Progress In Electromagnetics Research, PIER 73, 107–129, 2007

RADIATION CHARACTERISTICS OF THE WOOD LENS USING MASLOV'S METHOD

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Abstract—The field refracted by a Wood lens is determined analytically and numerically in the focal region by Maslov's method. Results are compared with those obtained by weak focusing approximation and Kirchhoff's diffraction integral. Agreement among them is fairly good when the parameters satisfy the conditions associated to the approximation.

1. INTRODUCTION

Asymptotic Ray Theory (ART) is widely used to describe the electromagnetic wave in both homogeneous and inhomogeneous media [1–4], but the field in caustics and shadow boundary where ART shows singularities have to be treated as separate problem. Unfortunately these singularities are often the points of great interest because of their usefulness in practical application. Usually, Kirchhoff integral is used to predict the fields at these points. There is an alternative method based on Maslov's theory. Maslov's asymptotic theory uses the idea that combines the simplicity of ART and generality of Fourier transform. A summary of this method has been given by Kravtsov [3] and Ziolkowski and Deschamps [4] in which they showed how to apply this method to propagation and radiation of waves in homogeneous and inhomogeneous media.

The Maslov's method is considered to be based on the following properties of ray and wave. **(a):** Ray trajectory in three dimensions may be described in terms of the Hamilton's canonical equation. The ray trajectory which is usually considered in spatial domain $\mathbf{X}$, but it can also be viewed from more generally in the six-dimensional phase space consisting of position vector $\mathbf{X}$ and wave vector $\mathbf{p}$. Therefore we can describe the ray in the mixed space in which part of components of position vector are replaced by that of the corresponding components of wave vector $\mathbf{p}$ since these are related by Hamilton's equation. The ray has singularity at caustics when we see it in the spectral domain $\mathbf{X}$, but generally it behaves regular in the mixed space since the location of corresponding singularity in the two spaces is usually different. Pictorial interpretation of the concept was given by Ziolkowski and Deschamps [4]. **(b):** The solution of the wave equation is expressed in terms of the superposition of plane wave spectrum through Fourier transform. It is known that we can derive asymptotic ray expression from Fourier integral of unknown integrand if we apply the stationary phase method of integration to the integral. The solution may be identified to the expression derived from Hamilton optics. From this process we can determine the formal expression of the integrand of the Fourier integral. **(c):** The ray derived in spatial domain which is associated with the integrand discussed in (b) is represented in appropriate mixed space using the solution of Hamilton's equation. If we apply the inverse Fourier transform to the expression we can return to the expression in spatial domain. Usually it has finite values even in caustic region because of the properties of Fourier transform. We can carry out this integral using higher order stationary phase method of integration or numerical quadrature. Of course if we apply the stationary phase method of integration to the Fourier integral, asymptotic ray expression is obtained.

Although the Maslov's method have attracted an attention of many investigators, but literature treating the application of the method to physical problems are relatively few. Chapman and Dummond [5] used it to construct the seismograms, Gorman and his associates [6] showed how to construct the asymptotic solution for various kinds of differential equation, Hongo and Ji [7–11] showed how to apply the method to predict the field in the focal region for various kinds of focusing systems. Hongo and Kobayashi [12] applied the method to study the radiation characteristics of plano-convex lens antenna. Aziz et al. determined the field in focal region of the two dimensional Cassegrain system [13] while Ghaffar et al. analyzed the three dimensional Cassegrain system [14]. In present work, field refracted by a Wood lens is determined analytically and numerically in the focal region by using the Maslov's method.

2. HAMILTON'S EQUATION AND SOLUTION

Consider the distribution of permittivity given by the following expression

$$\epsilon = \epsilon_c[1 - b^2(x^2 + y^2) + cb^4(x^2 + y^2)^2] \tag{1a}$$

where b and c are the constants. The Hamilton's equation for the medium described by Equation (1a) is given by

$$\frac{dx}{dt} = p_x, \qquad \frac{dy}{dt} = p_y, \qquad \frac{dz}{dt} = p_z \tag{1b}$$

$$\frac{dp_x}{dt} = -\frac{1}{2}\frac{\partial\epsilon}{\partial x}, \quad \frac{dp_y}{dt} = -\frac{1}{2}\frac{\partial\epsilon}{\partial y}, \quad \frac{dp_z}{dt} = -\frac{1}{2}\frac{\partial\epsilon}{\partial z} \tag{1c}$$

It is of interest to determine the solution of Hamilton's equation in medium defined by (1a) for two situations of permittivity. One situation deals when $c = 0$ while other situation deals with situation when value of c is very small in expression for permittivity. Throughout the discussion, we have labelled these situations as case 1 and case 2 respectively.

Case 1: For medium defined by Equation (1a) with $c = 0$, the solution of Hamilton's equation is given by

$$\begin{aligned} x &= \xi\cos\psi, \quad y = \eta\cos\psi, \quad z = p_z t \\ p_x &= -\beta\xi\sin\psi, \ p_y = -\beta\eta\sin\psi, \ p_z = \sqrt{\epsilon - p_x^2 - p_y^2} = \sqrt{\epsilon_c - \beta^2 r_0^2}, \\ \psi &= \beta t, \quad \beta = \sqrt{\epsilon_c}b, \quad r_0^2 = \xi^2 + \eta^2 \end{aligned} \tag{1d}$$

Case 2: When in Equation (1a), the parameter c is very small, we may write approximate solution for Hamilton's equation as

$$\begin{aligned} x &= \xi[(1+g)\cos\psi - g\cos 3\psi] \\ y &= \eta[(1+g)\cos\psi - g\cos 3\psi], \ z = p_z t \\ p_x &= -\beta\xi[(1+g)\sin\psi - 3g\sin 3\psi] \\ p_y &= -\beta\eta[(1+g)\sin\psi - 3g\sin 3\psi], \\ p_z &= \sqrt{\epsilon - p_x^2 - p_y^2} \\ &= \sqrt{\epsilon_c(1 - b^2 r_0^2 F_1 + cb^4 r_0^4[(1+g)\cos\psi - g\cos 3\psi]^4)} \end{aligned} \tag{1e}$$

where

$$F_1 = [(1+g)\cos\psi - g\cos 3\psi]^2 + [(1+g)\sin\psi - 3g\sin 3\psi]^2$$

In above equation p_z has been assumed in the form

$$p_z = \sqrt{\epsilon_c(1 - b'^{\,2} r_0^2 + c' b'^{\,4} r_0^4)}, \quad g = \frac{c' b'^{\,2} r_0^2}{4}, \quad b' = \lambda b, \quad c' = \kappa c \tag{1f}$$

Solution for small c may be considered as perturbation to the corresponding solution for $c = 0$ and g is perturbation parameter. The parameters λ and κ are determined so that (1e) give better solution and this point has been discussed in next section.

3. DERIVATION OF THE FIELD EXPRESSION FOR WOOD LENS

The Wood lens is shown in Figure 1. The distribution of the permittivity in region occupied by the Wood lens is given by Equation (1a) and the parameters b and c are related with the focal length of the Wood lens. The thickness of the lens is L. We consider both cases of permittivity one by one.

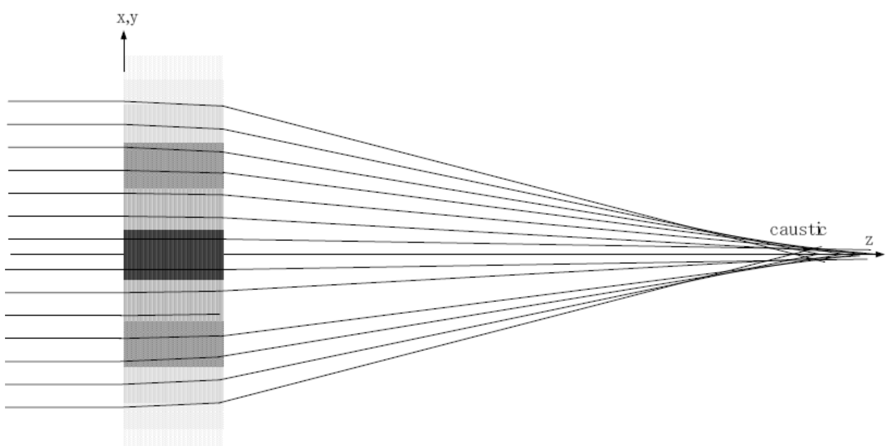

Figure 1. Profile of dielectric constant and ray trace of wood lens.

Case 1: First we discuss the situation considering $c = 0$. For $c = 0$, the Cartesian coordinates of refraction point at the rear face (ξ_1, η_1) and components of associated wave vector are given by

$$\xi_1 = \xi \cos\psi_1, \quad \eta_1 = \eta \cos\psi_1, \quad \zeta_1 = L \quad \psi_1 = \beta t_1$$
$$p_{x0} = -\beta\xi \sin\psi_1, \quad p_{y0} = -\beta\eta \sin\psi_1, \quad p_{z0} = \sqrt{\epsilon_c - \beta^2 r_0^2}, \quad p_{z0} t_1 = L \tag{2a}$$

t_1 is the arc length of the ray for region occupying the Wood lens. In Equation (2a), (ξ, η) are the Cartesian coordinates of refraction point of the front face of the Wood lens.

The coordinates of the ray after passing through the lens, that is $z > L$, are given by

$$x = \xi_1 + p_{x1}t, \quad y = \eta_1 + p_{y1}t, \quad z = \zeta_1 + p_{z1}t$$
$$p_{x1} = p_{x0}, \quad p_{y1} = p_{y0}, \quad p_{z1} = \sqrt{1 - p_{x1}^2 - p_{y1}^2} = \sqrt{1 - \beta^2 r_0^2 \sin^2 \psi_1} \tag{2b}$$

where t signifies the arc length of the ray after passing through the Wood lens. The GO solution is given by

$$u(x, y, z) = [\Xi J]^{-\frac{1}{2}} \exp[-jk(\Psi_0 + t)] \tag{3a}$$

where

$$\begin{aligned} J(t) \;=\; & \frac{1}{\Xi}\left[\frac{\beta^4 r_0^2 \sin^4 \psi_1}{p_{z1}^2} + \frac{\beta^7 r_0^4 L \sin^3 \psi_1 \cos \psi_1}{p_{z1}^2 p_{z0}^3}\right. \\ & \left. +\beta^2 \sin \psi_1 \left(\sin \psi_1 + \cos \psi_1 \frac{\beta^3 r_0^2 L}{p_{z0}^3}\right)\right] t^2 \\ & -\frac{1}{\Xi}\left\{\frac{\beta^3 r_0^2 \sin^3 \psi_1 \cos \psi_1}{p_{z1}^2} + \frac{\beta^6 L r_0^4 \sin^2 \psi_1 \cos^2 \psi_1}{p_{z1}^2 p_{z0}^3}\right. \\ & \left. + \left[2\beta \sin \psi_1 \cos \psi_1 + \cos 2\psi_1 \frac{\beta^4 r_0^2 L}{p_{z0}^3}\right]\right\} t + 1 \end{aligned} \tag{3b}$$

$$\Xi \;=\; \cos \psi_1 \left[\cos \psi_1 - \sin \psi_1 \frac{\beta^3 r_0^2 L}{p_{z0}^3}\right]$$

$$\Psi_0 = \int_0^{t_1} \epsilon(t) dt = \int_0^{t_1} \epsilon_c \left[1 - b^2 r_0^2 \cos^2 \beta t\right] dt = \epsilon_c \left(1 - \frac{b^2 r_0^2}{2}\right) t_1 - \frac{\beta r_0^2}{4} \sin 2\psi_1 \tag{3c}$$

In the above equation, Ψ_0 is the phase difference between front and rear faces of the Wood lens. For detail calculations of jacobian, see Appendix A.

The GO-field contains singularity at the focal point. Using Maslov's method our interest is to find the uniform field expression valid in focal region. The uniform expression which is valid in the focal region is given by

$$\begin{aligned} u(\mathbf{r}) \;=\; & \frac{k}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z(\xi_1, \eta_1) \left[\frac{1}{D(0)} \frac{\partial((p_{x1}, p_{y1}, z)}{\partial(\xi, \eta, t)}\right]^{-\frac{1}{2}} \\ & \exp\left[-jk\Psi_2(p_{x1}, p_{y1}, z) + jk[p_{x1}x + p_{y1}y]\right] dp_{x1} dp_{y1} \end{aligned} \tag{4}$$

$Z(\xi_1, \eta_1)$ is the initial value at the rare face of the lens. Phase $\Psi_2((p_{x1}, p_{y1}, z)$ is determined as

$$\begin{aligned}
\Psi_2(p_{x1}, p_{y1}, z) &= \Psi_0 + t_2 - p_{x1}x(p_{x1}, p_{y1}, t_2) - p_{y1}y(p_{x1}, p_{y1}, t_2) \\
&= \Psi_0 + t_2 - p_{x1}(\xi_1 + p_{x1}t_2) - p_{y1}(\eta_1 + p_{y1}t_2) \\
&= \Psi_0 + t_2 - p_{x1}\xi_1 - p_{y1}\eta_1 - (p_{x1}^2 + p_{y1}^2)t_2 \\
&= \Psi_0 + p_{z1}^2 t_2 - p_{x1}\xi_1 - p_{y1}\eta_1 \\
&= \Psi_0 + p_{z1}(z - L) - p_{x1}\xi_1 - p_{y1}\eta_1 \\
&= \Psi_0 + \beta r_0^2 \sin\psi_1 \cos\psi_1 + \sqrt{(1 - \beta^2 r_0^2 \sin^2\psi_1)}(z - L)
\end{aligned}$$

where $t_2 = z - L > 0$ is the horizontal distance from the rare face of the Wood lens.

In Equation (4), quantities in square bracket, are determined as

$$\begin{aligned}
\frac{\partial(p_{x1}, p_{y1}, z)}{\partial(\xi, \eta, t)} &= \begin{vmatrix} \dfrac{\partial p_{x1}}{\partial \xi} & \dfrac{\partial p_{y1}}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial p_{x1}}{\partial \eta} & \dfrac{\partial p_{y1}}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ 0 & 0 & p_{z1} \end{vmatrix} = p_{z1}\left[\frac{\partial p_{x1}}{\partial \xi}\frac{\partial p_{y1}}{\partial \eta} - \frac{\partial p_{y1}}{\partial \xi}\frac{\partial p_{x1}}{\partial \eta}\right] \\
&= \beta^2 \sin\psi_1 p_{z1}\left[\sin\psi_1 + \frac{\beta^3 L r_0^2}{p_{z0}^3}\cos\psi_1\right] \\
D(0) &= \cos\psi_1\left[\cos\psi_1 - \frac{\beta^3 L r_0^2}{p_{z0}^3}\sin\psi_1\right]p_{z1}
\end{aligned}$$

Therefore we have result

$$\begin{aligned}
\left[\frac{1}{D(0)}\frac{\partial(p_{x1}, p_{y1}, z)}{\partial(\xi, \eta, t)}\right]^{-\frac{1}{2}} &= \frac{1}{\beta}\sqrt{\frac{\cos\psi_1}{\sin\psi_1}}\left[\sin\psi_1 + \frac{\beta^3 L r_0^2}{p_{z0}^3}\cos\psi_1\right]^{-\frac{1}{2}} \\
&\quad \left[\cos\psi_1 - \frac{\beta^3 L r_0^2}{p_{z0}^3}\sin\psi_1\right]^{\frac{1}{2}} \qquad (5)
\end{aligned}$$

Transforming the integration variables (p_{x1}, p_{y1}) into (r_0, δ) related by $\xi = r_0\cos\delta$, $\eta = r_0 \sin\delta$, that is,

$$\begin{aligned}
p_{x1} &= -\beta r_0 \sin\psi_1 \cos\delta \\
p_{y1} &= -\beta r_0 \sin\psi_1 \sin\delta \\
p_{x1}x + p_{y1}y &= -\beta r_0 r \sin\psi_1 \cos(\delta - \phi)
\end{aligned}$$

where $x = r\cos\phi$ and $y = r\sin\phi$ are observation coordinates.

Equation (4) reduces to

$$\begin{aligned}
u(\mathbf{r}) \;&=\; \frac{k\beta}{2\pi}\int_0^a\int_0^{2\pi} T(r_0)\sqrt{\cos\psi_1\sin\psi_1}\left[\sin\psi_1+\frac{\beta^3 L r_0^2}{p_{z0}^3}\cos\psi_1\right]^{\frac{1}{2}} \\
&\quad\times\left[\cos\psi_1-\frac{\beta^3 L r_0^2}{p_{z0}^3}\sin\psi_1\right]^{\frac{1}{2}}\exp\Big[-jk\Psi_2(r_0)\Big] \\
&\quad\exp\Big[-jk\beta r r_0\sin\psi_1\cos(\delta-\phi)\Big]r_0 dr_0 d\delta \\
&=\; k\beta\int_0^a T(r_0)\sqrt{\cos\psi_1\sin\psi_1}\left[\sin\psi_1+\frac{\beta^3 L r_0^2}{p_{z0}^3}\cos\psi_1\right]^{\frac{1}{2}} \\
&\quad\left[\cos\psi_1-\frac{\beta^3 L r_0^2}{p_{z0}^3}\sin\psi_1\right]^{\frac{1}{2}} \\
&\quad\times J_0(k\beta r r_0\sin\psi_1)\exp\Big[-jk\Psi_2(r_0)\Big]r_0 dr_0
\end{aligned} \tag{6a}$$

where a is the radius of the lens. $T(r_0)$ and other related parameters are given by

$$\begin{aligned}
T(r_0) \;&=\; T_1T_2 \qquad T_1=\frac{2}{1+\sqrt{\epsilon_c-\beta^2 r_0^2}}, \\
T_2 \;&=\; \frac{2\sqrt{\epsilon_c-\beta^2 r_0^2\cos^2\psi_1}\cos\theta_{i2}}{\cos\theta_{i2}+\sqrt{\epsilon_c-\beta^2 r_0^2\cos^2\psi_1}\cos\theta_{t2}} \\
\theta_{i2} \;&=\; \tan^{-1}\frac{\sqrt{p_{x0}^2+p_{y0}^2}}{p_{z0}}=\tan^{-1}\frac{\beta r_0\sin\psi_1}{\sqrt{\epsilon_c-\beta^2 r_0^2}}, \\
\theta_{t2} \;&=\; \tan^{-1}\frac{\sqrt{p_{x1}^2+p_{y1}^2}}{p_{z1}}=\tan^{-1}\frac{\sqrt{p_{x0}^2+p_{y0}^2}}{\sqrt{1-(p_{x0}^2+p_{y0}^2)}} \\
&=\; \tan^{-1}\frac{\beta r_0\sin\psi_1}{\sqrt{1-\beta^2 r_0^2\sin^2\psi_1}} \\
t_1 \;&=\; \frac{L}{\sqrt{\epsilon_c-\beta^2 r_0^2}}
\end{aligned} \tag{6b}$$

and θ_{i2} and θ_{t2} are angles of the incidence and the refraction of the ray at the rear face of the lens.

In deriving the above expression we have used the following

Jacobian of the variable transformation

$$dp_{x1}dp_{y1}=\frac{\partial(p_{x1},p_{y1})}{\partial(\xi,\eta)}r_0dr_0d\delta=\beta^2\sin\psi_1\left[\sin\psi_1+\frac{\beta^3Lr_0^2}{p_{z0}^3}\cos\psi_1\right]r_0dr_0d\delta$$

and standard integral relation

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(-ja\cos(\delta-\phi))d\delta = J_0(a)$$

where J_0 is Bessel function of zeroth order. Transformation of observation point from cartesian to polar coordinates is $x = r\cos\phi$, $y = r\sin\phi$.

Case 2: Now consider the situation assuming that c is very small. The uniform expression which is valid in the focal region is given by

$$\begin{aligned} u(\mathbf{r}) \;=\; & \frac{k}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Z(\xi_1,\eta_1)\left[\frac{1}{D(0)}\frac{\partial((p_{x1},p_{y1},z)}{\partial(\xi,\eta,t)}\right]^{-\frac{1}{2}} \\ & \exp\left[-jk\Psi_3(p_{x1},p_{y1},z)+jk[p_{x1}x+p_{y1}y]\right]dp_{x1}dp_{y1} \end{aligned} \tag{7a}$$

Quantities in square bracket of Equation (7a) are

$$\begin{aligned} \frac{\partial(p_{x1},p_{y1},z)}{\partial(\xi,\eta,t)} \;=\; & \beta^2p_{z1}\Bigg\{\left[(1+2g)\sin^2\psi_1-6g\sin3\psi_1\sin\psi_1\right] \\ & +\cos\psi_1\sin\psi_1\frac{\beta L}{p_{z0}^3}r_0^2\left[b'^{\,2}\epsilon_c-2\epsilon_cc'b'^{\,4}r_0^2\right] \\ & +g\left[2\cos\psi_1\sin\psi_1-3\sin3\psi_1\cos\psi_1\right. \\ & \left.-9\cos3\psi_1\sin\psi_1\right]\frac{\beta\beta'^{\,2}L}{p_{z0}^3}r_0^2\Bigg\} \\ D(0) \;=\; & p_{z1}\Bigg\{\left[(1+2g)\cos^2\psi_1-2g\cos\psi\cos3\psi_1\right] \\ & -\sin\psi_1\cos\psi_1\frac{\beta L}{p_{z0}^3}(\epsilon_cb'^{\,2}-2\epsilon_cc'b'^{\,4}r_0^2)r_0^2 \\ & -g[2\sin\psi_1\cos\psi_1-\sin\psi_1\cos3\psi_1 \\ & -3\cos\psi_1\sin3\psi_1]\frac{\beta\beta'^{\,2}L}{p_{z0}^3}r_0^2\Bigg\} \end{aligned}$$

For detail calculation, see Appendix B and Appendix C. The phase function Ψ_3 and conversion of (p_{x1}, p_{y1}) into (r_0, δ) yields

$$\begin{aligned}
\Psi_3(p_{x1}, p_{y1}, z) &= \Psi_0' + p_{z1}(z-L) - p_{x1}\xi_1 - p_{y1}\eta_1 \\
&= \Psi_0' + \beta r_0^2 \left[(1+g)\cos\psi_1 - g\cos 3\psi_1\right] \\
&\quad \left[(1+g)\sin\psi_1 - 3g\sin 3\psi_1\right] \\
&\quad + \sqrt{\left\{1-\beta^2 r_0^2[(1+2g)\sin^2\psi_1 - 6g\sin\psi_1\sin 3\psi_1]\right\}}(z-L) \\
&= \Psi_0' + \beta r_0^2 \left[(1+2g)\sin\psi_1\cos\psi_1 - 3g\sin 3\psi_1\cos\psi_1\right. \\
&\quad \left. - g\sin\psi_1\cos 3\psi_1\right] \\
&\quad + \sqrt{\left\{1-\beta^2 r_0^2[(1+2g)\sin^2\psi_1 - 6g\sin\psi_1\sin 3\psi_1]\right\}}(z-L) \\
dp_{x1}dp_{y1} &= \frac{\partial(p_{x1}, p_{y1})}{\partial(\xi, \eta)} r_0 dr_0 d\delta \\
&= \beta^2 \left\{ \left[(1+2g)\sin^2\psi_1 - 6g\sin 3\psi_1\sin\psi_1\right] \right. \\
&\quad + \cos\psi_1\sin\psi_1 \frac{\beta L}{p_{z1}^3} r_0^2 \left[b'^{\,2}\epsilon_c - \epsilon_c c' b'^{\,4} r_0^2\right] \\
&\quad + g\left[(2\cos\psi_1\sin\psi_1 - 3\sin 3\psi_1\cos\psi_1\right. \\
&\quad \left.\left. - 9\cos 3\psi_1\sin\psi_1\right] \frac{\beta\beta'^2 L}{p_{z1}^3} r_0^2 \right\} r_0 dr_0 d\delta
\end{aligned}$$

Expression for Ψ_0' has been derived in Appendix D. It may be noted that

$$\begin{aligned}
p_{x1} &= -\beta r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\cos\delta \\
p_{y1} &= -\beta r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\sin\delta \\
p_{x1}x + p_{y1}y &= -\beta r_0 r[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\cos(\delta - \phi)
\end{aligned}$$

The uniform expression becomes

$$\begin{aligned}
u(r,z) &= k\int_0^a T(r_0) S^{\frac{1}{2}} J_0[k\beta r r_0\left((1+g)\sin\psi_1 - 3g\sin 3\psi_1\right)] \\
&\quad \exp[-jk\Psi_3] r_0 dr_0
\end{aligned} \tag{7b}$$

where S has been obtained as

$$\begin{aligned}
S &= \beta\left[(1+4g)\sin^2\psi_1\cos^2\psi_1 - 2g\sin^2\psi_1\cos\psi_1\cos 3\psi_1\right. \\
&\quad \left. -6g\sin\psi_1\cos^2\psi_1\sin 3\psi_1\right] - \frac{\beta L}{p_{z1}^3} r_0^2\left[b'^{\,2}\epsilon_c - \epsilon_c c' b'^{\,4} r_0^2\right]\sin^3\psi_1\cos\psi_1 \\
&\quad -2g\frac{\beta\beta'^{\,2} L}{p_{z0}^3} r_0^2(\sin^2\psi_1 - 6\sin 3\psi_1\sin\psi_1)\sin\psi_1\cos\psi_1
\end{aligned}$$

$$
\begin{aligned}
&-g\sin^2\psi_1[2\sin\psi_1\cos\psi_1-\sin\psi_1\cos 3\psi_1-3\cos\psi_1\sin 3\psi_1]\frac{\beta\beta'^{\,2}L}{p_{z0}^3}r_0^2\\
&+\frac{\beta L}{p_{z0}^3}r_0^2\left[b'^{\,2}\epsilon_c-\epsilon_c c' b'^{\,4}r_0^2\right]\cos^3\psi_1\sin\psi_1\\
&+g\frac{\beta\beta'^{\,2}L}{p_{z0}^3}r_0^2(\cos^2\psi_1-3\cos 3\psi_1\cos\psi_1)\cos\psi_1\sin\psi_1\\
&-\sin^2\psi_1\cos^2\psi_1\frac{\beta^2L^2}{p_{z0}^6}(\epsilon_c^2 b'^{\,4}-2\epsilon_c^2c'b'^{\,6}r_0^2)r_0^4\\
&-g\sin\psi_1\cos\psi_1[2\sin\psi_1\cos\psi_1-\sin\psi_1\cos 3\psi_1\\
&-3\cos\psi_1\sin 3\psi_1]\frac{\beta^2\beta'^{\,4}L^2}{p_{z0}^6}r_0^4+g[2\sin\psi_1\cos\psi_1-3\sin 3\psi_1\cos\psi_1\\
&-9\cos 3\psi_1\sin\psi_1]\frac{\beta\beta'^{\,2}L}{p_{z0}^3}r_0^2\left(\cos^2\psi_1-\sin\psi_1\cos\psi_1\frac{\beta\beta'^{\,2}L}{p_{z0}^3}r_0^2\right) \quad (7c)
\end{aligned}
$$

where

$$
\begin{aligned}
T(r_0) &= T_1T_2, \qquad T_1=\frac{2}{1+\sqrt{\epsilon_c-\beta^2r_0^2+c\beta^2b^2r_0^4}},\\
T_2 &= \frac{2\sqrt{\epsilon_c-\beta^2r_0^2\cos^2\psi_1+c\beta^2b^2r_0^4\cos^2\psi_1}\cos\theta_{i2}}{\cos\theta_{i2}+\sqrt{\epsilon_c-\beta^2r_0^2\cos^2\psi_1+c\beta^2b^2r_0^4\cos^2\psi_1}\cos\theta_{t2}}\\
\theta_{i2} &= \tan^{-1}\frac{\sqrt{p_{x0}^2+p_{y0}^2}}{p_{z0}}=\tan^{-1}\frac{\beta r_0[(1+g)\sin\psi_1-3g\sin 3\psi_1]}{\sqrt{\epsilon_c(1-b'^{\,2}r_0^2+c'b'^{\,4}r_0^4)}},\\
\theta_{t2} &= \tan^{-1}\frac{\beta r_0[(1+g)\sin\psi_1-3g\sin 3\psi_1]}{\sqrt{1-\beta^2r_0^2[(1+g)\sin\psi_1-3g\sin 3\psi_1]^2}}\\
t_1 &= \frac{L}{\sqrt{\epsilon_c(1-b'^{\,2}r_0^2+c'b'^{\,4}r_0^4)}}
\end{aligned}
$$

and θ_{i2} and θ_{t2} are angles of the incidence and the refraction of the ray at the rear face of the lens. It may be noted that in deriving Equation (7e), terms containing square and higher order of g, square and higher order of c', gc' and its higher orders, have been neglected.

3.1. Approximation for Weak Focusing

When the focal length of the Wood lens is relatively large, the condition $b^2(\xi^2+\eta^2)\ll 1$ holds. Under the condition, the derivation of the integral is simplified. The result is

$$\begin{aligned} u(r,z) &= k\beta bL \exp[-jk(z-L)] \int_0^a T(r_0)\left(1-\frac{1}{2}b^2r_0^2\right) \\ &\quad J_0(k\beta bLr_0r)\exp[-jk\Psi']r_0dr_0 \\ \Psi' &= \sqrt{\epsilon_c}L\left[1+\frac{b^2r_0^2}{2}\right]-\frac{\beta^2b^2L^2r_0^2}{2}(z-L) \end{aligned} \tag{8}$$

where $T(r_0)$ is the transmission coefficient given in (6b). It may be noted that we have used

$$p_{z0} \approx \sqrt{\epsilon_c}\left(1-\frac{1}{2}b^2r_0^2\right), \qquad b^2r_0^2 \ll 1$$

$$\sin\psi_1 \approx \psi_1 \approx bL\left(1+\frac{1}{2}b^2r_0^2\right)$$

$$\cos\psi_1 \approx 1$$

$$t_1 \approx \frac{L}{\sqrt{\epsilon_c}}\left(1+\frac{1}{2}b^2r_0^2\right)$$

$$\Psi_0 = \sqrt{\epsilon_c}L\left(1-\frac{1}{2}b^2r_0^2\right)$$

$$\begin{aligned} \Psi_2 &= \Psi_0+\sqrt{\epsilon_c}Lb^2r_0^2+\left(1-\frac{\beta^2b^2r_0^2L^2}{2}\right)(Z-L) \\ &= \sqrt{\epsilon_c}L\left(1+\frac{1}{2}b^2r_0^2\right)+\left(1-\frac{\beta^2b^2r_0^2L^2}{2}\right)(z-L) \\ &= \Psi'+(z-L) \end{aligned}$$

$$\sqrt{\sin\psi_1\cos\psi_1}\left[\sin\psi_1\cos\psi_1+\frac{\beta^3Lr_0^2}{2p_{z0}^3}\right]^{\frac{1}{2}} = bL\left(1+b^2r_0^2\right)$$

If we neglect the term proportional to $b^2r_0^2$, the parameters $T(r_0)$ and Ψ' become constants and (8) gives the well known Airy pattern

$$(k\beta bL)\exp[-jk(z-L)]\frac{J_1(k\beta bLar)}{r}$$

4. NUMERICAL RESULTS

4.1. Caustic Curve

The equation of the caustic is obtained by setting the Jacobian be zero and explicit expression is given by

$$x_{c1} = 0, \quad y_{c1} = 0, \quad z_{c1} = L + \frac{\sqrt{1-\beta^2 r_0^2 \sin^2\psi_1}}{\beta} \frac{\cos\psi_1}{\sin\psi_1} \tag{9a}$$

$$x_{c2} = \xi\cos\psi_1 - p_{z1}^2 - Qp_{z1}^2\xi\sin\psi_1, \quad y_{c2} = \eta\cos\psi_1 - Qp_{z1}^2\eta\sin\psi_1,$$

$$z_{c2} = L + \frac{Q}{\beta}p_{z1}^3 \tag{9b}$$

$$Q = \frac{C}{D}, \; C = \cos\psi_1 - \frac{\beta^3 r_0 2L}{p_{z0}^3}\sin\psi_1, \; D = \sin\psi_1 + \frac{\beta^3 r_0 2L}{p_{z0}^3}\cos\psi_1 \tag{9c}$$

where (x_{c1}, y_{c1}, z_{c1}) and (x_{c2}, y_{c2}, z_{c2}) are the rectangular coordinates of axial (or sagittal) caustic and tangential caustic, respectively.

4.2. Comparison to the Huygens-Kirchhoff's Expression

To verify the validity of the uniform expression which is valid near the caustic, we compare the numerical results with those computed from the Huygens-Kirchhoff's radiation integral given by

$$U(x,y,z) = \frac{1}{4\pi}\int\int \phi(x',y',z')\frac{\exp[-jkr]}{r}\cos\gamma dS,$$

$$r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{10a}$$

where ϕ is the field distribution behind the lens and $\cos\gamma$ is the inclination angle. When the focal length is relatively long, we can apply the Fresnel approximation. In this case the integration with respect to angular variable can be carried out resulting

$$u(r,z) = \frac{jk}{z-L}\exp\left[-jk\left(z-L+\frac{r^2}{2(z-L)}\right)\right]$$

$$\times\int_0^a J_0\left(\frac{krr_0\cos\psi_1}{z-L}\right)\Xi^{-\frac{1}{2}}\exp\left[-jk\left(\Psi_0+\frac{r_0^2\cos^2\psi_1}{2(z-L)}\right)\right]r_0 dr_0 \tag{10b}$$

4.3. Effect of Quadratic Term of Dielectric Profile

We will discuss the validity of approximate solution (1e) of the Hamilton's Equations (1b) and (1c). Hamiltonian equations can be transformed into inhomogeneous second order differential equation and its solution can be solved by iterative process. This equation has also been solved by another method [15]. Comparisons among them shows two iterations are sufficient for the parameters which we are of interest here. The solutions by iteration and by [15] are not useful to construct the radiation integral since their expressions are too lengthy. So we use more simpler (1f) to derive the radiation integral (7b). When the value of c is small, (1e) gives very precise results, but discrepancy increases as c increases. We change the factor λ and κ slightly and determine the values of λ and κ so that gives close values to the iteration solution. Figure 2 shows the ray trajectories inside the lens computed from (1f) (dotted line) and more precise iterative method (cross). The figure shows the ray trajectory inside the lens for incident ray at $r_0 = 0.5a$. The ordinate is the values of x (or y), $\epsilon_c = 2.25$, $L = a$, $c = 0.666$ and $ba = 0.4$ for (a) and $ba = 0.6$ with more focusing effect for

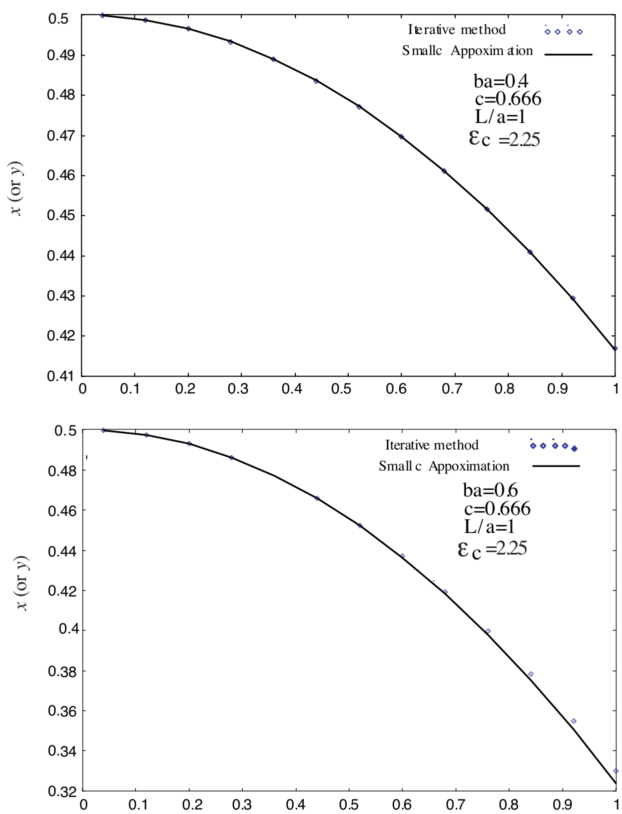

Figure 2. Ray trajectory inside the lens.

(b). The results show that the approximation (1e) is sufficient for the parameters treated here. Figure 3 is the pattern computed by four different methods along z-axis around the focal point. The four methods are (1): effect of the quatic term of the dielectric profile is taken into account (+), (2): effect of quatic term is neglected ($\diamond$), week focusing approximation (Box) and Krichhoff approximation(solid line).

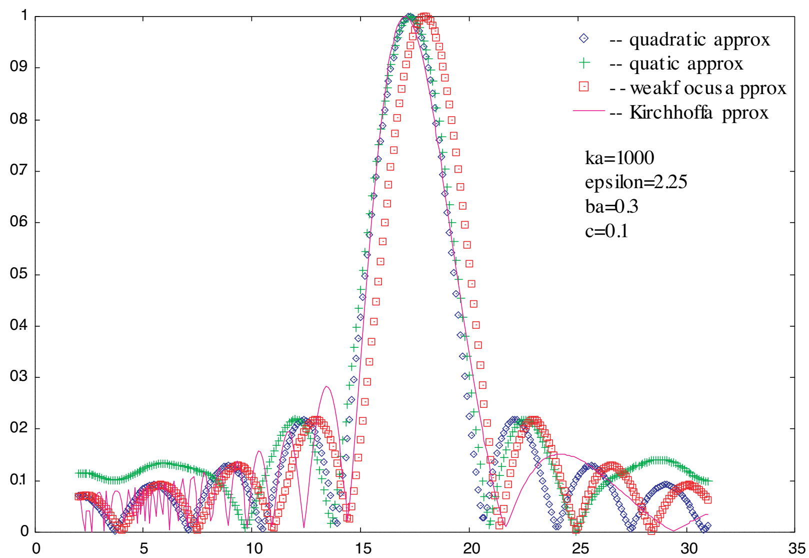

Figure 3. Field pattern around the focal region of Wood lens computed by four different methods.

ACKNOWLEDGMENT

Authors would like to thank National Center for Physics (NCP) for cooperation.

APPENDIX A. EXPRESSION FOR JACOBIAN WHEN $C = 0$

$$x = \xi \cos \psi_1 + p_{x1} t, \quad y = \eta \cos \psi_1 + p_{y1} t, \quad z = \zeta_1 + p_{z1} t = L + p_{z1} t$$

$$
\begin{aligned}
D(t) &= \frac{\partial(x,y,z)}{\partial(\xi,\eta,t)} \\
&= \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \\
&= \begin{vmatrix} \cos\psi_1 - \xi\sin\psi_1 \frac{\partial \psi_1}{\partial \xi} + \frac{\partial p_{x1}}{\partial \xi} t & -\eta\sin\psi_1 \frac{\partial \psi_1}{\partial \xi} + \frac{\partial p_{y1}}{\partial \xi} t & \frac{\partial p_{z1}}{\partial \xi} t \\ -\xi\sin\psi_1 \frac{\partial \psi_1}{\partial \eta} + \frac{\partial p_{x1}}{\partial \eta} t & \cos\psi_1 - \eta\sin\psi_1 \frac{\partial \psi_1}{\partial \eta} + \frac{\partial p_{y1}}{\partial \eta} t & \frac{\partial p_{z1}}{\partial \eta} t \\ p_{x1} & p_{y1} & p_{z1} \end{vmatrix} \\
&= At^2 + Bt + C \\
&= \left(\cos\psi_1 - \xi\sin\psi_1 \frac{\partial \psi_1}{\partial \xi} + \frac{\partial p_{x1}}{\partial \xi} t\right) \\
&\quad \left[p_{z1}\left(\cos\psi_1 - \eta\sin\psi_1 \frac{\partial \psi_1}{\partial \eta} + \frac{\partial p_{y1}}{\partial \eta} t\right) - p_{y1}\frac{\partial p_{z1}}{\partial \eta} t\right] + \\
&\quad \left(-\eta\sin\psi_1 \frac{\partial \psi_1}{\partial \xi} + \frac{\partial p_{y1}}{\partial \xi} t\right) \\
&\quad \left[p_{x1}\frac{\partial p_{z1}}{\partial \eta} t - p_{z1}\left(-\xi\sin\psi_1 \frac{\partial \psi_1}{\partial \eta} + \frac{\partial p_{x1}}{\partial \eta} t\right)\right] + \\
&\quad \frac{\partial p_{z1}}{\partial \xi} t \left[p_{y1}\left(-\xi\sin\psi_1 \frac{\partial \psi_1}{\partial \eta} + \frac{\partial p_{x1}}{\partial \eta} t\right)\right. \\
&\quad \left. - p_{x1}\left(\cos\psi_1 - \eta\sin\psi_1 \frac{\partial \psi_1}{\partial \eta} + \frac{\partial p_{y1}}{\partial \eta} t\right)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \left(\frac{\partial p_{x1}}{\partial \xi}\frac{\partial p_{y1}}{\partial \eta} - \frac{\partial p_{y1}}{\partial \xi}\frac{\partial p_{x1}}{\partial \eta}\right) p_{z1} + \left(\frac{\partial p_{y1}}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta} - \frac{\partial p_{y1}}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi}\right) p_{x1} \\
&\quad + \left(\frac{\partial p_{x1}}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi} - \frac{\partial p_{x1}}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta}\right) p_{y1}
\end{aligned}
$$

$$
\begin{aligned}
B &= \left(-\eta\sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta} + \eta\sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi} - \cos\psi_1 \frac{\partial p_{z1}}{\partial \xi}\right) p_{x1} \\
&\quad - \left[\left(\cos\psi_1 - \xi\sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\right)\frac{\partial p_{z1}}{\partial \eta} + \xi\sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi}\right] p_{y1} \\
&\quad + \left[\left(\cos\psi_1 - \xi\sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\right)\frac{\partial p_{y1}}{\partial \eta} + \left(\cos\psi_1 - \eta\sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\right)\frac{\partial p_{x1}}{\partial \xi}\right.
\end{aligned}
$$

$$\begin{aligned}
&+\xi \sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{y1}}{\partial \xi} + \eta \sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{x1}}{\partial \xi}\Bigg] p_{z1} \\
C \;=\; &\left[\left(\cos\psi_1 - \xi \sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\right)\left(\cos\psi_1 - \eta \sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\right)\right. \\
&\left. -\xi\eta \sin^2\psi_1 \frac{\partial \psi_1}{\partial \xi}\frac{\partial \psi_1}{\partial \eta}\right] p_{z1}
\end{aligned}$$

By using the relations

$$\begin{aligned}
\frac{\partial p_{x1}}{\partial \xi} &= -\beta \sin\psi_1 - \beta\xi \cos\psi_1 \frac{\partial \psi_1}{\partial \xi} \qquad \frac{\partial p_{x1}}{\partial \eta} = -\beta\xi \cos\psi_1 \frac{\partial \psi_1}{\partial \eta}, \\
p_{x1} &= -\beta\xi \sin\psi_1 \\
\frac{\partial p_{y1}}{\partial \xi} &= -\beta\eta \cos\psi_1 \frac{\partial \psi_1}{\partial \xi}, \qquad \frac{\partial p_{y1}}{\partial \eta} = -\beta \sin\psi_1 - \beta\eta \cos\psi_1 \frac{\partial \psi_1}{\partial \eta} \\
p_{y1} &= -\beta\eta \sin\psi_1 \\
\frac{\partial \psi_1}{\partial \xi} &= \frac{\beta^3 L\xi}{p_{z0}^3} \qquad \frac{\partial \psi_1}{\partial \eta} = \frac{\beta^3 L\eta}{p_{z0}^3} \qquad \begin{array}{l} p_{z0} = \sqrt{\epsilon_c - \beta^2 r_0^2} \\ p_{z1} = \sqrt{1 - \beta^2 r_0^2 \sin^2\psi_1} \end{array}
\end{aligned}$$

and

$$\begin{aligned}
\xi \frac{\partial \psi_1}{\partial \xi} + \eta \frac{\partial \psi_1}{\partial \eta} &= \frac{\beta^3 L r_0^2}{p_{z0}^3} \\
\frac{\partial p_{z1}}{\partial \xi} &= -\frac{\beta^2 \sin\psi_1}{p_{z1}}\left(\xi \sin\psi_1 + r_0^2 \cos\psi_1 \frac{\partial \psi_1}{\partial \xi}\right) \\
\frac{\partial p_{z1}}{\partial \eta} &= -\frac{\beta^2 \sin\psi_1}{p_{z1}}\left(\eta \sin\psi_1 + r_0^2 \cos\psi_1 \frac{\partial \psi_1}{\partial \eta}\right) \\
\xi \frac{\partial p_{z1}}{\partial \xi} + \eta \frac{\partial p_{z1}}{\partial \eta} &= -\frac{\beta^2 r_0^2 \sin^2\psi_1}{p_{z1}} - \frac{\beta^2 r_0^2 \sin\psi_1 \cos\psi_1}{p_{z1}}\left(\xi \frac{\partial \psi_1}{\partial \xi} + \eta \frac{\partial \psi_1}{\partial \eta}\right) \\
&= -\frac{\beta^2 r_0^2 \sin^2\psi_1}{p_{z1}} - \frac{\beta^5 L r_0^4 \sin\psi_1 \cos\psi_1}{p_{z1} p_{z0}^3}
\end{aligned}$$

A, B, C are evaluated as follows.

$$\begin{aligned}
A_1 &= \left(\frac{\partial p_{x1}}{\partial \xi}\frac{\partial p_{y1}}{\partial \eta} - \frac{\partial p_{y1}}{\partial \xi}\frac{\partial p_{x1}}{\partial \eta}\right) p_{z1} \\
&= p_{z1}\beta^2 \sin\psi_1 \left[\sin\psi_1 + \cos\psi_1 \left(\xi \frac{\partial \psi_1}{\partial \xi} + \eta \frac{\partial \psi_1}{\partial \eta}\right)\right] \\
&= p_{z1}\beta^2 \sin\psi_1 \left[\sin\psi_1 + \cos\psi_1 \frac{\beta^3 r_0^2 L}{p_{z0}^3}\right]
\end{aligned}$$

$$
\begin{aligned}
A_2 &= \left(\frac{\partial p_{y1}}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta} - \frac{\partial p_{y1}}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi}\right) p_{x1} + \left(\frac{\partial p_{x1}}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi} - \frac{\partial p_{x1}}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta}\right) p_{y1} \\
&= -\beta^2 \sin^2\psi_1 \left(\xi \frac{\partial p_{z1}}{\partial \xi} + \eta \frac{\partial p_{z1}}{\partial \eta}\right) \\
&\quad +\beta^2 \xi\eta \sin\psi_1 \cos\psi_1 \left(\frac{\partial \psi_1}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta} + \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi}\right) \\
&\quad -\beta^2 \xi\eta \sin\psi_1 \cos\psi_1 \left(\frac{\partial \psi_1}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta} + \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi}\right) \\
&= -\beta^2 \sin^2\psi_1 \left(\xi \frac{\partial p_{z1}}{\partial \xi} + \eta \frac{\partial p_{z1}}{\partial \eta}\right) \\
&= \frac{\beta^4 r_0^2 \sin^4\psi_1}{p_{z1}} + \frac{\beta^4 r_0^2 \sin^3\psi_1 \cos\psi_1}{p_{z1}} \left(\xi \frac{\partial \psi_1}{\partial \xi} + \eta \frac{\partial \psi_1}{\partial \eta}\right) \\
&= \frac{\beta^4 r_0^2 \sin^4\psi_1}{p_{z1}} + \frac{\beta^7 r_0^4 L \sin^3\psi_1 \cos\psi_1}{p_{z1} p_{z0}^3} \\
A &= A_1 + A_2 = \frac{\beta^4 r_0^2 \sin^4\psi_1}{p_{z1}} + \frac{\beta^7 r_0^4 L \sin^3\psi_1 \cos\psi_1}{p_{z1}^4} \\
&\quad + p_{z1}\beta^2 \sin\psi_1 \left[\sin\psi_1 + \cos\psi_1 \frac{\beta^3 r_0^2 L}{p_{z0}^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= \left(-\eta \sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\frac{\partial p_{z1}}{\partial \eta} + \eta \sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi} - \cos\psi_1 \frac{\partial p_{z1}}{\partial \xi}\right) p_{x1} \\
&\quad - \left[\left(\cos\psi_1 - \xi \sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\right)\frac{\partial p_{z1}}{\partial \eta} + \xi \sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{z1}}{\partial \xi}\right] p_{y1} \\
&= \beta \sin\psi_1 \cos\psi_1 \left(\xi \frac{\partial p_{z1}}{\partial \xi} + \eta \frac{\partial p_{z1}}{\partial \eta}\right) \\
&= -\frac{\beta^3 r_0^2 \sin^3\psi_1 \cos\psi_1}{p_{z1}} - \frac{\beta^6 L r_0^4 \sin^2\psi_1 \cos^2\psi_1}{p_{z1} p_{z0}^3}
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= \left[\left(\cos\psi_1 - \xi \sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\right)\frac{\partial p_{y1}}{\partial \eta} + \left(\cos\psi_1 - \eta \sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\right)\frac{\partial p_{x1}}{\partial \xi}\right. \\
&\quad \left. + \xi \sin\psi_1 \frac{\partial \psi_1}{\partial \eta}\frac{\partial p_{y1}}{\partial \xi} + \eta \sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\frac{\partial p_{x1}}{\partial \eta}\right] p_{z1} \\
&= \left[\left(\cos\psi_1 - \xi \sin\psi_1 \frac{\partial \psi_1}{\partial \xi}\right)\left(-\beta \sin\psi_1 - \beta\eta \cos\psi_1 \frac{\partial \psi_1}{\partial \eta}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\cos\psi_1-\eta\sin\psi_1\frac{\partial\psi_1}{\partial\eta}\right)\left(-\beta\sin\psi_1-\beta\xi\cos\psi_1\frac{\partial\psi_1}{\partial\xi}\right)\\
&+\xi v\sin\psi_1\frac{\partial\psi_1}{\partial\eta}\left(-\beta\eta\cos\psi_1\frac{\partial\psi_1}{\partial\xi}\right)+\eta\sin\psi_1\frac{\partial\psi_1}{\partial\xi}\left(-\beta\xi\cos\psi_1\frac{\partial\psi_1}{\partial\eta}\right)\Bigg]p_{z1}\\
=&\left[-2\beta\sin\psi_1\cos\psi_1-\beta(\cos^2\psi_1-\sin^2\psi_1)\left(\xi\frac{\partial\psi_1}{\partial\xi}+\eta\frac{\partial\psi_1}{\partial\eta}\right)\right]p_{z1}\\
=&\left[-2\beta\sin\psi_1\cos\psi_1-\beta(\cos^2\psi_1-\sin^2\psi_1)\frac{\beta^3r_0^2L}{p_{z0}^3}\right]p_{z1}
\end{aligned}
$$

$$
\begin{aligned}
C &= \left[\cos^2\psi_1-\sin\psi_1\cos\psi_1\left(\xi\frac{\partial\psi_1}{\partial\xi}+\eta\frac{\partial\psi_1}{\partial\eta}\right)\right]p_{z1}\\
&= \cos\psi_1\left[\cos\psi_1-\sin\psi_1\frac{\beta^3r_0^2L}{p_{z0}^3}\right]p_{z1}
\end{aligned}
$$

By using these results we have

$$
\begin{aligned}
J(t) &= \frac{A}{C}t^2+\frac{B}{C}t+1\\
&= \frac{1}{\Xi}\left[\frac{\beta^4r_0^2\sin^4\psi_1}{p_{z1}^2}+\frac{\beta^7r_0^4L\sin^3\psi_1\cos\psi_1}{p_{z1}^2p_{z0}^3}\right.\\
&\quad\left.+\beta^2\sin\psi_1\left(\sin\psi_1+\cos\psi_1\frac{\beta^3r_0^2L}{p_{z0}^3}\right)\right]t^2\\
&\quad-\frac{1}{\Xi}\left\{\frac{\beta^3r_0^2\sin^3\psi_1\cos\psi_1}{p_{z1}^2}+\frac{\beta^6Lr_0^4\sin^2\psi_1\cos^2\psi_1}{p_{z1}^2p_{z0}^3}+\right.\\
&\quad\left.\left[2\beta\sin\psi_1\cos\psi_1+\beta(\cos^2\psi_1-\sin^2\psi_1)\frac{\beta^3r_0^2L}{p_{z0}^3}\right]\right\}t+1
\end{aligned}
$$

where

$$\Xi=\cos\psi_1\left[\cos\psi_1-\sin\psi_1\frac{\beta^3r_0^2L}{p_{z0}^3}\right]$$

APPENDIX B. $C \neq 0$, EVALUATION OF $\frac{\partial(p_{x1},p_{y1},z)}{\partial(\xi,\eta,t)}$

$$
\frac{\partial(p_{x1},p_{y1},z)}{\partial(\xi,\eta,t)}=\begin{vmatrix}\frac{\partial p_{x1}}{\partial\xi} & \frac{\partial p_{y1}}{\partial\xi} & \frac{\partial z}{\partial\xi}\\ \frac{\partial p_{x1}}{\partial\eta} & \frac{\partial p_{y1}}{\partial\eta} & \frac{\partial z}{\partial\eta}\\ \frac{\partial p_{x1}}{\partial t} & \frac{\partial p_{y1}}{\partial t} & \frac{\partial z}{\partial t}\end{vmatrix}=\begin{vmatrix}\frac{\partial p_{x1}}{\partial\xi} & \frac{\partial p_{y1}}{\partial\xi} & \frac{\partial z}{\partial\xi}\\ \frac{\partial p_{x1}}{\partial\eta} & \frac{\partial p_{y1}}{\partial\eta} & \frac{\partial z}{\partial\eta}\\ 0 & 0 & p_{z1}\end{vmatrix}
$$

$$
\begin{aligned}
&= p_{z1}\left(\frac{\partial p_{x1}}{\partial \xi}\frac{\partial p_{y1}}{\partial \eta}-\frac{\partial p_{x1}}{\partial \eta}\frac{\partial p_{y1}}{\partial \xi}\right)\\
&= p_{z1}\left[\left(A+B\xi\frac{\partial \psi_1}{\partial \xi}\right)\left(A+B\eta\frac{\partial \psi_1}{\partial \eta}\right)-\left(B\eta\frac{\partial \psi_1}{\partial \xi}\right)\left(B\xi\frac{\partial \psi_1}{\partial \eta}\right)\right]\\
&= p_{z1}A^2+p_{z1}AB\left[\left(\xi\frac{\partial \psi_1}{\partial \xi}+\eta\frac{\partial \psi_1}{\partial \eta}\right)\right]\\
&= \beta^2\left[(1+2g)\sin^2\psi_1-6g\sin 3\psi_1\sin\psi_1\right]p_{z1}+\\
&\quad \beta^2\left[(1+g)\sin\psi_1-3g\sin 3\psi_1\right]\left[(1+g)\cos\psi_1\right.\\
&\quad \left.-9g\cos 3\psi_1\right]p_{z1}\left[\left(\xi\frac{\partial \psi_1}{\partial \xi}+\eta\frac{\partial \psi_1}{\partial \eta}\right)\right]\\
&= \beta^2\left[(1+2g)\sin^2\psi_1-6g\sin 3\psi_1\sin\psi_1\right]p_{z1}+\\
&\quad \beta^2\left[(1+2g)\cos\psi_1\sin\psi_1-3g\sin 3\psi_1\cos\psi_1\right.\\
&\quad \left.-9g\cos 3\psi_1\sin\psi_1\right]p_{z1}\left[\left(\xi\frac{\partial \psi_1}{\partial \xi}+\eta\frac{\partial \psi_1}{\partial \eta}\right)\right]\\
&= \beta^2 p_{z1}\left\{\left[(1+2g)\sin^2\psi_1-6g\sin 3\psi_1\sin\psi_1\right]+\right.\\
&\quad \left[(1+2g)\cos\psi_1\sin\psi_1-3g\sin 3\psi_1\cos\psi_1\right.\\
&\quad \left.\left.-9g\cos 3\psi_1\sin\psi_1\right]\frac{\beta L}{p_{z0}^3}r_0^2\left[b'^{\,2}\epsilon_c-2\epsilon_c c' b'^{\,4}r_0^2\right]\right\}
\end{aligned}
$$

Neglecting the terms containing c^2 yields the following

$$
\begin{aligned}
\frac{\partial(p_{x1},p_{y1},z)}{\partial(\xi,\eta,t)} &= \beta^2 p_{z1}\left\{\left[(1+2g)\sin^2\psi_1-6g\sin 3\psi_1\sin\psi_1\right]\right.\\
&\quad +\cos\psi_1\sin\psi_1\frac{\beta L}{p_{z0}^3}r_0^2\left[b'^{\,2}\epsilon_c-2\epsilon_c c' b'^{\,4}r_0^2\right]\\
&\quad +g\left[2\cos\psi_1\sin\psi_1-3\sin 3\psi_1\cos\psi_1\right.\\
&\quad \left.\left.-9\cos 3\psi_1\sin\psi_1\right]\frac{\beta\beta'^{\,2}L}{p_{z0}^3}r_0^2\right\}
\end{aligned}
$$

where $\beta'=\sqrt{\epsilon_c}b'$.

APPENDIX C. EXPRESSION FOR CAUSTIC WHEN $C \neq 0$

$$
\begin{aligned}
x &= \xi\left[(1+g)\cos\psi_1-g\cos 3\psi_1\right]+p_{x1}t\\
y &= \eta\left[(1+g)\cos\psi_1-g\cos 3\psi_1\right]+p_{y1}t\\
z &= \zeta_1+p_{z1}t=L+p_{z1}t
\end{aligned}
$$

$$
\begin{aligned}
D(t) &= \frac{\partial(x,y,z)}{\partial(\xi,\eta,t)} \\
&= \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \\
&= \begin{vmatrix} D_{11} & \begin{array}{l} -\eta[(1+g)\sin\psi_1 \\ -3g\sin 3\psi_1] \\ \frac{\partial \psi_1}{\partial \xi} + \frac{\partial p_{y1}}{\partial \xi} t \end{array} & \frac{\partial p_{z1}}{\partial \xi} t \\ \begin{array}{l} -\xi[(1+g)\sin\psi_1 \\ -3g\sin 3\psi_1] \\ \frac{\partial \psi_1}{\partial \eta} + \frac{\partial p_{x1}}{\partial \eta} t \end{array} & D_{22} & \frac{\partial p_{z1}}{\partial \eta} t \\ p_{x1} & p_{y1} & p_{z1} \end{vmatrix} \\
&= At^2 + Bt + C
\end{aligned}
$$

$$
\begin{aligned}
D(0) = &\left([(1+g)\cos\psi_1 - g\cos 3\psi_1] - \xi[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\frac{\partial \psi_1}{\partial \xi}\right) \\
&\times [p_{z1}\,([(1+g)\cos\psi_1 - g\cos 3\psi_1] \\
&-\eta[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\frac{\partial \psi_1}{\partial \eta}\Big)\Big] \\
&+ \left(-\eta[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\frac{\partial \psi_1}{\partial \xi}\right) \\
&\times \left[-p_{z1}\left(-\xi[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\frac{\partial \psi_1}{\partial \eta}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
D(0) &= p_{z1}[(1+g)\cos\psi_1 - g\cos 3\psi_1]^2 \\
&\quad -p_{z1}[(1+g)\cos\psi_1 - g\cos 3\psi_1][(1+g)\sin\psi_1 - 3g\sin 3\psi_1] \\
&\quad \times \left(\eta\frac{\partial \psi_1}{\partial \eta} + \xi\frac{\partial \psi_1}{\partial \xi}\right) \\
D(0) &= p_{z1}[(1+2g)\cos^2\psi_1 - 2g\cos\psi\cos 3\psi_1] \\
&\quad -p_{z1}[(1+2g)\sin\psi_1\cos\psi_1 - g\sin\psi_1\cos 3\psi_1 \\
&\quad -3g\cos\psi_1\sin 3\psi_1]\left(\eta\frac{\partial \psi_1}{\partial \eta} + \xi\frac{\partial \psi_1}{\partial \xi}\right) \\
\psi_1 &= \beta t_1 = \frac{\beta L}{p_{z0}} = \frac{\beta L}{\sqrt{\epsilon_c(1 - b'^{\,2} r_0^2 + c' b'^{\,4} r_0^4)}} \\
\frac{\partial \psi_1}{\partial \xi} &= \frac{\epsilon_c \beta L (b'^{\,2} - 2c' b'^{\,4} r_0^2)\xi}{p_{z0}^3} \\
&\quad \times \left(\eta\frac{\partial \psi_1}{\partial \eta} + \xi\frac{\partial \psi_1}{\partial \xi}\right) = \frac{\beta L}{p_{z0}^3}(\epsilon_c b'^{\,2} - 2\epsilon_c c' b'^{\,4} r_0^2) r_0^2 \\
D(0) &= p_{z1}[(1+2g)\cos^2\psi_1 - 2g\cos\psi\cos 3\psi_1] \\
&\quad -p_{z1}[(1+2g)\sin\psi_1\cos\psi_1 - g\sin\psi_1\cos 3\psi_1 \\
&\quad -3g\cos\psi_1\sin 3\psi_1]\frac{\beta L}{p_{z0}^3}(\epsilon_c b'^{\,2} - 2\epsilon_c c' b'^{\,4} r_0^2) r_0^2
\end{aligned}
$$

where

$$
\begin{aligned}
D_{11} &= [(1+g)\cos\psi_1 - g\cos 3\psi_1] - \xi[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\frac{\partial\psi_1}{\partial\xi} \\
&\quad + \frac{\partial p_{x1}}{\partial\xi}t \\
D_{22} &= [(1+g)\cos\psi_1 - g\cos 3\psi_1] - \eta[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\frac{\partial\psi_1}{\partial\eta} \\
&\quad + \frac{\partial p_{x1}}{\partial\eta}t
\end{aligned}
$$

Neglecting terms containing c^2 yields the following

$$
\begin{aligned}
D(0) &= p_{z1}\Big[(1+2g)\cos^2\psi_1 - 2g\cos\psi\cos 3\psi_1 \\
&\quad - \sin\psi_1\cos\psi_1\frac{\beta L}{p_{z0}^3}(\epsilon_c b'^{\,2} - 2\epsilon_c c' b'^{\,4} r_0^2) r_0^2\Big] \\
&\quad - p_{z1}[2g\sin\psi_1\cos\psi_1 - g\sin\psi_1\cos 3\psi_1 \\
&\quad - 3g\cos\psi_1\sin 3\psi_1]\frac{\beta\beta'^{\,2} L}{p_{z0}^3} r_0^2
\end{aligned}
$$

It may be noted that $D(0)$ given in above equation reduces to corresponding expressions of $D(0)$ for $c = 0$, if we take $g = 0$ and replace b' with b.

APPENDIX D.

$$
\begin{aligned}
\Psi_0' &= \int_0^{t_1} \epsilon_c\left[1 - b^2(x^2+y^2) + cb^4(x^2+y^2)^2\right]dt \\
&= \int_0^{t_1} \epsilon_c\Big[1 - b^2 r_0^2[(1+g)\cos\psi - g\cos 3\psi]^2 \\
&\quad + cb^4 r_0^2[(1+g)\cos\psi - g\cos 3\psi]^4\Big]dt \\
&= \int_0^{t_1} \epsilon_c\Big[1 - b^2 r_0^2[(1+2g)\cos^2\psi - 2g\cos 3\psi\cos\psi] \\
&\quad + cb^4 r_0^2[(1+2g)\cos^2\psi - 2g\cos 3\psi\cos\psi]^2\Big]dt \\
\Psi_0' &= \int_0^{t_1} \epsilon_c\left[1 - b^2 r_0^2\cos^2\psi\right]dt \\
&\quad - 2\epsilon_c g b^2 r_0^2 \int_0^{t_1}[\cos^2\psi - \cos 3\psi\cos\psi]dt
\end{aligned}
$$

$$
+\epsilon_c c b^4 r_0^2 \int_0^{t_1} [(1+2g)\cos^2\psi - 2g\cos 3\psi\cos\psi]^2 dt
$$

$$
\begin{aligned}
\Psi_0' \; = \; & \int_0^{t_1} \epsilon_c \left[1 - b^2 r_0^2 \cos^2\psi\right] dt \\
& -\epsilon_c g b^2 r_0^2 \int_0^{t_1} [1 - \cos 4\psi]\, dt \\
& +\epsilon_c c b^4 r_0^2 \int_0^{t_1} [(1+2g)\cos^2\psi - 2g\cos 3\psi\cos\psi]^2 dt
\end{aligned}
$$

$$
\begin{aligned}
\Psi_0' \; = \; & \int_0^{t_1} \epsilon_c \left[1 - b^2 r_0^2 \cos^2\psi\right] dt - \epsilon_c g b^2 r_0^2 \left[t_1 - \frac{\sin 4\psi_1}{4\beta}\right] \\
& +\epsilon_c c b^4 r_0^2 \left[\frac{3}{8} t_1 + \frac{2}{16\beta}\sin 2\psi_1 + \frac{1}{32\beta}\sin 4\psi_1\right]
\end{aligned}
$$

$$
\begin{aligned}
\Psi_0' \; = \; & \Psi_0 - \epsilon_c g b^2 r_0^2 \left[t_1 - \frac{\sin 4\psi_1}{4\beta}\right] \\
& +\epsilon_c c b^4 r_0^2 \left[\frac{3}{8} t_1 + \frac{2}{16\beta}\sin 2\psi_1 + \frac{1}{32\beta}\sin 4\psi_1\right]
\end{aligned}
$$

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