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ANALYSIS OF THE FIELDS IN TWO DIMENSIONAL CASSEGRAIN SYSTEM

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Abstract—High frequency field expressions are derived around feed point of a two dimensional cassegrain system using the Maslov's method. Maslov's method is a systematic procedure for predicting the field in the caustic region combining the simplicity of ray theory and generality of the transform method. Numerical computations are made for the analysis of field pattern around the caustic of a cassegrain system.

1. INTRODUCTION

Asymptotic ray theory (ART) or the geometrical optics approximation is widely used to study various kinds of problems in the areas of electromagnetics, acoustics waves, seismic waves, etc. [1–5]. It is also well known that the geometrical optics fails in the vicinity of caustic. So, in order to study the field behavior near caustic [6–9], other approach is required. Maslov proposed a method to predict the field in the caustic region [10]. Maslov method combines the simplicity of asymptotic ray theory and the generality of the Fourier transform method. This is achieved by representing the geometrical optics fields in terms of mixed coordinates consisting of space coordinates and wave vector coordinates. That is by representing the field in terms of six coordinates. It may be noted that information of ray trajectories is included in both space coordinates $\mathbf{R} = (x, y, z)$ and wave vector coordinates $\mathbf{P} = (p_x, p_y, p_z)$. In this way, conventional ray expression may be considered as projection into space coordinates.

Similarly one can replace a part of the components of (x, y, z) by corresponding components of (p_x, p_y, p_z), e.g., (x, p_y, p_z), (x, y, p_z), (x, p_y, z) to describe a ray. The reason for considering the solution in mixed or hybrid domain is that, in general, the singularities in different domains do not coincide. This means that there exists always a domain which can give bounded solution.

Solving the Hamiltonian equations under the prescribed initial conditions one can construct the geometrical optics field in space R, which is valid except in the vicinity of caustic. Near the caustic, the expression for the geometrical optics field in spatial space is rewritten in mixed domain. The expression in mixed domain is related to the original domain R through the asymptotic Fourier transform.

Applications of Maslov's method in an inhomogeneous medium and continuation problems have been discussed by Kravtsov [6] and Gorman [11]. The physical interpretation of the mathematics of Maslov's method and its relation to other ART methods have been discussed by Ziolkowski and Dechamps [7]. Hongo and Co-workers applied Maslov's method to derive the high frequency solutions for field generated by a phase transformer and a cylindrical reflector [8, 9].

Two dimensional Cassegrain system consists of two cylindrical reflectors, that is, parabolic and hyperbolic reflectors. The aim of this paper is to derive the field distribution around the feed point using the Maslov's method when it is used as the receiving antenna. Before we discuss the analysis of the field in the caustic of a cassegrain system we consider the field reflected by a single arbitrary cylindrical reflector.

2. DERIVATION OF THE FIELD IN A SINGLE CYLINDRICAL REFLECTOR

First we consider the field reflected by an arbitrarily shaped single cylindrical reflector, whose contour is described by

$$\zeta = f(\xi) \tag{1}$$

where (ξ, ζ) are the Cartesian coordinates on the reflector. An incident plane wave is given by

$$E^i = \exp\Big[-j\Big(k_x x + k_z z\Big)\Big], \qquad k_x = k_0 \sin\phi_0, \quad k_z = k_0 \cos\phi_0 \tag{2}$$

and the wave vector of the incident wave is given by $\mathbf{p}^i = \sin\phi_0 \mathbf{i}_x + \cos\phi_0 \mathbf{i}_z$. Then the wave vector of the reflected wave is obtained from the formula $\mathbf{p}^r = \mathbf{p}^i - 2(\mathbf{p}^i \cdot \mathbf{n})\mathbf{n}$, which is derived from Snell's law, where $\mathbf{n}$ is the unit normal of the surface given by

$$\mathbf{n} = \sin\theta \mathbf{i}_x + \cos\theta \mathbf{i}_z, \qquad \sin\theta = \frac{-f'(\xi)}{\sqrt{1 + [f'(\xi)]^2}}, \quad \cos\theta = \frac{1}{\sqrt{1 + [f'(\xi)]^2}} \tag{3}$$

where $f'(\xi)$ is the derivative of the function with respect to ξ. By using these relations we derive $\mathbf{p}^r$ as

$$\begin{aligned} \mathbf{P}^r &= \Big[\sin\phi_0 - 2\sin\theta\cos(\theta - \phi_0)\Big]\mathbf{i}_x + \Big[\cos\phi_0 - 2\cos\theta\cos(\theta - \phi_0)\Big]\mathbf{i}_z \\ &= -\sin(2\theta - \phi_0)\mathbf{i}_x - \cos(2\theta - \phi_0)\mathbf{i}_z = p_x^r \mathbf{i}_x + p_z^r \mathbf{i}_z \end{aligned} \tag{4}$$

The coordinates of the point on the reflected ray is given by [11]

$$x = \xi + p_x^r t, \qquad z = f(\xi) + p_z^r t \tag{5}$$

and the geometrical optics expression of field associated with the ray is given by [12]

$$U(\mathbf{r}) = A_0(\xi, \zeta) \left[\frac{D(t)}{D(0)}\right]^{-\frac{1}{2}} \exp\Big\{-jk\Big[\xi\sin\phi_0 + f(\xi)\cos\phi_0 + t\Big]\Big\} \tag{6}$$

where $A_0(\xi, \zeta)$ is the amplitude of the incident wave at the reflected point (ξ, ζ), and t represents the distance along the ray from a certain reference point. The value $\xi\sin\phi_0 + f(\xi)\cos\phi_0$ represents the initial value of the phase function. $D(t)$ is the Jacobian of the transformation from the Cartesian to the ray coordinates, and it is given by

$$\begin{aligned} D(t) &= \frac{\partial(x, z)}{\partial(\xi, t)} = -\cos(2\theta - \phi_0) + f'(\xi)\sin(2\theta - \phi_0) + 2\frac{\partial\theta}{\partial\xi}t \\ &= -\frac{\cos(\theta - \phi_0)}{\cos\theta} - 2\cos^2\theta f''(\xi)t \end{aligned} \tag{7a}$$

$$J(t) = \frac{D(t)}{D(0)} = 1 + \frac{2\cos^3\theta}{\cos(\theta - \phi_0)} f''(\xi)t \tag{7b}$$

The caustic of this ray is given by the point satisfying $D(t) = 0$ and (5), more explicitly,

$$x_c = \xi + \frac{\sin(2\theta - \phi_0)\cos(\theta - \phi_0)}{2\cos^3\theta f''(\xi)}, \quad z_c = f(\xi) + \frac{\cos(2\theta - \phi_0)\cos(\theta - \phi_0)}{2\cos^3\theta f''(\xi)} \tag{8}$$

It is seen that the ray reflected from the singular point $f''(\xi) = 0$ can not form the caustics. At the point satisfying (8), the ray becomes infinity. According to the Maslov's method, the ray expression covering the caustics can be derived from the formula

$$\begin{aligned} U(\mathbf{r}) \; &= \; \sqrt{\frac{k}{j2\pi}} \int_{-\infty}^{\infty} A_0(\xi) \left[\frac{D(t)}{D(0)} \frac{\partial p_z}{\partial z} \right]^{-\frac{1}{2}} \\ &\quad \times \exp\Big\{ -jk \Big[S_0 + t - z(x, p_z) p_z + p_z z \Big] \Big\} \, dp_z \qquad (9) \end{aligned}$$

where $S_0 = \xi \sin\phi_0 + f(\xi)\cos\phi_0$ is the initial phase. In (9) $z(x, p_z)$ means that the coordinate z should be expressed in terms of mixed coordinates (x, p_z) by using the solution. The same is true for t and it is given by $t = \frac{x-\xi}{p_x}$. The phase function $S(p_z)$ is given by

$$\begin{aligned} S(p_z) \; &= \; \xi \sin\phi_0 + f(\xi)\cos\phi_0 + \frac{x-\xi}{p_x} - f(\xi) p_z - (p_z)^2 \frac{x-\xi}{p_x} + p_z z \\ &= \; \xi \Big[\sin\phi_0 + \sin(2\theta - \phi_0) \Big] + f(\xi) \Big[\cos\phi_0 + \cos(2\theta - \phi_0) \Big] + p_x x + p_z z \\ &= \; 2 \Big[\xi \sin\theta + f(\xi)\cos\theta \Big] \cos(\theta - \phi_0) - \rho\cos(2\theta - \phi - \phi_0) \qquad (10) \end{aligned}$$

and the amplitude of the integrand is evaluated in Appendix A. In the above equation, we introduce the polar coordinates

$$x = \rho \sin\phi, \qquad z = \rho\cos\phi \qquad (11)$$

Substituting these results in (9), we have the field expression valid in caustic region as

$$\begin{aligned} U(\mathbf{r}) \; &= \; \sqrt{\pi} \exp\left(-j\frac{\pi}{4} \right) \int_{-\Theta}^{\Theta} A_0(\xi) \left[\frac{\cos(\theta - \phi_0)}{\cos^3\theta f''(\xi)} \right]^{\frac{1}{2}} \\ &\quad \times \exp\Big\{ -j2k \Big[\xi\sin\theta + f(\xi)\cos\theta \Big] \cos(\theta - \phi_0) \Big\} \\ &\quad \times \exp\Big[jk\rho\cos(2\theta - \phi - \phi_0) \Big] d\theta \qquad (12) \end{aligned}$$

where Θ is the half angle of θ at the edge of the reflector and we have changed the integration variable from p_z to θ.

In a region far from the caustics, (12) can be evaluated approximately by applying the stationary phase method of integration [11] and the result should agree with the GO expression derived in (6) with (7b). This serves as an important check of the validity of the expression (12). The stationary point is determined from

$$\begin{aligned} S'(\theta_s) \; &= \; 2 \Big[\xi\cos(2\theta - \phi_0) - f(\xi)\sin(2\theta - \phi_0) \Big] \\ &\quad \times \cos(\theta - \phi_0) + 2\rho\sin(2\theta - \phi - \phi_0) \\ &= \; 0 \qquad (13) \end{aligned}$$

The second derivative of the phase function is

$$\begin{aligned}S''(\theta_s) &= -4\Big[\xi\sin(2\theta-\phi_0)+f(\xi)\cos(2\theta-\phi_0)\Big]\\&\quad+2\Big[\cos(2\theta-\phi_0)-f(\xi)\sin(2\theta-\phi_0)\Big]\frac{d\xi}{d\theta}+4\rho\cos(2\theta-\phi-\phi_0)\\&\quad-2\frac{\cos(\theta-\phi_0)}{\cos^3 f''(\xi)}\left[1+2\frac{\cos^3\theta}{\cos(\theta-\phi_0)}f''(\xi)t\right]\\&= -2\frac{\cos(\theta-\phi_0)}{\cos^3 f''(\xi)}J(t)\end{aligned}\tag{14}$$

and

$$\begin{aligned}S(\theta_s) &= 2\Big[\xi\sin\theta+f(\xi)\cos\theta\Big]\cos(\theta-\phi_0)-\rho\cos(2\theta-\phi-\phi_0)\\&= \xi\Big[\sin(2\theta-\phi_0)+\sin\phi_0\Big]+f(\xi)\Big[\cos(2\theta-\phi_0)+\cos\phi_0\Big]\\&\quad-x\sin(2\theta-\phi_0)-z\cos(2\theta-\phi_0)\\&= \xi\sin\phi_0+f(\xi)\cos\phi_0+t\end{aligned}\tag{15}$$

We substitute (14) and (15) into (12) and carry out the integration, then we reproduce (6).

3. RECEIVING CHARACTERISTIC OF CYLINDRICAL CASSEGRAIN REFLECTOR

Cassegrain reflector consists of two reflectors, one is parabolic main reflector and another is hyperbolic subreflector. This system has many advantages over a single parabolic reflector. We consider here a receiving characteristic of this system by applying Maslov's method. The equation of each surface is given by [see Fig. 1]

$$\zeta_1=\frac{\xi_1^2}{4f}-f+c,\qquad \zeta_2=a\left[\frac{\xi_2^2}{b^2}+1\right]^{\frac{1}{2}},\qquad c^2=a^2+b^2 \tag{16}$$

where (ξ_1,ζ_1) and (ξ_2,ζ_2) are the Cartesian coordinates of the point on the parabolic and hyperbolic reflectors, respectively. Incident wave is given by

$$E^i=\exp(jkz) \tag{17}$$

The wave vector of the wave reflected by the parabolic cylinder is given by

$$\mathbf{p}_1^r=-\sin 2\alpha\mathbf{i}_x+\cos 2\alpha\mathbf{i}_z \tag{18a}$$

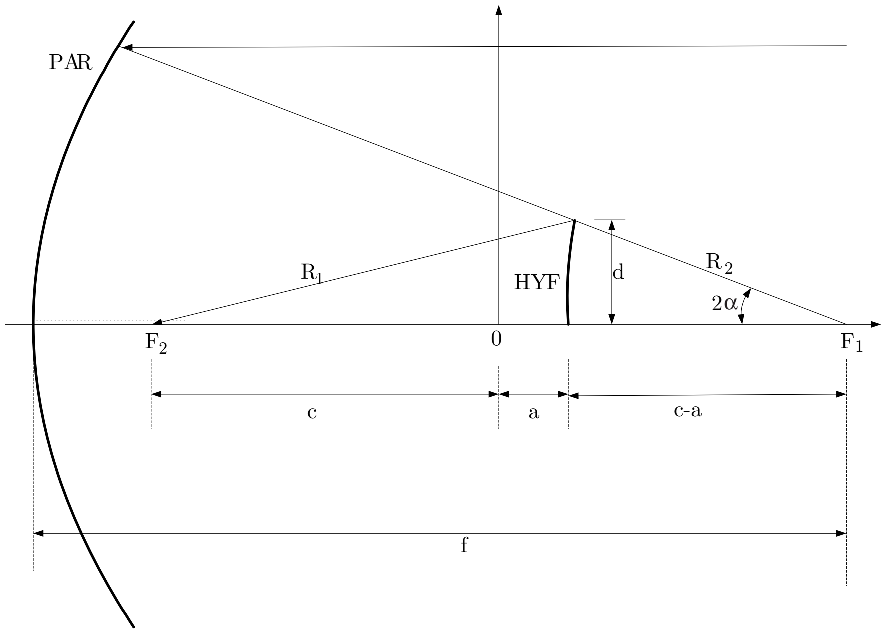

Figure 1. Cassegrain system.

and the wave vector of the wave reflected by the hyperbolic cylinder is

$$\mathbf{p}_2^r = -\sin(2\alpha - 2\psi)\mathbf{i}_x - \cos(2\alpha - 2\psi)\mathbf{i}_z \tag{18b}$$

where

$$\sin\alpha = \frac{\xi_1}{\sqrt{\xi_1^2 + 4f^2}}, \quad \cos\alpha = \frac{2f}{\sqrt{\xi_1^2 + 4f^2}}, \quad \mathbf{n}_1 = -\sin\alpha\mathbf{i}_x + \cos\alpha\mathbf{i}_z \tag{19a}$$

$$\sin\psi = -\frac{1}{\sqrt{R_1 R_2}}\frac{a}{b}\xi_2, \; \cos\psi = \frac{1}{\sqrt{R_1 R_2}}\frac{b}{a}\zeta_2, \; \mathbf{n}_2 = -\sin\psi\mathbf{i}_x + \cos\psi\mathbf{i}_z \tag{19b}$$

In the above equation R_1 and R_2 are the distances from the point (ξ_2, ζ_2) to the focal points $z = -c$ and $z = c$, respectively with $c^2 = a^2 + b^2$. The Jacobian associated with the wave reflected by the parabolic cylinder is obtained by applying (7b) with $\phi_0 = \pi$, $f''(\xi) = \frac{1}{2f}$, $\theta = -\alpha$. The result is

$$J_p(t) = \frac{D_p(t)}{D_p(0)} = 1 - \frac{\cos^2\alpha}{f}t \tag{20}$$

The Cartesian coordinates of the ray reflected by the hyperbolic cylinder is given by

$$x = \xi_2 + p_{x2}t = \xi_1 + p_{x1}t_1 + p_{x2}t \tag{21}$$
$$z = \zeta_2 + p_{z2}t = \zeta_1 + p_{z1}t_1 + p_{z2}t \tag{22}$$

where $t_1 = \sqrt{(\xi_1 - \xi_2)^2 + (\zeta_1 - \zeta_2)^2}$. In the above equation (p_{x1}, p_{z1}) and (p_{x2}, p_{z2}) are the rectangular components of $\mathbf{p}_1^r$ and $\mathbf{p}_2^r$, respectively

Now we consider the field after the reflection from the hyperbolic cylinder. The transformation from the Cartesian coordinates (x, z) to the ray fixed coordinates (ξ_1, t) is given by [see Appendix B]

$$D(t) = \frac{\cos^2\alpha}{f} R_2 \left[1 - \frac{t}{R_1}\right] \tag{23}$$

Thus the geometrical ray expression of the reflected wave is

$$E^r = E_0^r \left[1 - \frac{t}{R_1}\right]^{-\frac{1}{2}} \exp\Big[-jk\Big(S_0 + t_1 + t\Big)\Big] \tag{24}$$

where E_0^r is the amplitude of the incident wave at the reflection point on the parabolic cylinder and

$$S_0 = -\zeta_2 = 2f\frac{\cos 2\alpha}{1 + \cos 2\alpha} - c, \quad t_1 = \sqrt{(\xi_2 - \xi_1)^2 + (\zeta_2 - \zeta_1)^2},$$
$$t = \sqrt{(x - \xi_2)^2 + (z - \zeta_2)^2} \tag{25}$$

It is readily seen that the GO expression of the reflected wave becomes infinity at the point F_2 as is expected. We can derive the refined expression which is valid at the focal point according to (9). The value of $\left[J(t)\frac{\partial p_{z2}}{\partial z}\right]^{-\frac{1}{2}}$ is given by [see Appendix C]

$$\left[J(t)\frac{\partial p_{z2}}{\partial z}\right]^{-\frac{1}{2}} = \frac{\sqrt{R_1}}{\sin(2\alpha - 2\psi)} \tag{26}$$

The phase function is given by

$$S = S_0 + t_1 + t - z(x, p_{z2})p_{z2} + p_{z2}z \tag{27}$$

where $S_0 + t_1$ is given by (25). The extra term is given by

$$\begin{aligned} S_{ex} &= t - z(x, p_{z2})p_{z2} + p_{z2}z \\ &= t - \Big[\zeta_2 + p_{z2}t\Big]p_{z2} + p_{z2}z = (p_{x2})^2 t + (z - \zeta_2)p_{z2} \\ &= p_{x2}(x - \xi_2) + p_{z2}(z - \zeta_2) \\ &= -\rho\cos(2\alpha - 2\psi - \phi) + \Big[\sin(2\alpha - 2\psi)\xi_2 + \cos(2\alpha - 2\psi)\zeta_2\Big] \end{aligned} \tag{28}$$

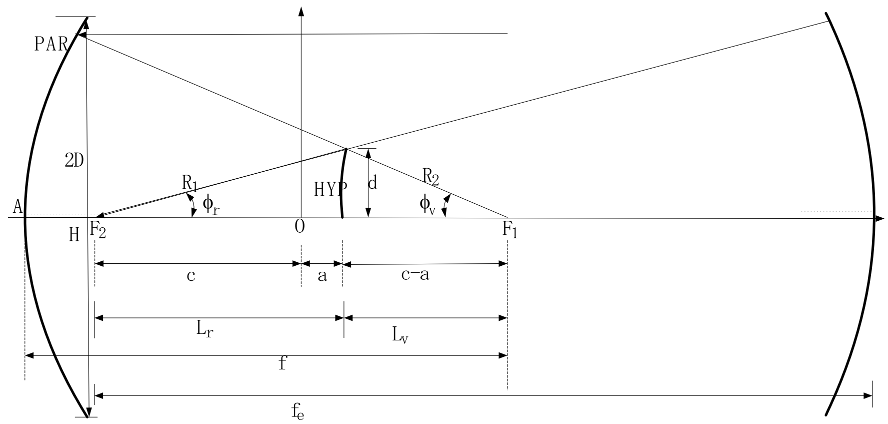

Figure 2. Parameters in Cassegrain system.

We substitute (26)–(28) into (9), then we have

$$E^r(x,z) = \sqrt{\frac{k}{j2\pi}} \left[\int_{A_1}^{A_2} + \int_{-A_2}^{-A_1} \right] \sqrt{R_1} \exp\Big[-jk\Big(S_0 + t_1 + S_{ex}\Big)\Big] d(2\alpha) \tag{29}$$

In the above equation R_1, S_0, t_1 and S_{ex} are expressed in terms of α and A_1 and A_2 are the subtention angles 2α at the edges of the parabolic and hyperbolic cylinders.

4. RESULT AND DISCUSSION

Field pattern around the caustic of a cassegrain system are determined using equation (29) by performing the integration numerically. Figure 3 contains contour plots (equi-amplitude plots) of the field around the focal region located between the two cylindrical re ectors, that point F_2 in figures. It is assumed that $kf = 100$, $a = 25$, $b = 45$, $d = 30$, $D = 130$. The location of the caustic may be observed and verified easily. It may be noted that limits of the integrals in equation (29) are selected using the following relations [see Appendix D]

$$A_1 = \phi_\nu = 2\arctan\left(\frac{D}{2f}\right)$$
$$A_2 = \arctan\left(\frac{d}{2c}\right)$$

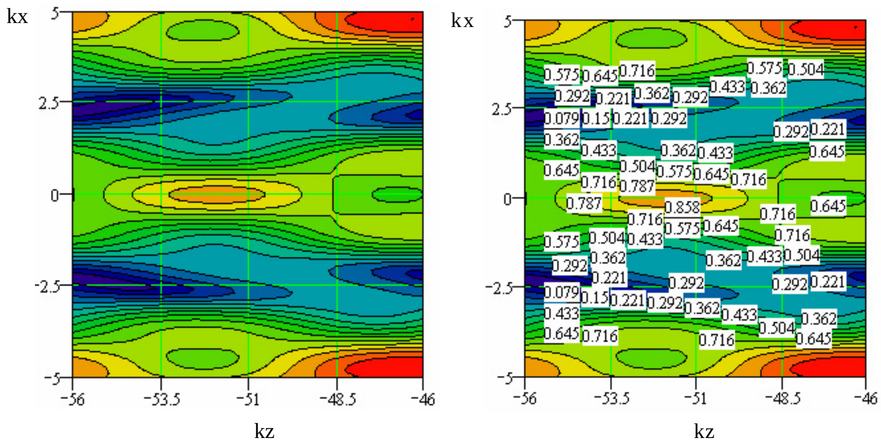

Figure 3. Contour plot for a Cassegrain antenna.

The results are compared with the results of an equivalent parabola, whose focal length is determined using the following relation [see Appendix D]

$$f_e = \left(\frac{c+a}{c-a}\right) f$$

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APPENDIX A. EVALUATION OF $F = J(t)\frac{\partial p_z}{\partial z}$

First we evaluate $\frac{\partial p_z}{\partial z}$. From (5) and (4) z can be written as

$$z = f(\xi) + \cot(2\theta - \phi_0)(x - \xi) \tag{A1}$$

We differentiate (A1) with respect to θ, then we have

$$\begin{aligned} \frac{\partial z}{\partial \theta} &= f'(\xi)\frac{\partial \xi}{\partial \theta} - \frac{2(x-\xi)}{\sin^2(2\theta - \phi_0)} - \cot(2\theta - \phi_0)\frac{\partial \xi}{\partial \theta} \\ &= -\Big[\tan\theta + \cot(2\theta - \phi_0)\Big]\frac{\partial \xi}{\partial \theta} + \frac{2t}{\sin(2\theta - \phi_0)} \\ &\quad - \frac{\cos(\theta - \phi_0)}{\sin(2\theta - \phi_0)}\frac{1}{\cos\theta}\frac{\partial \xi}{\partial \theta}\left[1 - \frac{2\cos\theta}{\cos(\theta - \phi_0)}\frac{\partial \theta}{\partial \xi}t\right] \end{aligned} \tag{A2}$$

Hence the derivative $\frac{\partial z}{\partial p_z}$ can be derived as follows.

$$
\begin{aligned}
\frac{\partial z}{\partial p_z} &= \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial p_z} = \frac{1}{2\sin(2\theta-\phi_0)}\frac{\partial z}{\partial \theta} \\
&= -\frac{\cos(\theta-\phi_0)}{2\sin^2(2\theta-\phi_0)\cos\theta}\frac{\partial \xi}{\partial \theta}\left[1+\frac{2f''(\xi)\cos^3\theta}{\cos(\theta-\phi_0)}t\right] \qquad \text{(A3)}
\end{aligned}
$$

Using (A3) and (7b) yields the final result

$$
\begin{aligned}
F &= J(t)\frac{\partial p_z}{\partial z} \\
&= \left[1+\frac{2f''(\xi)\cos^3\theta}{\cos(\theta-\phi_0)}t\right]\left[\frac{2\cos^3\theta\sin^2(2\theta-\phi_0)}{\cos(\theta-\phi_0)}f''(\xi)\right] \\
&\quad \times\left[1+\frac{2f''(\xi)\cos^3\theta}{\cos(\theta-\phi_0)}t\right]^{-1}\frac{2\cos^3\theta\sin^2(2\theta-\phi_0)}{\cos(\theta-\phi_0)}f''(\xi) \qquad \text{(A4)}
\end{aligned}
$$

APPENDIX B. DERIVATION OF THE JACOBIAN

We evaluate the Jacobian of coordinate transformation (x, z) to (ξ_1, t) with (x, z) given by (21).

$$
\begin{aligned}
D(t) &= \frac{\partial(x,z)}{\partial(\xi_1,t)} = \begin{vmatrix} \frac{\partial \xi_2}{\partial \xi_1}+\frac{\partial p_{x2}}{\partial \xi_1}t & \frac{\partial \zeta_2}{\partial \xi_1}+\frac{\partial p_{z2}}{\partial \xi_1}t \\ p_{x2} & p_{z2} \end{vmatrix} \\
&= 2\frac{\partial(\alpha-\psi)}{\partial \xi_1}t-\frac{\cos(2\alpha-\psi)}{\cos\psi}\frac{\partial \xi_2}{\partial \xi_1} \qquad \text{(B1)}
\end{aligned}
$$

where we have used the relation $\frac{\partial \zeta_2}{\partial \xi_1} = \frac{\partial \zeta_2}{\partial \xi_2}\frac{\partial \xi_2}{\partial \xi_1} = \tan\psi\frac{\partial \xi_2}{\partial \xi_1}$. The relation between (ξ_1, ζ_1) and (ξ_2, ζ_2) is given by

$$
\xi_2-\xi_1 = -\tan 2\alpha(\zeta_2-\zeta_1) \qquad \text{(B2)}
$$

and we differentiate the both sides with respect to ξ_1. Then we have

$$
\begin{aligned}
\frac{\partial \xi_2}{\partial \xi_1} &= \frac{\cos\psi}{\cos(2\alpha-\psi)}\left[1-\frac{\zeta_2-\zeta_1}{f}\frac{\cos^2\alpha}{\cos 2\alpha}\right] \\
&= \frac{\cos\psi}{\cos(2\alpha-\psi)}\frac{R_2\cos^2\alpha}{f} \qquad \text{(B3)}
\end{aligned}
$$

Furthermore we have

$$
\frac{\partial \alpha}{\partial \xi_1} = \frac{1}{2f}\cos^2\alpha, \qquad \frac{\partial \psi}{\partial \xi_1} = \frac{\partial \psi}{\partial \xi_2}\frac{\partial \xi_2}{\partial \xi_1} = \cos^2\psi\frac{a^4}{b^2}\frac{1}{\zeta_2^3}\frac{\partial \xi_2}{\partial \xi_1} \qquad \text{(B4)}
$$

Substituting (B3) and (B4) in (B1) yields

$$\begin{aligned} D(t) \;=\; & 2t\left\{\frac{\cos^2\alpha}{2f}-\frac{\cos^3\psi}{\cos(2\alpha-\psi)}\frac{a^4}{b^2\zeta_2^3}\left[1-\frac{\zeta_2-\zeta_1}{f}\frac{\cos^2\alpha}{\cos 2\alpha}\right]\right\} \\ & -\left[1-\frac{\zeta_2-\zeta_1}{f}\frac{\cos^2\alpha}{\cos 2\alpha}\right] \end{aligned} \tag{B5}$$

From Fig. 1 and simple calculation we readily find that the following relations hold

$$\begin{aligned} \zeta_2=c-R_2\cos 2\alpha, \quad \zeta_1=\frac{\xi_1^2}{4f}-f+c=-f\frac{\cos 2\alpha}{\cos^2\alpha}+c; \\ 1-\frac{\zeta_2-\zeta_1}{f}\frac{\cos^2\alpha}{\cos 2\alpha}=\frac{R_2}{f}\cos^2\alpha \end{aligned} \tag{B6}$$

Hence $D(t)$ can be written as

$$D(t)=\frac{\cos^2\alpha}{f}\left\{\left[1-\frac{2\cos^3\psi}{\cos(2\alpha-\psi)}\frac{a^4}{b^2\zeta_2^3}R_2\right]t-R_2\right\} \tag{B7}$$

By using the relations

$$\cos 2\alpha=\frac{c-\zeta_2}{R_2}, \qquad \sin 2\alpha=\frac{\xi_2}{R_2} \tag{B8}$$

$\cos(2\alpha-\psi)$ in (B7) can be expressed by

$$\begin{aligned} \cos(2\alpha-\psi) \;&=\; \frac{c-\zeta_2}{R_2}\cos\psi+\frac{\xi_2}{R_2}\sin\psi \\ &=\; \frac{1}{R_2\sqrt{R_1R_2}}\left[\frac{b}{a}\zeta_2(c-\zeta_2)+\frac{a}{b}\xi_2^2\right] \\ & \quad\; \frac{1}{R_2\sqrt{R_1R_2}}\left[\frac{b}{a}c\zeta_2-ab\left(\frac{\zeta_2^2}{a^2}-\frac{\xi_2^2}{b^2}\right)\right] \\ &=\; \frac{1}{R_2\sqrt{R_1R_2}}\frac{b}{a}(c\zeta_2-a^2)=\frac{b}{\sqrt{R_1R_2}} \end{aligned} \tag{B9}$$

where we have used the relation $R_2=\sqrt{(c-\zeta_2)^2+\xi_2^2}=\frac{c\zeta_2-a^2}{a}$. Then the coefficient of t in (B7) is simplified to

$$\begin{aligned} U \;=\; & 1-\frac{2\cos^3\psi}{\cos(2\alpha-\psi)}\frac{a^4}{b^2\zeta_2^3}R_2=1-\frac{2abR_2}{(R_1R_2)^{\frac{3}{2}}}\frac{aR_2\sqrt{R_1R_2}}{b(c\zeta_2-a^2)} \\ =\; & 1-\frac{2a}{R_1}=\frac{R_2}{R_1} \end{aligned} \tag{B10}$$

Hence $D(t)$ in (B7) becomes

$$D(t) = \frac{\cos^2\alpha}{f} R_2 \left[1 - \frac{t}{R_1}\right] \tag{B11}$$

This shows that the ray is focused at point F_2.

APPENDIX C. EVALUATION OF $F = J(t)\frac{\partial p_{z2}}{\partial z}$

We now evaluate the integrand of (9) to derive the expression which is valid at the focal point F. From the relation

$$z = \zeta_2 + \frac{p_{z2}}{p_{x2}}(x - \xi_2) = \zeta_2 + \cot(2\alpha - 2\psi)(x - \xi_2) \tag{C1}$$

we have

$$\begin{aligned}
\frac{\partial z}{\partial \xi_2} &= \frac{\partial \zeta_2}{\partial \xi_2} - \frac{2(x-\xi_2)}{\sin^2(2\alpha - 2\psi)} \frac{\partial(\alpha - \psi)}{\partial \xi_2} - \cot(2\alpha - 2\psi) \\
&= -\frac{\cos(2\alpha - \psi)}{\cos\psi \sin(2\alpha - 2\psi)} + \frac{2t}{\sin(2\alpha - 2\psi)} \frac{\partial(\alpha-\psi)}{\partial \xi_2} \\
&\quad \frac{1}{\sin(2\alpha - 2\psi)} \frac{\partial \xi_1}{\partial \xi_2}\left[2t\frac{\partial(\alpha-\psi)}{\partial \xi_1} - \frac{\cos(2\alpha-\psi)}{\cos\psi}\frac{\partial \xi_2}{\partial \xi_1}\right]
\end{aligned} \tag{C2}$$

By using the relations

$$\begin{aligned}
\frac{\partial z}{\partial p_{z2}} &= \frac{\partial z}{\partial \xi_2}\frac{\partial \xi_2}{\partial p_{z2}}, \qquad \frac{\partial p_{z2}}{\partial z} = \frac{\partial \xi_2}{\partial z}\frac{\partial p_{z2}}{\partial \xi_2}; \\
\frac{\partial p_{z2}}{\partial \xi_2} &= 2\sin(2\alpha - 2\psi)\frac{\partial(\alpha-\psi)}{\partial \xi_2} \\
&= 2\sin(2\alpha - 2\psi)\frac{\partial \xi_1}{\partial \xi_2}\frac{\partial(\alpha-\psi)}{\partial \xi_1}
\end{aligned} \tag{C3}$$

we have

$$\begin{aligned}
\frac{\partial p_{z2}}{\partial z} &= \frac{\partial \xi_2}{\partial z}\frac{\partial p_{z2}}{\partial \xi_2} \\
&= 2\sin(2\alpha - 2\psi)\frac{\partial \xi_1}{\partial \xi_2}\frac{\partial(\alpha-\psi)}{\partial \xi_1}\sin(2\alpha - 2\psi)\frac{\partial \xi_2}{\partial \xi_1} \\
&\quad \times\left[2t\frac{\partial(\alpha-\psi)}{\partial \xi_1} - \frac{\cos(2\alpha-\psi)}{\cos\psi}\frac{\partial \xi_2}{\partial \xi_1}\right]^{-1}
\end{aligned} \tag{C4}$$

There we have

$$\begin{aligned}\frac{D(t)}{D(0)}\frac{\partial p_{z2}}{\partial z} &= \left[2t\frac{\partial(\alpha-\psi)}{\partial\xi_1} - \frac{\cos(2\alpha-\psi)}{\cos\psi}\frac{\partial\xi_2}{\partial\xi_1}\right]\left[\frac{\cos\psi}{\cos(2\alpha-\psi)}\frac{\partial\xi_2}{\partial\xi_2}\right]\frac{\partial p_{z2}}{\partial z}\\ &= 2\sin^2(2\alpha-2\psi)\frac{\partial(\alpha-\psi)}{\partial\xi_1}\frac{\cos\psi}{\cos(2\alpha-\psi)}\frac{\partial\xi_1}{\partial\xi_2} \qquad \text{(C5)}\end{aligned}$$

From the results of Appendix B we have

$$\begin{aligned}\frac{D(t)}{D(0)}\frac{\partial p_{z2}}{\partial z} &= 2\sin^2(2\alpha-2\psi)\frac{\cos^2\alpha}{2f}\frac{R_2}{R_1}\frac{\cos\psi}{\cos(2\alpha-\psi)}\frac{\cos(2\alpha-\psi)}{\cos\psi}\frac{f}{R_2\cos^2\alpha}\\ &= \frac{\sin^2(2\alpha-2\psi)}{R_1} \qquad \text{(C6)}\end{aligned}$$

where

$$\begin{aligned}R_1 &= \frac{c\zeta_2+a^2}{a} = a\frac{c\cos\psi+\sqrt{a^2\cos^2\psi-b^2\sin^2\psi}}{\sqrt{a^2\cos^2\psi-b^2\sin^2\psi}},\\ \cos\psi &= \frac{c+a\cos2\alpha}{\sqrt{a^2+c^2+2ac\cos2\alpha}}, \quad \sin\psi = \frac{a\sin2\alpha}{\sqrt{a^2+c^2+2ac\cos2\alpha}},\\ \xi_1 &= 2f\tan\alpha, \qquad \zeta_1 = c-\frac{2f\cos2\alpha}{1+\cos2\alpha}\\ \xi_2 &= \frac{b^2\sin\psi}{\sqrt{a^2\cos^2\psi-b^2\sin^2\psi}}, \quad \zeta_2 = \frac{a^2\cos\psi}{\sqrt{a^2\cos^2\psi-b^2\sin^2\psi}} \qquad \text{(C7)}\end{aligned}$$

APPENDIX D. PARAMETERS OF CASSEGRAIN ANTENNA

D.1. $\tan\frac{\phi_v}{2}=\frac{D}{2f}$ and $\tan\frac{\phi_r}{2}=\frac{D}{2f_e}$

The equation of the parabolic cylinder is given by (16). Hence we have

$$OA = -f+c, \qquad AH = \frac{D^2}{4f}, \qquad FH = f-\frac{D^2}{4f} \qquad \text{(D1)}$$

and

$$\tan\phi_v = \frac{D}{f-\dfrac{D^2}{4f}} = \frac{2\cdot\dfrac{D}{2f}}{1-\left(\dfrac{D}{2f}\right)^2} = \frac{2\tan\dfrac{\phi_v}{2}}{1-\tan^2\dfrac{\phi_v}{2}} \qquad \text{(D2)}$$

From (D2) we have $\tan\frac{\phi_v}{2} = \frac{D}{2f}$. It may be noted that D is height of the edge of the parabolic cylinder from horizontal axis.

Similarly we have

$$\tan\frac{\phi_r}{2} = \frac{D}{2f_e} \tag{D3}$$

where f_e is the focal length of the equivalent parabola.

D.2. $e = \frac{\sin\frac{1}{2}(\phi_v+\phi_r)}{\sin\frac{1}{2}(\phi_v-\phi_r)}$

From the similarities of the triangles we have

$$\frac{f_e}{f} = \frac{L_r}{L_v} = \frac{c+a}{c-a} = \frac{e+1}{e-1} = \frac{\tan\frac{1}{2}\phi_v}{\tan\frac{1}{2}\phi_r} \tag{D4}$$

The last term is obtained from the results in (D1). From the above equation e is obtained as

$$e = \frac{\tan\frac{1}{2}\phi_v + \tan\frac{1}{2}\phi_r}{\tan\frac{1}{2}\phi_v - \tan\frac{1}{2}\phi_r} = \frac{\sin\frac{1}{2}(\phi_v+\phi_r)}{\sin\frac{1}{2}(\phi_v-\phi_r)} = \frac{L_r+L_v}{L_r-L_v} \tag{$D5$}$$

and

$$1-\frac{1}{e} = \frac{2L_v}{L_r+L_v} = 1 - \frac{\sin\frac{1}{2}(\phi_v+\phi_r)}{\sin\frac{1}{2}(\phi_v-\phi_r)} \tag{$D6$}$$

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