible if a hole develops in the center of this inclusion. This can be appreciated on the formal level by studying the polarizability of a hollow circular negative-permittivity layer, whence Equation (5) gives us

$$\alpha = 2\frac{(\epsilon_1 - 1)(\epsilon_1 + 1)(1 - \beta)}{(\epsilon_1 + 1)^2 - \beta(\epsilon_1 - 1)^2} \tag{11}$$

and this is obviously zero for $\epsilon_1 = -1$, except if also β equals zero, in which case it goes to infinity. In other words, the case $\epsilon_1 \to -1$ and $\beta \to 0$ is a singular limit.

Something similar if not as dramatic happens in the three-dimensional case, too. Consider the case when the permittivity of the sphere is $\epsilon = -1/2$. Such a homogeneous sphere has polarizability $\alpha = -3$. Surprisingly, if this particle had a hollow core, no matter how small, the polarizability would suddenly vanish (Equation (4)) and the scatterer would become invisible.

4. CONCLUSIONS

With the advent of new metamaterials, the peculiar responses of small scatterers discussed above may find use in high-frequency engineering and optics. After all, the invisibility and Fröhlich responses require negative permittivities. It is obvious that transparency or low observability of dielectric objects is a very desired property in many application domains. But the phenomenon discussed in this paper, the very narrow boundary between invisibility and infinite reflectivity, is perhaps even more extreme and interesting, and may certainly find application potential in several other related fields. A great sensitivity of the response on the internal structure of the inclusion could be exploited in the design of efficient measuring and evaluation algorithms of material characteristics.

It is worth emphasizing the spherical geometry which was essential in the analysis since in that case we were able to concentrate of the dipole level response. Vanishing of the dipolarizability is sufficient for invisibility because for small spherically symmetric structures only the dipole moment survives in the multipole series.

But the analysis also opens up possibilities to understand unexpected phenomena for non-spherical scatterers, too. The complementary relation (9), applied for scatterers of any shape with permittivity $\epsilon = -1$, tells that the orthogonal polarizabilities are opposite numbers. Hence a random isotropic two-dimensional mixture of such inclusions (at least in the dilute Maxwell Garnett limit) will be transparent because the average polarization caused by the inclusions adds up to zero.

Finally, one may wonder whether the inevitable dielectric losses will soften or even wash away the extreme properties discussed above. In the frequency scale, the negative values of permittivity are band-limited, and because of the dispersion, there has to be a non-vanishing spectrum of the imaginary part of permittivity. However, due to the global integral character of the Kramers–Kronig relations, it is in principle not impossible to construct material response that combines negative real part and very small imaginary part of the permittivity at a certain frequency.

REFERENCES

1. Bohren, C. F. and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, John Wiley, New York, 1983.
2. Sihvola, A., "Character of surface plasmons in layered spherical structures," *Progress In Electromagnetics Research,* PIER 62, 317–331, EMW Publishing, Cambridge, Massachusetts, 2006.
3. Fredkin, D. R. and I. D. Mayergoyz, "Resonant behavior of dielectric objects (electrostatic resonances)," *Physica Review Letters*, Vol. 91, No. 25, Article 253902, 2003.
4. Pendry, J. B., "Negative refraction makes a perfect lens," *Physical Review Letters*, Vol. 85, 3966–3969, 2000.
5. Smith, D. R., J. B. Pendry, and M. C. K. Wiltshire, "Metamaterials and negative refractive index," *Science*, Vol. 305, 788–792, 2004.
6. Veselago, V. G., "The electrodynamics of substances with simultaneously negative values of ϵ and μ," *Soviet Physics Uspekhi*, Vol. 10, No. 4, 509–514, 1968.
7. Grzegorczyk, T. M. (ed.), Special Issue on Metamaterials exhibiting left-handed properties and negative refraction, *Progress In Electromagnetics Research*, PIER 51, 2005.
8. Alù, A. and N. Engheta, "Achieving transparency with plasmonic and metamaterial coatings," *Physical Review E*, Vol. 72, 016623, 2005.
9. Pendry, J. B., D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," *Science*, 10.1126/science.1125907, May 25, 2006.
10. Leonhardt, U., "Optical conformal mapping," *Science*, 10.1126/science.1126493, May 25, 2006.
11. Schurig, D., J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, "Metamaterial

electromagnetic cloak at microwave frequencies," *Sciencexpress*, 10.1126/science.1133628, October 19, 2006.

12. Greenleaf, A., Y. Kurylev, M. Lassas, and G. Uhlmann, "Full-wave invisibility of active devices at all frequencies," *ArXiv*, arXiv: math.AP/0611185v1, November 7, 2006.
13. Sihvola, A., *Electromagnetic Mixing Formulas and Applications*, IEE Publishing, London, 1999.
14. Keller, J. B., "Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders," *Journal of Applied Physics*, Vol. 34, No. 4, Part I, 991–993, April 1963.
15. Mendelson, K. S., "A theorem on the effective conductivity of a two-dimensional heterogeneous medium," *Journal of Applied Physics*, Vol. 46, No. 11, 4740–4741, November 1975.
16. Sihvola, A. and J. Venermo, "Boosting numerical accuracy in calculation of polarizability of two-dimensional scatterers," *IEEE Antennas and Propagation Magazine*, Vol. 47, No. 5, 60–65, October 2005.
17. Sihvola, A. and I. V. Lindell, "Transmission line analogy for calculating the effective permittivity of mixtures with spherical multilayer scatterers," *Journal of Electromagnetic Waves and Applications*, Vol. 2, No. 8, 741–756, 1988.