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THE SMALL-SLOPE APPROXIMATION METHOD APPLIED TO A THREE-DIMENSIONAL SLAB WITH ROUGH BOUNDARIES

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1. INTRODUCTION

The analysis of the electromagnetic field scattering by random rough surfaces has been a subject of intensive research in recent decades [1–4]. Theoretical and numerical approaches have received a wide interest, we mention: the small-perturbation method (SPM) [5–10], the Kirchhoff (or tangent plane) approximation method [1, 11, 12]. However, some restrictions limit the domain of their applicability, the perturbation method is only valid for surfaces with small roughness and the Kirchhoff approximation is applicable to surfaces with long correlation length. Their combination gives the two-scale model, which is inaccurate for grazing angles [13, 14]. Besides these methods, new approaches were suggested, like: the full-wave method analysis [15], the surface-field phase-perturbation technique [16, 17], the quasislope approximation [18].

In the mid-1980s, Voronovich [19–24] proposed a new method called the small-slope approximation (SSA) which is valid for arbitrary roughness provided that the slopes of the surface are smaller than the angles of incidence and scattering, and irrespective of the wavelength of the incident radiation. The SSA is in fact making a bridge between two classical approaches, namely: the Kirchhoff approximation and the small-perturbation method. An extension to situations in which multiple scattering from points situated at significant distance becomes important is known as the non-local small-slope [23].

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In this paper we will focused on the SSA method in view to study different rough structures like a slab or a film, considering the effects of higher orders in a perturbative expansion. The problem of one rough surface up to the order 3 is treated in [25, 31]. The [25] and [31] consider the second order of the SSA and the one that includes the next-order correction to it. The [25] proposed simplified forms for the first three SSA terms in the case of penetrable surfaces under the assumption of a Gaussian random process with an isotropic Gaussian correlation function. In [31] we find results up to the third SSA term for incoherent scattering from dielectric and metallic surfaces with Gaussian and non-Gaussian correlation functions. The main point is to investigate the case where a dielectric slab is bounded by two rough surfaces [29]. Since the SSA method involves components of the SPM in the calculations, we have used results of our previous works [26–28] developed under the Rayleigh hypothesis.

The organization of the paper is as follows. In Section 2, we give a description of a random rough surface and the notations used for the electromagnetic field in a vectorial basis. In Section 3, we define the scattering matrix as an expansion in terms of the surface height. In Section 4, we summarize the main features of the small-perturbation method and give an example in the case of a single rough surface showing the relation with the operators of the SPM in the formalism of [26]. Section 5 is devoted to the calculation of the bistatic cross-section where an explicit example is given. In Section 6, we give several examples of application of the SSA method in the case of a single rough surface between two semi-infinite media, and make a comparison with the results obtained by the SPM. Section 7, treats the scattering by a slab with a rough surface on the bottom side, and applications are given. In Section 8, we are interested in a slab where the upper boundary is a rough surface, some applications are considered. Section 9, deals with the general case of a slab with two rough boundaries. A detailed development of the SSA method is presented up to the order 4 with respect to the heights. We give an example of application and compare with the results we have obtained in the SPM case [27]. Appendices collect some formulas derived in [26, 27] and needed to make the paper self-contained.

2. PRELIMINARY DEFINITIONS AND NOTATIONS

The structure we consider is shown in Fig. 1, where the two rough surfaces separate three media. The three media are characterized by an isotropic, homogeneous dielectric constant ϵ_0, ϵ_1 and ϵ_2 respectively. The two boundaries of the rough surfaces are located at the height

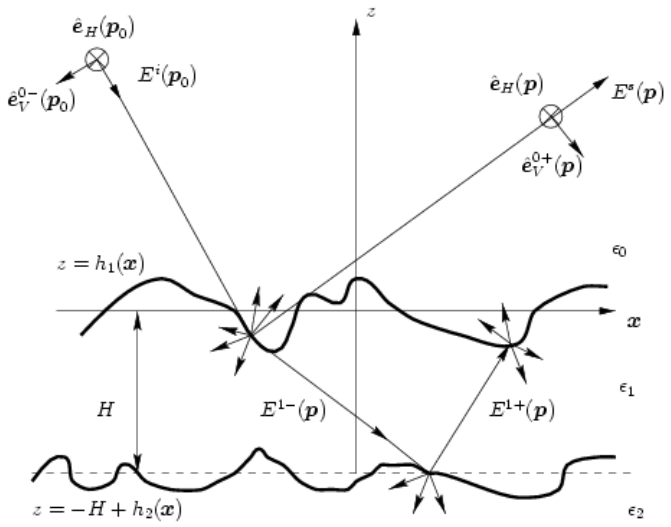


Figure 1. An incident wave coming from medium 0 and scattered by a slab with two rough surfaces.

$z = h_1(\boldsymbol{x})$, $z = -H + h_2(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y)$. The two rough surfaces are described statistically, more precisely, we assume that $h_1(\boldsymbol{x})$ and $h_2(\boldsymbol{x})$ are stationary, isotropic uncorrelated Gaussian random processes defined by their moments:

$$< h_i(\boldsymbol{x}) > = 0, \tag{1}$$

$$< h_i(\boldsymbol{x}) h_i(\boldsymbol{x}') > = W_i(\boldsymbol{x} - \boldsymbol{x}'), \tag{2}$$

$$< h_1(\boldsymbol{x}) h_2(\boldsymbol{x}') > = 0, \tag{3}$$

where $i = 1, 2$, and the angle brackets denote an average over the ensemble of realizations of the function $h_1(\boldsymbol{x})$ and $h_2(\boldsymbol{x})$. In this work we will use a Gaussian form for the surface-height correlation functions $W_1(\boldsymbol{x})$ and $W_2(\boldsymbol{x})$:

$$W_i(\boldsymbol{x}) = \sigma_i^2 \exp(-\boldsymbol{x}^2 / l_i^2), \tag{4}$$

where σ_i is the rms height of the surface $h_i(\boldsymbol{x})$, and l_i is the transverse correlation length. The corresponding expressions in momentum space are given by:

$$< h_i(\boldsymbol{p}) > \; = 0, \tag{5}$$

$$< h_i(\boldsymbol{p}) h_i(\boldsymbol{p}') > \; = (2\pi)^2 \delta(\boldsymbol{p} + \boldsymbol{p}') W_i(\boldsymbol{p}), \tag{6}$$

$$< h_1(\boldsymbol{p}) h_2(\boldsymbol{p}') > = 0, \tag{7}$$

where (We use the same symbol for a function and its Fourier transform, they are differentiated by their arguments.)

$$W_i(\boldsymbol{p}) \equiv \int \mathrm{d}^2\boldsymbol{x} W_i(\boldsymbol{x}) \exp(-i\boldsymbol{p}\cdot\boldsymbol{x}), \tag{8}$$

$$= \pi\sigma_i^2 l_i^2 \exp(-\boldsymbol{p}^2 l_i^2/4). \tag{9}$$

For the electromagnetic field we consider that each wave propagates with a pulsation ω and the time dependence is assumed to be $\exp(-i\omega t)$. The electric fields $\boldsymbol{E}^i$ satisfy in the different media an Helmholtz equation

$$\left(\nabla^2 + \epsilon_i K_0^2\right) \boldsymbol{E}^i(\boldsymbol{r}) = 0. \tag{10}$$

In the medium 0, $\boldsymbol{E}^0$ can be written as a superposition of an incident and scattered fields:

$$\begin{aligned} \boldsymbol{E}^0(\boldsymbol{x}, z) \;=\; & \boldsymbol{E}^i(\boldsymbol{p}_0) \exp(\mathrm{i}\boldsymbol{p}_0\cdot\boldsymbol{x} - \mathrm{i}\alpha_0(\boldsymbol{p}_0) z) \\ & + \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} \boldsymbol{E}^s(\boldsymbol{p}) \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} + \mathrm{i}\alpha_0(\boldsymbol{p}) z), \end{aligned} \tag{11}$$

where (see Fig. 2)

$$\alpha_0(\boldsymbol{p}) = (\epsilon_0 K_0^2 - \boldsymbol{p}^2)^{\frac{1}{2}}, \tag{12}$$

$$K_0 = \omega/c, \tag{13}$$

$$\boldsymbol{E}^i(\boldsymbol{p}_0) = E_V^i(\boldsymbol{p}_0)\hat{\boldsymbol{e}}_V^{0-}(\boldsymbol{p}_0) + E_H^i(\boldsymbol{p}_0)\hat{\boldsymbol{e}}_H(\boldsymbol{p}_0), \tag{14}$$

$$\boldsymbol{E}^s(\boldsymbol{p}) = E_V^s(\boldsymbol{p})\hat{\boldsymbol{e}}_V^{0+} + E_H^s(\boldsymbol{p})\hat{\boldsymbol{e}}_H(\boldsymbol{p}). \tag{15}$$

The subscript H refers to the horizontal polarization (TE), and V to the vertical polarization (TM), they are defined by the two vectors:

$$\hat{\boldsymbol{e}}_H(\boldsymbol{p}) = \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{p}}, \tag{16}$$

$$\hat{\boldsymbol{e}}_V^{0\pm}(\boldsymbol{p}) = \pm\frac{\alpha_0(\boldsymbol{p})}{\sqrt{\epsilon_0}K_0}\hat{\boldsymbol{p}} - \frac{||\boldsymbol{p}||}{\sqrt{\epsilon_0}K_0}\hat{\boldsymbol{e}}_z, \tag{17}$$

where the minus sign corresponds to incident wave and the plus sign to the scattered wave. It has to be noticed that the vector $\boldsymbol{E}^s(\boldsymbol{p})$ and $\boldsymbol{E}^i(\boldsymbol{p}_0)$ are expressed in a different basis due the fact that $\hat{\boldsymbol{e}}_V^{0\pm}(\boldsymbol{p})$ and $\hat{\boldsymbol{e}}_V^{1\pm}(\boldsymbol{p})$ depend on $\boldsymbol{p}$. In medium 1, we get a similar expression namely:

$$\begin{aligned} \boldsymbol{E}^1(\boldsymbol{r}) \;=\; & \int \frac{\mathrm{d}^2\boldsymbol{p}}{(2\pi)^2} \boldsymbol{E}^{1-}(\boldsymbol{p}) \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} - \mathrm{i}\alpha_1(\boldsymbol{p}) z) \\ & + \int \frac{\mathrm{d}^2\boldsymbol{p}}{(2\pi)^2} \boldsymbol{E}^{1+}(\boldsymbol{p}) \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} + \mathrm{i}\alpha_1(\boldsymbol{p}) z), \end{aligned} \tag{18}$$

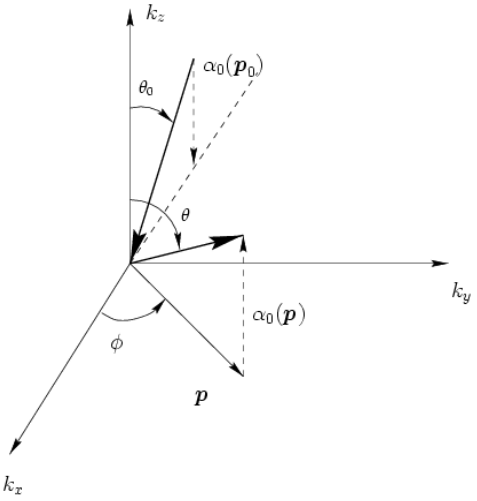


Figure 2. Definition of the scattering vectors.

where

$$\alpha_1(\boldsymbol{p}) \equiv (\epsilon_1 K_0^2 - \boldsymbol{p}^2)^{\frac{1}{2}}. \tag{19}$$

The field $\boldsymbol{E}^{1-}$ is decomposed in the basis $(\hat{\boldsymbol{e}}_V^{1-}(\boldsymbol{p}), \hat{\boldsymbol{e}}_H(\boldsymbol{p}))$, and $\boldsymbol{E}^{1+}$ in the basis $(\hat{\boldsymbol{e}}_V^{1+}(\boldsymbol{p}), \hat{\boldsymbol{e}}_H(\boldsymbol{p}))$ with

$$\hat{\boldsymbol{e}}_H(\boldsymbol{p}) = \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{p}}, \tag{20}$$

$$\hat{\boldsymbol{e}}_V^{1\pm}(\boldsymbol{p}) = \pm\frac{\alpha_1(\boldsymbol{p})}{\sqrt{\epsilon_1}K_0}\hat{\boldsymbol{p}} - \frac{||\boldsymbol{p}||}{\sqrt{\epsilon_1}K_0}\hat{\boldsymbol{e}}_z. \tag{21}$$

3. THE SCATTERING MATRIX

We define the scattering matrix connecting the incident field to the scattered field by the following expression

$$\boldsymbol{E}^s(\boldsymbol{p}) \equiv \overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) \cdot \boldsymbol{E}^i(\boldsymbol{p}_0), \tag{22}$$

where $\overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0)$ is a two-dimensional matrix where the components depend on the polarizations V and H

$$\overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) = \begin{pmatrix} R_{VV}(\boldsymbol{p}|\boldsymbol{p}_0) & R_{VH}(\boldsymbol{p}|\boldsymbol{p}_0) \\ R_{HV}(\boldsymbol{p}|\boldsymbol{p}_0) & R_{HH}(\boldsymbol{p}|\boldsymbol{p}_0) \end{pmatrix}.$$

We will consider a perturbative development of $\overline{\boldsymbol{R}}$ in powers of the height h of the form

$$\overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) = \overline{\boldsymbol{R}}^{(0)}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\boldsymbol{R}}^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\boldsymbol{R}}^{(2)}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\boldsymbol{R}}^{(3)}(\boldsymbol{p}|\boldsymbol{p}_0) + \cdots . \tag{23}$$

We have proven [26] in the case of the small-perturbation method that the development takes the form

$$\begin{aligned}
\overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) \;=\; & (2\pi)^2\delta(\boldsymbol{p}-\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(0)}(\boldsymbol{p}_0) + \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_0) \\
& +\alpha_0(\boldsymbol{p}_0)\int \frac{\mathrm{d}^2\boldsymbol{p}_1}{(2\pi)^2}\overline{\boldsymbol{X}}^{(2)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_1)h(\boldsymbol{p}_1-\boldsymbol{p}_0) \\
& +\alpha_0(\boldsymbol{p}_0)\iint \frac{\mathrm{d}^2\boldsymbol{p}_1}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{p}_2}{(2\pi)^2}\overline{\boldsymbol{X}}^{(3)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_2|\boldsymbol{p}_0) \\
& \times h(\boldsymbol{p}-\boldsymbol{p}_1)h(\boldsymbol{p}_1-\boldsymbol{p}_2)h(\boldsymbol{p}_2-\boldsymbol{p}_0),
\end{aligned} \tag{24}$$

where $h(\boldsymbol{p})$ is the Fourier transform of $\hat{\boldsymbol{x}}$:

$$h(\boldsymbol{p}) \equiv \int \mathrm{d}^2\boldsymbol{x}\exp(-i\boldsymbol{p}\cdot\boldsymbol{x})h(\boldsymbol{x}). \tag{25}$$

The expression of the scattered field represents the general solution of the Maxwell equations which satisfy the radiation condition. For instance, in medium 0, the scattered field reads

$$\boldsymbol{E}^s(\boldsymbol{r}) = \boldsymbol{E}^i(\boldsymbol{p}_0)\exp(\mathrm{i}\boldsymbol{k}_{\boldsymbol{p}_0}^{0-}\cdot\boldsymbol{r}) + \int \frac{\mathrm{d}^2\boldsymbol{p}}{(2\pi)^2}\overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0)\cdot\boldsymbol{E}^i(\boldsymbol{p}_0)\exp(\mathrm{i}\boldsymbol{k}_{\boldsymbol{p}}^{0+}\cdot\boldsymbol{r}). \tag{26}$$

where $\boldsymbol{k}_{\boldsymbol{p}}^{0\pm} \equiv \boldsymbol{p} \pm \alpha_0(\boldsymbol{p})\hat{e}_z$.

In order to determine the scattering matrix we have to satisfy the boundary conditions on the rough surfaces by writing the continuity of the tangential components of the electric and magnetic fields, in the case of the upper surface we obtain

$$\boldsymbol{n}(\boldsymbol{x}) \times \left[\boldsymbol{E}^0(\boldsymbol{x}, h_1(\boldsymbol{x})) - \boldsymbol{E}^1(\boldsymbol{x}, h_1(\boldsymbol{x}))\right] = 0, \tag{27}$$

$$\boldsymbol{n}(\boldsymbol{x}) \cdot \left[\epsilon_0\boldsymbol{E}^0(\boldsymbol{x}, h_1(\boldsymbol{x})) - \epsilon_1\boldsymbol{E}^1(\boldsymbol{x}, h_1(\boldsymbol{x}))\right] = 0, \tag{28}$$

$$\boldsymbol{n}(\boldsymbol{x}) \times \left[\boldsymbol{B}^0(\boldsymbol{x}, h_1(\boldsymbol{x})) - \boldsymbol{B}^1(\boldsymbol{x}, h_1(\boldsymbol{x}))\right] = 0, \tag{29}$$

$$\boldsymbol{n}(\boldsymbol{x}) \equiv \hat{e}_z - \nabla h_1(\boldsymbol{x}) \ .$$

For the lower surface we can write equivalent conditions by making the replacements, $0 \to 1$, $1 \to 2$, and h_1 by h_2.

4. THE SMALL-SLOPE APPROXIMATION FOR A ROUGH SURFACE

In his approach Voronovich [21] remarks that the unitary of the scattering matrix implies a reciprocity theorem leading to the following properties:

$$\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) = \overline{\boldsymbol{R}}(\boldsymbol{p}_0,-\boldsymbol{p}), \tag{30}$$

for an horizontal translation of the rough boundary $h(\boldsymbol{r}) \to h(\boldsymbol{r}-\boldsymbol{a})$

$$\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) \to \overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0)\exp\left[-i(\boldsymbol{p}-\boldsymbol{p}_0)\cdot\boldsymbol{a}\right], \tag{31}$$

and for a vertical translation $h(\boldsymbol{r}) \to h(\boldsymbol{r}) + H$

$$\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) \to \overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0)\exp\left[-i(\alpha(\boldsymbol{p})+\alpha(\boldsymbol{p}_0))H\right]. \tag{32}$$

Using these results Voronovich proposes the following expression of the scattering matrix

$$\begin{aligned}\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) \;=\; & \int \frac{d^2\boldsymbol{r}}{(2\pi)^2}\exp\left[-i(\boldsymbol{p}-\boldsymbol{p}_0)\cdot\boldsymbol{r} - i(\alpha(\boldsymbol{p})+\alpha(\boldsymbol{p}_0))h(\boldsymbol{r})\right] \\ & \times\overline{\boldsymbol{\Phi}}\left[\boldsymbol{p},\boldsymbol{p}_0;\ \boldsymbol{r};\ [h]\right],\end{aligned} \tag{33}$$

in the case of a rough surface located between media 0 and 1. The functional $\overline{\boldsymbol{\Phi}}$ which depends on h has to be determined. The translation conditions (31) and (32), lead to some properties on $\overline{\boldsymbol{\Phi}}$ (here it is more convenient to work with the Fourier transform $\overline{\boldsymbol{\Phi}}\left[\boldsymbol{p},\boldsymbol{p}_0;\ \boldsymbol{r};\ [h]\right]$ with respect to the variable r). The first condition (31) reads:

$$\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) = \int \frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\mathrm{d}^2\boldsymbol{r}\,\exp^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h(\boldsymbol{r})}\,\overline{\boldsymbol{\Phi}}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}), \tag{34}$$

and the second (32)

$$\overline{\boldsymbol{\Phi}}_{\boldsymbol{x}\to h(\boldsymbol{x}-\boldsymbol{a})}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}) = \exp^{\mathrm{i}\xi\cdot\boldsymbol{a}}\,\overline{\boldsymbol{\Phi}}_{\boldsymbol{x}\to h(\boldsymbol{x})}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}), \tag{35}$$

for all vector $\boldsymbol{a}$. In the framework of a perturbative development, $\overline{\boldsymbol{\Phi}}$ is expanded as an integral-power series of h namely:

$$\begin{aligned}\overline{\boldsymbol{\Phi}}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}) &= \delta(\boldsymbol{\xi})\tilde{\boldsymbol{\Phi}}^{(0)}(\boldsymbol{p},\boldsymbol{p}_0) + \int \frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_1)\tilde{\boldsymbol{\Phi}}^{(1)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1)h(\boldsymbol{\xi}_1) \\ &+ \iint \frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\tilde{\boldsymbol{\Phi}}^{(2)}(\boldsymbol{p},\boldsymbol{p}_0,\xi_1,\xi_2)h(\boldsymbol{\xi}_1)h(\boldsymbol{\xi}_2)+\ldots\end{aligned} \tag{36}$$

The condition (32) imposes:

$$\overline{\mathbf{\Phi}}_{\boldsymbol{x}\to h(\boldsymbol{x})+H}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}) = \overline{\mathbf{\Phi}}_{\boldsymbol{x}\to h(\boldsymbol{x})}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}). \tag{37}$$

In the Fourier space, the transformation $\boldsymbol{x} \to h(\boldsymbol{x}) + H$ corresponds to $\boldsymbol{x} \to h(\boldsymbol{p}) + (2\pi)^2\delta(\boldsymbol{p})H$. So for the order 1 in H, the condition (37) reads

$$(2\pi)^2\delta(\boldsymbol{p})H\tilde{\mathbf{\Phi}}^{(1)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}) = 0, \tag{38}$$

or $\tilde{\mathbf{\Phi}}^{(1)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}=0) = 0$. In the same way, one can prove that

$$\tilde{\mathbf{\Phi}}^{(1)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_k=0,\ldots,\boldsymbol{\xi}_n) = 0 \quad \forall k \in [1,n]. \tag{39}$$

Now, using a finite expansion with respect to the variables $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n$ it follows that:

$$\tilde{\mathbf{\Phi}}^{(n)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n) = \sum_{\alpha_1,\ldots,\alpha_n=x,y} \xi_{1\alpha_1}\ldots\xi_{n\alpha_n}\tilde{\mathbf{\Phi}}^{(n)\alpha_{1\ldots n}}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n), \tag{40}$$

where $\boldsymbol{\xi}_i = (\xi_{ix},\xi_{iy})$. This expansion justifies the name of small-slope approximation when the effects due to the frontiers are neglected in the integration

$$\mathrm{i}\xi_\alpha h(\xi) = \int \mathrm{d}^2\boldsymbol{x}\frac{\partial h(\boldsymbol{x})}{\partial x_\alpha}\exp^{-\mathrm{i}\xi\cdot\boldsymbol{x}}. \tag{41}$$

The coefficients $\overline{\mathbf{\Phi}}^{(n)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n)$ are not unique and independent. However, Voronovich [21, 23] showed that $\overline{\mathbf{\Phi}}^{(n)}$ can be expanded as:

$$\tilde{\mathbf{\Phi}}^{(n)} = \tilde{\mathbf{\Phi}}^{(n)}\big|_{\boldsymbol{\xi}_n=\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}_1-\ldots-\boldsymbol{\xi}_{n-1}} + \left[\tilde{\mathbf{\Phi}}^{(n)} - \tilde{\mathbf{\Phi}}^{(n)}\big|_{\boldsymbol{\xi}_n=\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}_1-\ldots-\boldsymbol{\xi}_{n-1}}\right], \tag{42}$$

the first term in the right handside can be transformed into a term of order $n-1$ which is analogous to $\tilde{\mathbf{\Phi}}^{(n-1)}$, and the term between brackets is transformed into a term of order $n+1$. This important relation will be called a reduction formula in the following.

Taking as an example the first terms in an expansion of Eq. (36), and using Eq. (42) the computation of the term $\overline{\mathbf{\Phi}}^{(0)}$ should involve $\overline{\mathbf{\Phi}}^{(1)}$ but its coefficient can be related to $\overline{\mathbf{\Phi}}^{(0)}$ and $\overline{\mathbf{\Phi}}^{(2)}$ and then

replaced, we obtain the formula

$$\begin{aligned}\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) \;=\; & \int \mathrm{d}^2\boldsymbol{r}\exp^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\cdot\boldsymbol{r}-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h(\boldsymbol{r})}\,\overline{\boldsymbol{\Phi}}^{(0)}(\boldsymbol{p},\boldsymbol{p}_0) \\ & + \int \frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\mathrm{d}^2\boldsymbol{r}\exp^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\cdot\boldsymbol{r}-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h(\boldsymbol{r})} \\ & \times(2\pi)^2\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\overline{\boldsymbol{\Phi}}^{(2)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)h(\boldsymbol{\xi}_1)h(\boldsymbol{\xi}_2). \quad (43)\end{aligned}$$

In this expression, if we take the term of order 1 in h, we get

$$\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) = \overline{\boldsymbol{\Phi}}^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\left[(2\pi)^2\delta(\boldsymbol{p}-\boldsymbol{p}_0) - \mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h(\boldsymbol{p}-\boldsymbol{p}_0)\right]. \quad (44)$$

Voronovich [21, 20] has proposed to identify the expression (43) with the small perturbation method (see [26] Eq. (53)) we obtain (The upper indices 10 in (45) must be read from right to left, indicating the order of the successive media. The same notation will be used in the following.):

$$\begin{aligned}\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0) = & (2\pi)^2\delta(\boldsymbol{p}-\boldsymbol{p}_0)\overline{\boldsymbol{X}}_s^{(0)}(\boldsymbol{p}_0) + \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_0) \\ & +\alpha_0(\boldsymbol{p}_0)\int\frac{\mathrm{d}^2\boldsymbol{p}_1}{(2\pi)^2}\overline{\boldsymbol{X}}_s^{(2)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_1)h(\boldsymbol{p}_1-\boldsymbol{p}_0), \quad (45)\end{aligned}$$

it results the equations:

$$\overline{\boldsymbol{\Phi}}^{(0)}(\boldsymbol{p}_0,\boldsymbol{p}_0) = \overline{\boldsymbol{X}}_s^{(0)}(\boldsymbol{p}_0), \quad (46)$$

$$-\mathrm{i}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\overline{\boldsymbol{\Phi}}^{(0)}(\boldsymbol{p},\boldsymbol{p}_0) = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0), \quad (47)$$

or

$$\overline{\boldsymbol{\Phi}}^{(0)}(\boldsymbol{p},\boldsymbol{p}_0) = \frac{\alpha_0(\boldsymbol{p}_0)}{-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0), \quad (48)$$

$$-2\mathrm{i}\overline{\boldsymbol{X}}^{(0)}(\boldsymbol{p}_0) = \overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}_0|\boldsymbol{p}_0). \quad (49)$$

The first equation gives the coefficient $\overline{\boldsymbol{\Phi}}^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)$ whose the corresponding scattering matrix becomes

$$\begin{aligned}\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0) \;=\; & \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) \\ & \times\int\mathrm{d}^2\boldsymbol{r}\exp^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\cdot\boldsymbol{r}-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h(\boldsymbol{r})}, \quad (50)\end{aligned}$$

where $\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)$ is given by A2, (see also [26] Eq. (61)). Following the same procedure, the order two approximation $\overline{\boldsymbol{\Phi}}^{(2)}$ can be written in term of a order 1 and 3, leading to the expression:

$$\begin{aligned}\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0) = & \frac{\alpha_0(\boldsymbol{p}_0)}{-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))} \cdot \int \frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\mathrm{d}^2\boldsymbol{r} \\ & \times \exp^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h(\boldsymbol{r})}\Big\{(2\pi)^2\delta(\boldsymbol{\xi})\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) \\ & +\frac{\mathrm{i}}{2}\Big[\overline{\boldsymbol{X}}_s^{(2)}(\boldsymbol{p}|\boldsymbol{p}-\boldsymbol{\xi}|\boldsymbol{p}_0)+\overline{\boldsymbol{X}}_s^{(2)}(\boldsymbol{p}|\boldsymbol{p}_0+\boldsymbol{\xi}|\boldsymbol{p}_0) \\ & +\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\Big]\, h(\boldsymbol{\xi})\Big\}, \end{aligned} \tag{51}$$

where $\overline{\boldsymbol{X}}_s^{(2)}$ is given by A3, (see also Eq. (62) [26]). We immediately deduce

$$\begin{aligned}\tilde{\boldsymbol{\Phi}}^{(1)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}) = & \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\frac{1}{2}\Big[\overline{\boldsymbol{X}}_s^{(2)}(\boldsymbol{p}|\boldsymbol{p}-\boldsymbol{\xi}|\boldsymbol{p}_0) \\ & +\overline{\boldsymbol{X}}_s^{(2)}(\boldsymbol{p}|\boldsymbol{p}_0+\boldsymbol{\xi}|\boldsymbol{p}_0)+\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\Big]. \end{aligned} \tag{52}$$

The small-slope approximation method contains following the construction procedure a perturbative term of order 1: Eq. (50), and of order two: Eq. (52). It contains also a phase factor coming from the tangent plane approximation. In addition, Voronovich has shown in the scalar case with boundary Diriclet conditions that the Kirchhoff tangent plane approximation was included in the small-slope method for the order 2 (Eq. (52)).

5. COMPUTATION OF THE CROSS-SECTION

The scattered field is related to the incident field by:

$$\boldsymbol{E}^s(\boldsymbol{x},z) = \frac{\exp(\mathrm{i}K_0||\boldsymbol{r}||)}{||\boldsymbol{r}||}\overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0)\cdot\boldsymbol{E}^i(\boldsymbol{p}_0), \tag{53}$$

with

$$\overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) \equiv \frac{K_0\cos\theta}{2\pi\mathrm{i}}\overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0), \tag{54}$$

$$\boldsymbol{p} = K_0\frac{\boldsymbol{x}}{||\boldsymbol{r}||}, \tag{55}$$

where θ is the angle between $\hat{\boldsymbol{e}}_z$ and the scattering direction. Introducing the Muller matrix $\overline{\boldsymbol{M}}(\boldsymbol{p}|\boldsymbol{p}_0)$, the bistatic matrix is defined by the relations

$$\overline{\gamma}(\boldsymbol{p}|\boldsymbol{p}_0) \equiv \frac{1}{A\cos(\theta_0)}\overline{\boldsymbol{M}}(\boldsymbol{p}|\boldsymbol{p}_0), \tag{56}$$

$$= \frac{1}{A\cos(\theta_0)}\overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) \odot \overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0), \tag{57}$$

$$= \frac{K_0^2\cos^2(\theta)}{(2\pi)^2 A\cos(\theta_0)}\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) \odot \overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0). \tag{58}$$

where the product $\odot$ of two matrices $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{g}}$ is defined by

$$\overline{\boldsymbol{f}} \odot \overline{\boldsymbol{g}} \equiv \begin{pmatrix} f_{VV} & f_{VH} \\ f_{HV} & f_{HH} \end{pmatrix} \odot \begin{pmatrix} g_{VV} & g_{VH} \\ g_{HV} & g_{HH} \end{pmatrix} = \tag{59}$$

$$\begin{pmatrix} f_{VV}g_{VV}^* & f_{VH}g_{VH}^* & Re(f_{VV}g_{VH}^*) & -Im(f_{VV}g_{VH}^*) \\ f_{HV}g_{HV}^* & f_{HH}g_{HH}^* & Re(f_{HV}g_{HH}^*) & -Im(f_{HV}g_{VH}^*) \\ 2Re(f_{VV}g_{HV}^*) & 2Re(f_{VH}g_{HH}^*) & Re(f_{VV}g_{VV}^*+f_{HV}g_{VH}^*) & -Im(f_{VV}g_{HH}^*-f_{VH}g_{HV}^*) \\ 2Im(f_{VV}g_{HV}^*) & 2Im(f_{VH}g_{HH}^*) & Im(f_{VV}g_{VV}^*+f_{HV}g_{VH}^*) & Re(f_{VV}g_{HH}^*-f_{VH}g_{HV}^*) \end{pmatrix}.$$

The scattering from a randomly rough surface is a stochastic process, so the computations of radar or laser cross-section for the coherent and incoherent parts involve an average over the surfaces realizations. The definition of the coherent bistatic matrix reads

$$\overline{\gamma}^{coh} \equiv \frac{1}{A\cos\theta_0} < \overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) > \odot < \overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) >$$

$$= \frac{K_0^2\cos^2\theta}{A(2\pi)^2\cos\theta_0} < \overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) > \odot < \overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) >, \tag{60}$$

and the incoherent bistatic matrix

$$\begin{aligned}\overline{\gamma}^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0) \equiv & \frac{1}{A\cos\theta_0}\left[< \overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) \odot \overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) > \right. \\ & \left. - < \overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) > \odot < \overline{\boldsymbol{f}}(\boldsymbol{p}|\boldsymbol{p}_0) >\right], \\ = & \frac{K_0^2\cos^2\theta}{A(2\pi)^2\cos\theta_0}\left[< \overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) \odot \overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) > \right. \\ & \left. - < \overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) > \odot < \overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) >\right]. \end{aligned} \tag{61}$$

If we consider the case of a single rough surface Eq. (51) where we set

$$\begin{aligned}\overline{\boldsymbol{\Sigma}}_s^0(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi}) = & \overline{\boldsymbol{X}}_s^{(2)}(\boldsymbol{p}|\boldsymbol{p}-\boldsymbol{\xi}|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}_s^{(2)}(\boldsymbol{p}|\boldsymbol{p}_0+\boldsymbol{\xi}|\boldsymbol{p}_0) \\ & + \mathrm{i}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0), \end{aligned} \tag{62}$$

this matrix does not comply with the reciprocity condition, so we will define a reciprocal matrix by the relation

$$\overline{\boldsymbol{\Sigma}}_s(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi}) = \frac{1}{2}\left[\overline{\boldsymbol{\Sigma}}_s^0(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi}) + \overline{\boldsymbol{\Sigma}}_s^0(-\boldsymbol{p}_0|-\boldsymbol{p}|-\boldsymbol{\xi})^{aT}\right], \tag{63}$$

where aT means the anti-transpose of a matrix, with the definition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{aT} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}. \tag{64}$$

Taking the statistical average of the matrix $\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)$ one obtains for the coherent part

$$\begin{aligned} <\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)> \equiv\ & \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}\exp^{-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\sigma^2/2}. \\ & \times\int \mathrm{d}^2\boldsymbol{r}e^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\cdot\boldsymbol{r}}\left\{\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\right. \\ & \left. + \frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}W(\boldsymbol{\xi})\overline{\boldsymbol{\Sigma}}_s(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\right\}, \end{aligned} \tag{65}$$

and for the incoherent part the expression

$$\begin{aligned} \overline{\gamma}^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0) =& \frac{K_0^2\cos^2\theta}{A(2\pi)^2\cos\theta_0}\left[<\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)>\right. \\ & \left. -<\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)>\odot<\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)>\right], \end{aligned} \tag{66}$$

where

$$\begin{aligned} <\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}_s^{10}(\boldsymbol{p}|\boldsymbol{p}_0)> =& \frac{\alpha_0(\boldsymbol{p}_0)\alpha_0(\boldsymbol{p})}{-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}\exp^{-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\sigma^2/2}. \\ & \times\int\mathrm{d}^2\boldsymbol{r}\int\mathrm{d}^2\boldsymbol{r}'\exp^{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2W(\boldsymbol{r}-\boldsymbol{r}')}\exp^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')}. \\ & \left\{\left[\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p},\boldsymbol{p}_0) - \frac{\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}W(\boldsymbol{\xi})\right.\right. \\ & \left.\times(\exp^{\mathrm{i}\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}')}-1)\overline{\boldsymbol{\Sigma}}_s(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\right] \\ & \odot\left[\overline{\boldsymbol{X}}_s^{(1)}(\boldsymbol{p},\boldsymbol{p}_0) - \frac{\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}W(\boldsymbol{\xi})\right. \\ & \left.\left.\times(\exp^{\mathrm{i}\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}')}-1)\overline{\boldsymbol{\Sigma}}_s(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\right]^*\right. \\ & \left.+\frac{1}{4}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}W(\boldsymbol{\xi})\overline{\boldsymbol{\Sigma}}_s(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\odot\overline{\boldsymbol{\Sigma}}_s(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\right\}. \end{aligned} \tag{67}$$

6. A ROUGH SURFACE BETWEEN TWO SEMI-INFINITE MEDIA

In this section we apply the above derivation to the simple case of a rough surface between two semi-infinite media. It will serve as test of the small-slope method described in our formalism by making a comparison with well-know examples. We take as first (incident) medium the vacuum followed by a dielectric medium ($n_1 = 1.62 + \mathrm{i}0.001$), the frontier being a rough surface with a rms height $\sigma = 0.223\,\mu$m and a correlation length $l = 1.42\,\mu$m (structure no 1). The incident wave length $\lambda = 632.8\,$nm, the angle of incidence $\theta_i = 20°$, and the azimuthal plane $\phi_i = 0°$.

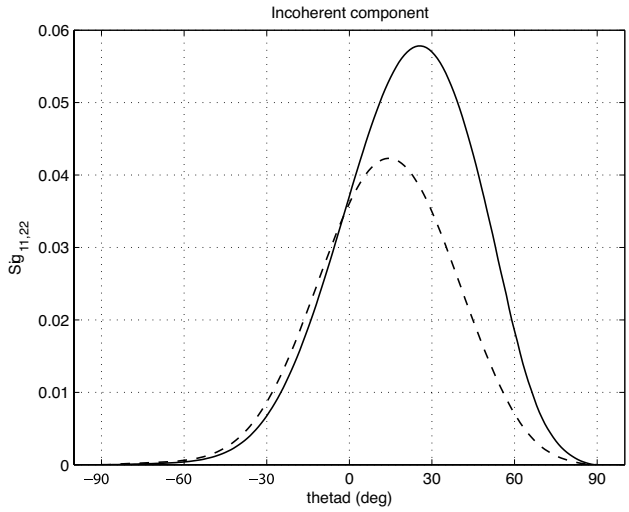


Figure 3. Incoherent components $\gamma^{incoh}(\theta_d)$ to the second order approximation as a function of the scattering angle θ_d VV (dashed line), HH (solid line). Medium characteristics: height $\sigma = 0.223\,\mu$m, correlation length $l = 1.42\,\mu$m, index $n_0 = 1$, $n_1 = 1.62 + \mathrm{i}\,0.001$. Incident angles: $\theta_i = 20°$, $\phi_i = 0°$, wavelength $\lambda = 632.8\,$nm.

The incoherent components $\gamma^{incoh}(\theta_d)$ are shown in Figs. 3 and 4 as a function of the scattering angle θ_d in the order 2 approximation (All the calculations are performed with MATLAB, The MathWorks, Inc.). The scattering intensity for the coherent part with 4 polarization components is presented in Fig. 5. The results agree well with those obtained in [30, 31]. As a second example (structure no 2), we consider a rough surface made of aluminium with relative permittivity

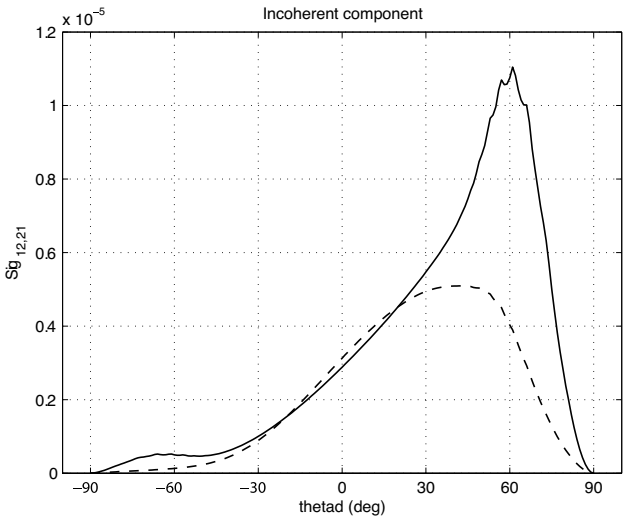


Figure 4. Same characteristics as the previous figure, components VH (dashed line), HV (solid line).

$\epsilon_1 = -40 - \mathrm{i}1.1$. The rough surface is supposed to be homogeneous and isotrope with rms height ($\sigma = 0.3/K_0$) of Gaussian nature, and with a correlation length ($l = 3/K_0$). The incident wave length $\lambda = 632.8\,\mathrm{nm}$. The angle of incidence $\theta_i = 20°$, and the azimuthal plane of incidence $\phi_i = 0$. The incoherent components $\gamma^{incoh}(\theta_d)$ are drawn in Figs. 6 and 7 as a function of the scattering angle θ_d, calculated to the second order approximation. The scattered intensity for the coherent part including 4 polarization components is shown in Fig. 8.

Starting from the previous structure (no 2) we modify the statistical parameters in such a way that neither the Kirchoff method nor the small-perturbation method are valid, taking for instance $\sigma = 1/K_0$, et $l = 1/K_0$ (structure no 3). In this case we obtain the results shown in Figs. 9–11, and we observe a very different behavior for the incoherent components, there exists for VV and VH two maxima around $\theta_d = \pm 70°$, while for HV a maximum occurs for $\theta_d = 0°$ and the order of magnitude of the cross-section is reduced by a factor 2.

In a last example (structure no 4) we take the case of a calculation made with small-perturbation method we have published in [26], see Fig. 8. For this structure, $K_0\sigma = 0.068$ and $1/K_0 l = 0.73$. The results with the SSA method are shown in Figs. 12–14. For the four polarization components we agree with the order of magnitude and the shape of the intensity, however, small oscillations are present, their origin is certainly due to the FFT integration method we have used.

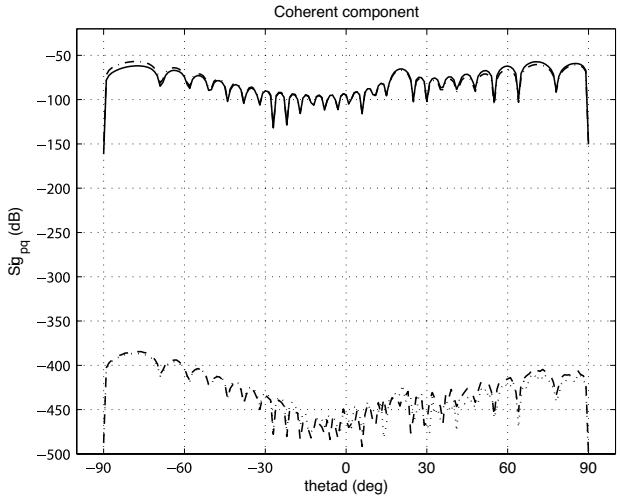


Figure 5. Same medium characteristics as in Fig. 3. Coherent components $\gamma^{coh}(\theta_d)$ as a function of the scattering angle θ_d, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

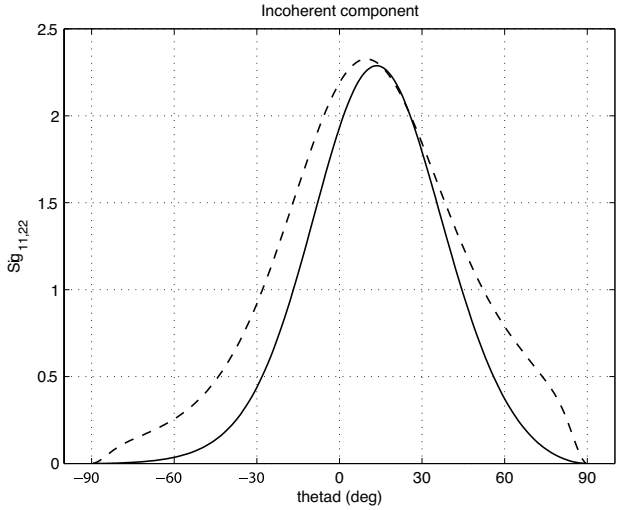


Figure 6. Incoherent components $\gamma^{incoh}(\theta_d)$ in the order 2 approximation as a function of the scattering angle θ_d. VV (dashed line), HH (solid line). Incident wavelength $\lambda = 632.8\,\text{nm}$, height $\sigma = 0.3/K_0$, correlation length $l = 3/K_0$. Angles: $\theta_i = 20°$, $\phi_i = 0°$, permittivity: $\epsilon_0 = 1$, $\epsilon_1 = -40 - \text{i}\,1.1$.

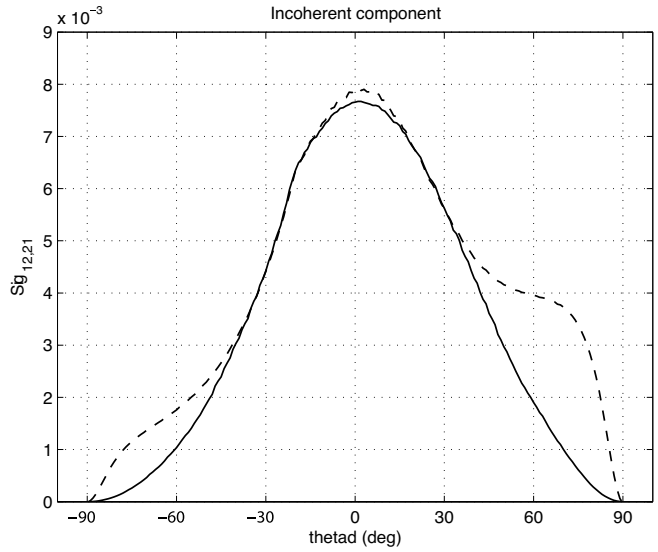


Figure 7. Same characteristics as the previous figure, components VH (dashed line), HV (solid line).

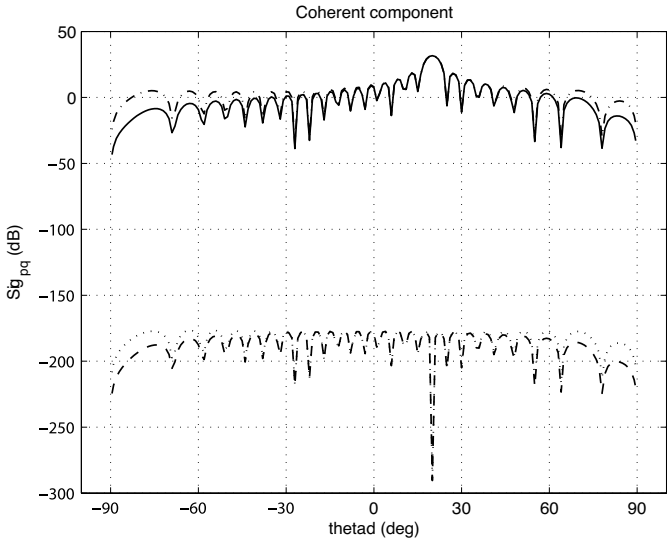


Figure 8. Medium characteristics of Fig. 6. Coherent components $\gamma^{coh}(\theta_d)$ as a function of the scattering angle θ_d, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

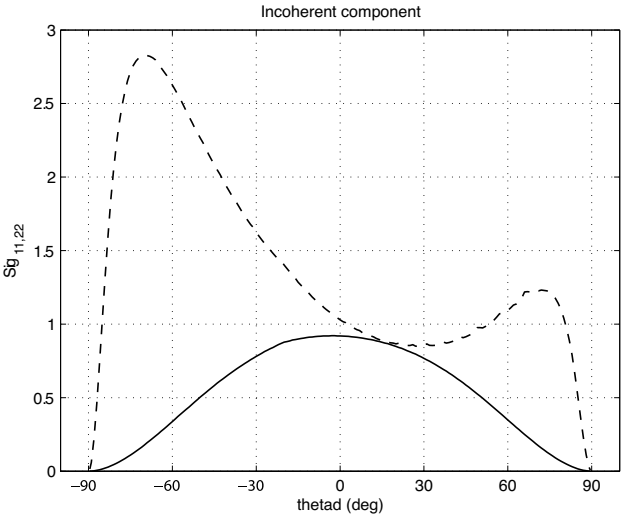


Figure 9. Incoherent components $\gamma^{incoh}(\theta_d)$ in the order 2 approximation as a function of the scattering angle θ_d. VV (dashed line), HH (solid line), Incident wavelength $\lambda = 632.8\,\text{nm}$, surface height $\sigma = 1/K_0$, correlation length: $l = 1/K_0$. Angles: $\theta_i = 20°$, $\phi_i = \phi = 0°$. Permittivity: $\epsilon_0 = 1$, $\epsilon_1 = -40 - \text{i}\,1.1$.

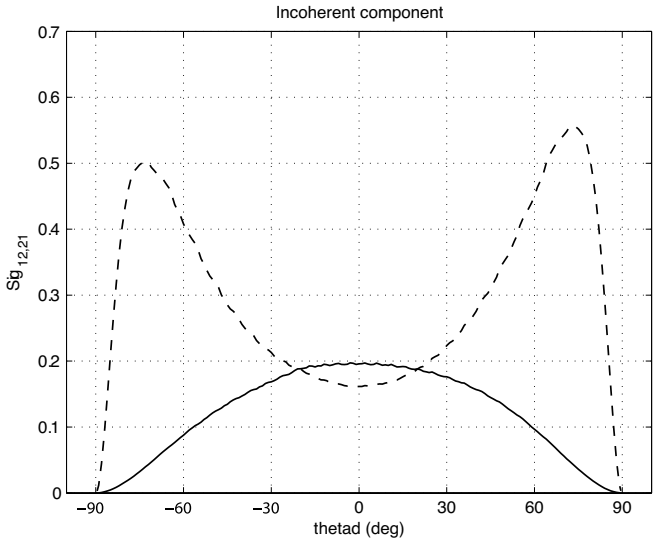


Figure 10. Medium characteristics identical to the previous figure. Incoherent components VH (dashed line), HV (solid line).

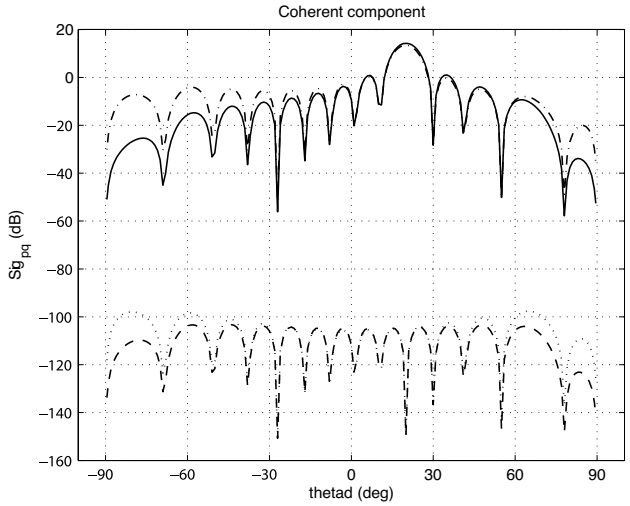


Figure 11. Medium characteristics identical to Fig. 9. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

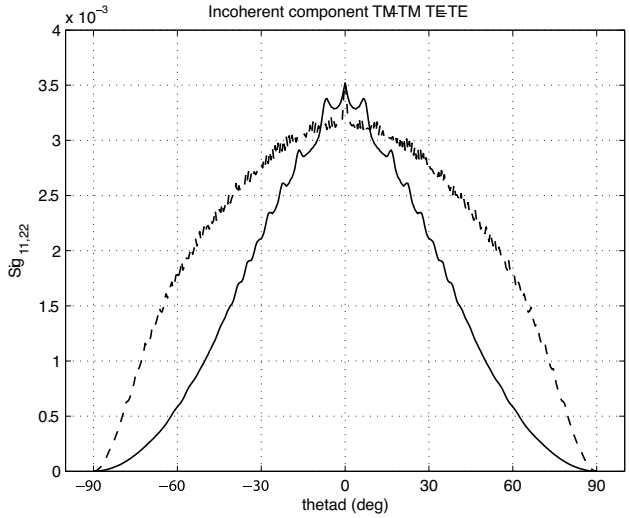


Figure 12. Incoherent components $\gamma^{incoh}(\theta_d)$ with SSA to the order 2. VV (dashed line), HH (solid line). Incident wavelength $\lambda = 457.9\,nm$, surface height $\sigma = 5\,\text{nm}$, correlation length: $l = 100\,\text{nm}$. Angles: $\theta_i = 0°$, $\phi_i = 0°$, permittivity: $\epsilon_0 = 1$, $\epsilon_1 = -7.5 - \text{i}\,0.24$.

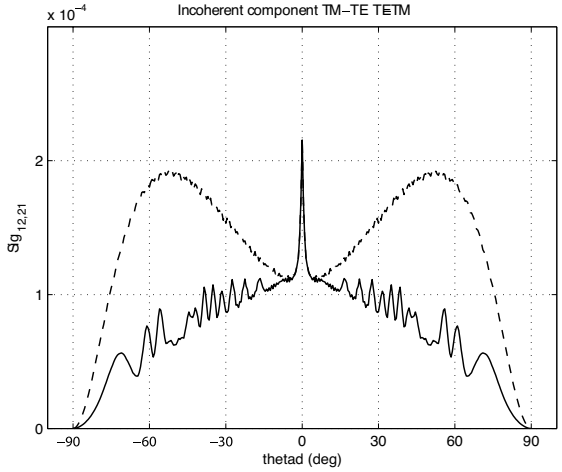


Figure 13. Same characteristics as the previous figure. Components VH (dashed line), HV (solid line).

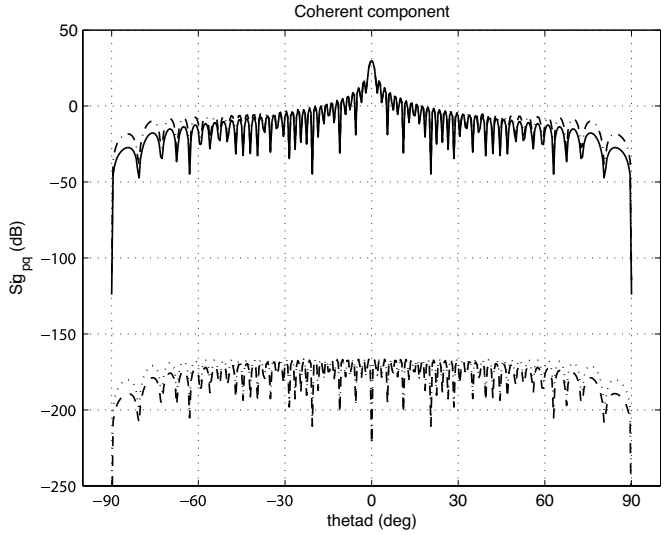


Figure 14. Same characteristics as Fig. 12. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

7. A SLAB WITH A ROUGH SURFACE ON THE BOTTOM SIDE

In this section we start with main object of the paper namely to compute a scattering process generated by a slab. Here, we consider a

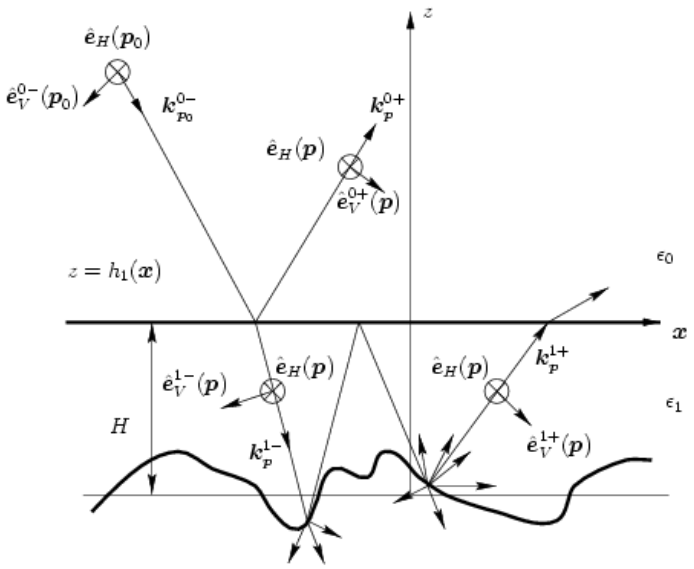


Figure 15. A slab with a rough surface at the lower boundary and planar surface on the upper side.

slab whose lower boundary is a two dimensional rough surface and the upper boundary is a planar surface. A schematic view of the geometry and the different waves propagating in the structure is given in Fig. 15. Making the observation that in medium 2 no wave is coming in the upward direction, the scattering matrices obtained in the previous section are still valid.

In [26] Section B, we have shown in the case of the small-perturbation method that the scattering matrix is given up to the order 2 in h by the expression

$$\begin{aligned}\overline{\boldsymbol{R}}(\boldsymbol{p}|\boldsymbol{p}_0) \;=\; & (2\pi)^2\delta(\boldsymbol{p}-\boldsymbol{p}_0)\overline{\boldsymbol{X}}_d^{(0)}(\boldsymbol{p}_0) + \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_0) \\ & +\alpha_0(\boldsymbol{p}_0)\int \frac{\mathrm{d}^2\boldsymbol{p}_1}{(2\pi)^2}\overline{\boldsymbol{X}}_d^{(2)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_1)h(\boldsymbol{p}_1-\boldsymbol{p}_0), \quad (68)\end{aligned}$$

where the matrices $\overline{\boldsymbol{X}}_d^{(i)}$ are given in the Appendix B. Following the method proposed by Voronovich and applied in the previous section, we identify the terms obtained by the small-perturbation method with those of the SSA method, this procedure leads to the expression of the

scattering matrix

$$\overline{\boldsymbol{R}}_d^{10}(\boldsymbol{p}|\boldsymbol{p}_0) = \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))} \cdot \int \frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\mathrm{d}^2\boldsymbol{r} \times \exp^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h(\boldsymbol{r})} \left\{ (2\pi)^2\delta(\boldsymbol{\xi})\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) + \frac{\mathrm{i}}{2}\left[\overline{\boldsymbol{X}}_d^{(2)}(\boldsymbol{p}|\boldsymbol{p}-\boldsymbol{\xi}|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}_d^{(2)}(\boldsymbol{p}|\boldsymbol{p}_0+\boldsymbol{\xi}|\boldsymbol{p}_0) + \mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\right] h(\boldsymbol{\xi})\right\}. \quad (69)$$

7.1. Applications

We begin with the structure (structure no 5) taken from [26] see Fig. 14. The slab is characterized by the parameters: rms height $\sigma = 5\,\mathrm{nm}$, correlation length $l = 500\,\mathrm{nm}$, and a slab thickness $H = 500\,\mathrm{nm}$. The permittivities of the successive media are: $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \mathrm{i}0.0075$, $\epsilon_2 = -18.3 + \mathrm{i}0.55$. The incident wavelength $\lambda = 632.8\,\mathrm{nm}$, and the angle of incidence $\theta_i = 0°$. The intensity curves are shown in Figs. 16–18. We observe for the incoherent components that the magnitude is the same as in the small-perturbation method (SPM), with a maximum of intensity for $\theta_d = 0$. We notice the presence of small oscillations for the polarization VV, and for the polarization HV the appearance of a structure around $\theta_d = \pm 50°$ which does not show up in the former method, and the absence of satellite peaks for the VV component.

At this point we can make two remarks: the order 2 approximation of the SSA method is a linear combination of the order 1 and 2 of the SPM, see Eqs. (62) and (63), it implies that the fine structure observed for the order 2 in SPM is probably masked by the global effect due to the SSA order 2. Moreover, our numerical experience in the SPM case, shows that the functions to be integrated contain very narrow peaks needing a special treatment (see [28] for a discussion), in the case of the SSA method where we integrate by a FFT, even an increase of the number of points is not sufficient to recover the peaks. In the next example, we take the parameters of structure no 1, where we introduce above the rough surface a slab of thickness $H = 500\,\mathrm{nm}$ and permittivity $\epsilon_1 = 2.6896 + \mathrm{i}0.0075$ (structure no 6). The effect of the absorbing dielectric slab shows (as expected) a decrease of the reflected intensities for all the polarization states, however, the shape of the curves remains the same for the polarizations VV and HH, the results are shown in Figs. 19–21. In a last example, we take a rough surface made of aluminium, the parameters are the same as in structure no 2, and we add above the surface an absorbing dielectric slab of permittivity $\epsilon_2 = 2.6896 - \mathrm{i}0.0075$ (structure no 7). The results are

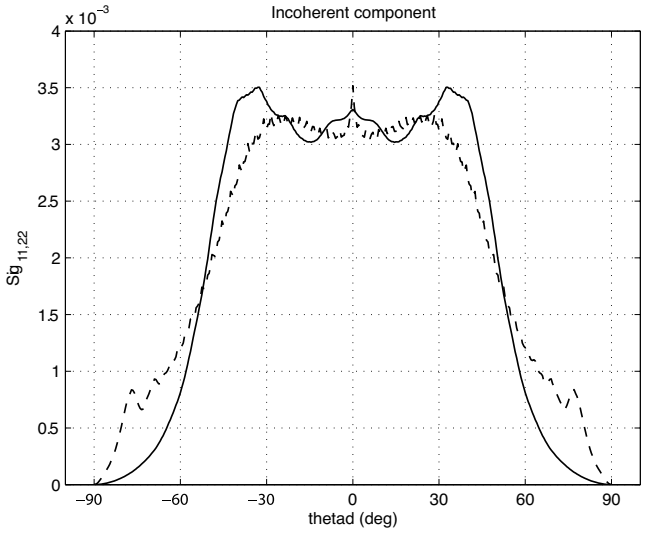


Figure 16. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2, VV (dashed line), HH (solid line), Surface height $\sigma = 5\,\mathrm{nm}$, correlation length $l = 100\,\mathrm{nm}$, slab thickness 500 nm. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \mathrm{i}\,0.0075$, $\epsilon_2 = -18.3 + \mathrm{i}0.55$. Incident angles: $\theta_i = 0°$, $\phi_i = 0°$, wavelength $\lambda = 632.8\,\mathrm{nm}$.

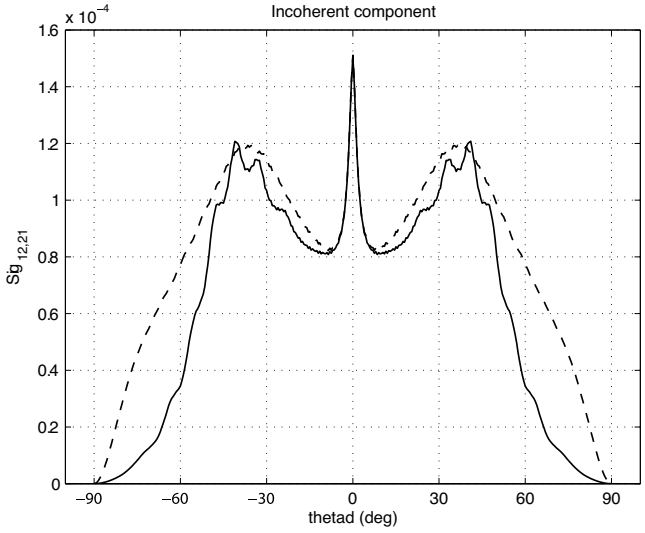


Figure 17. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

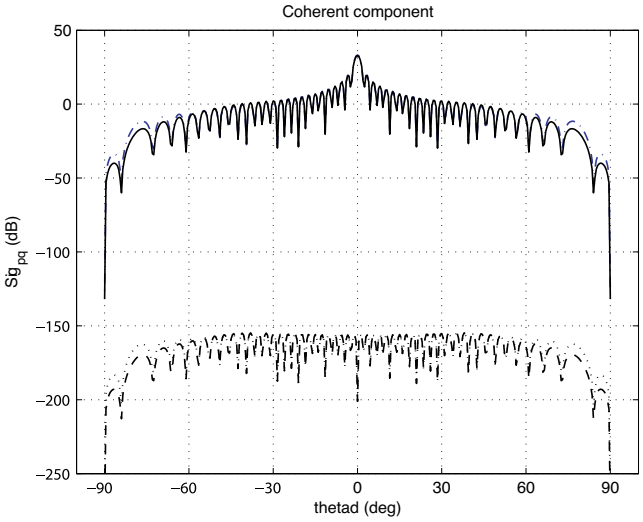


Figure 18. Characteristics of Fig. 16. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

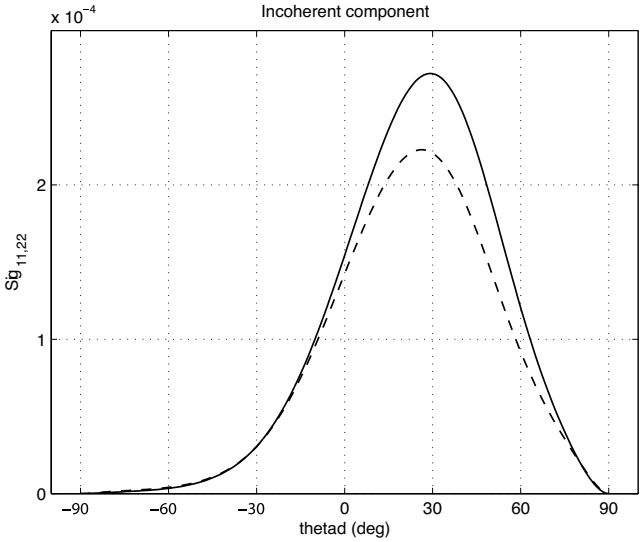


Figure 19. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2. VV (dashed line), HH (solid line), Surface height $\sigma = 0.223\,\mu$m, correlation length $l = 1.42\,\mu$m, slab thickness 500 nm. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \mathrm{i}\,0.0075$, $\epsilon_2 = 1.62 + \mathrm{i}\,0.001$. Incident angles: $\theta_i = 20°$, $\phi_i = 0°$, wavelength $\lambda = 632.8$ nm.

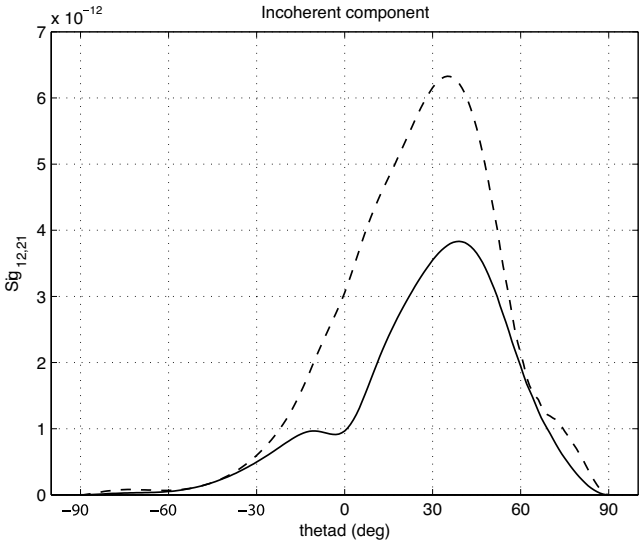


Figure 20. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

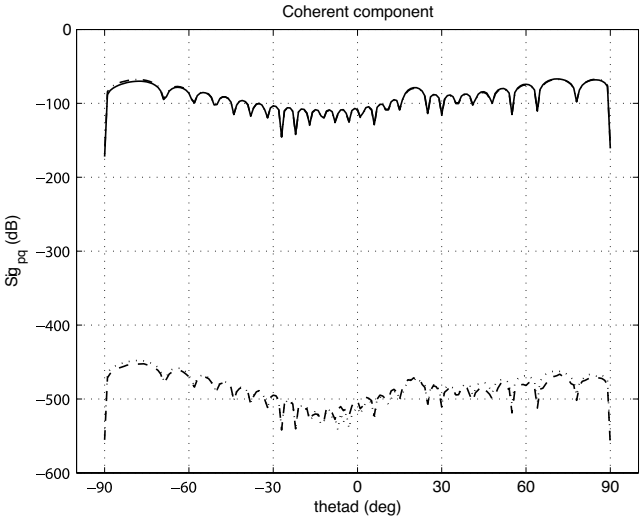


Figure 21. Characteristics of Fig. 19. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

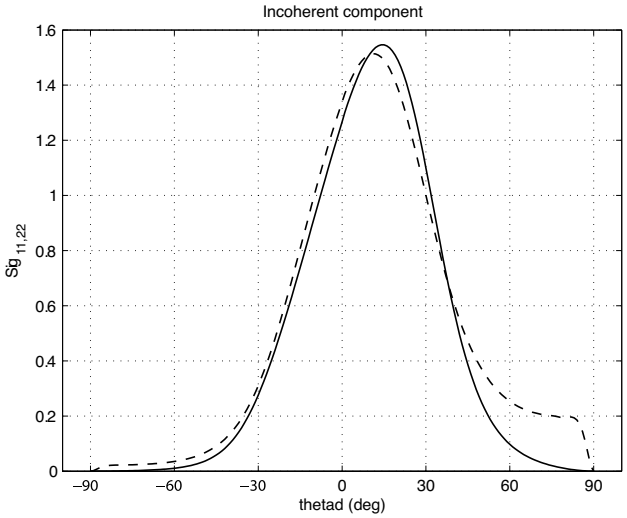


Figure 22. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2. VV (dashed line), HH (solid line), Surface height $\sigma = 0.3/K_0$, correlation length $l = 3/K_0$, slab thickness 500 nm. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \mathrm{i}\,0.0075$, $\epsilon_2 = -40 - \mathrm{i}\,1.1$. Incident angles: $\theta_i = 20°$, $\phi_i = 0°$, wavelength $\lambda = 632.8$ nm.

presented in Figs. 22–24. The addition of an absorbing slab decreases the intensity for the polarizations VV and HH while the shape remains the same, but for the polarisation VH we observe two maxima instead of one in structure no 1.

8. A SLAB WITH A ROUGH SURFACE ON THE UPPER SIDE

We consider a dielectric slab of permittivity ϵ_1 inserted between two semi-infinite media of permittivity ϵ_0 and ϵ_2. The upper part of slab is a rough surface, the lower part is a planar one, see Fig. 25. In order to compute the scattering matrix $\overline{\boldsymbol{R}}_u(\boldsymbol{p}|\boldsymbol{p}_0)$, we need first to determine the scattering matrix in the small-perturbation method that we summarized. We will start from the two reduced Rayleigh equations

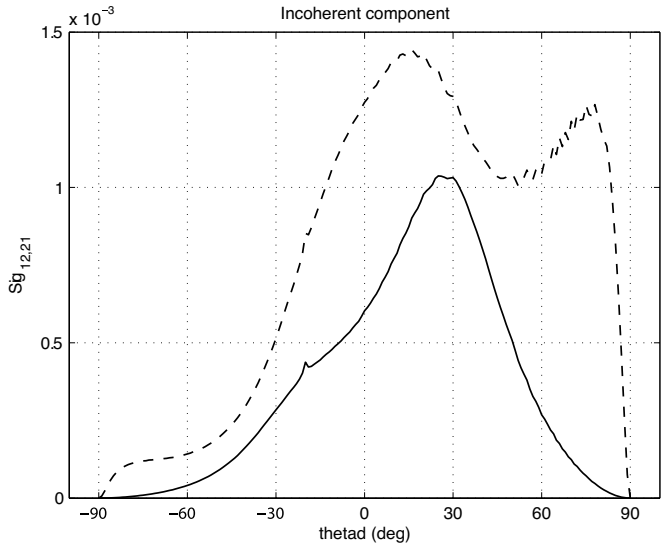


Figure 23. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

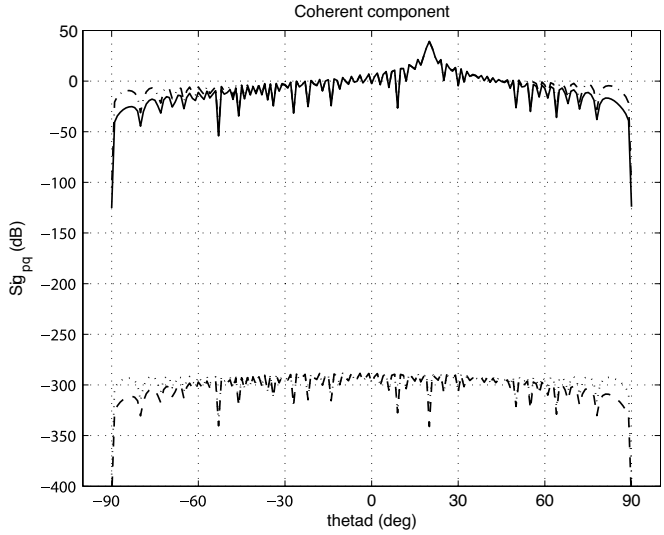


Figure 24. Characteristics of Fig. 22. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

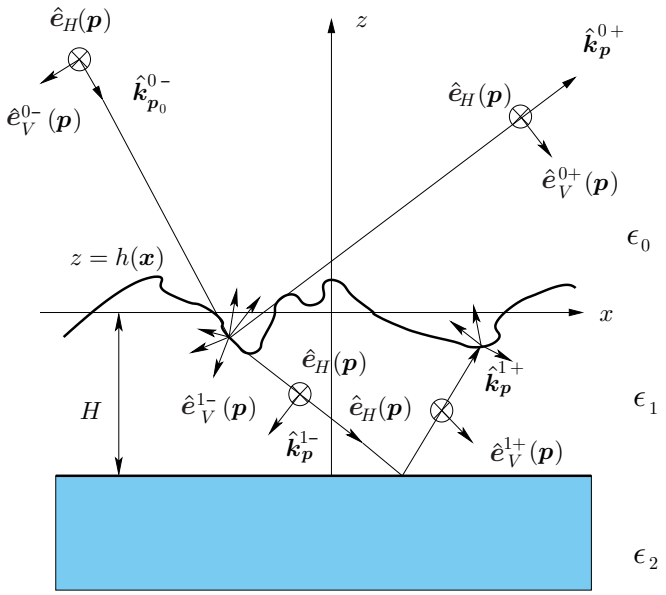


Figure 25. A slab of permittivity ϵ_1 and thickness H between two semi-infinite media of permittivity ϵ_0 and ϵ_2.

obtained in [26] Eqs. (100) and (101), namely:

$$\int \frac{\mathrm{d}^2\boldsymbol{u}}{(2\pi)^2}\overline{\boldsymbol{M}}_h^{1+,0+}(\boldsymbol{p}|\boldsymbol{u})\cdot\overline{\boldsymbol{R}}_u(\boldsymbol{u}|\boldsymbol{p}_0)\cdot\boldsymbol{E}^i(\boldsymbol{p}_0)+\overline{\boldsymbol{M}}_h^{1+,0-}(\boldsymbol{p}|\boldsymbol{p}_0)\cdot\boldsymbol{E}^i(\boldsymbol{p}_0)$$
$$=\frac{2(\epsilon_0\epsilon_1)^{\frac{1}{2}}\alpha_1(\boldsymbol{p})}{(\epsilon_1-\epsilon_0)}\boldsymbol{E}^{1+}(\boldsymbol{p}), \tag{70}$$

$$\int \frac{\mathrm{d}^2\boldsymbol{u}}{(2\pi)^2}\overline{\boldsymbol{M}}_h^{1-,0+}(\boldsymbol{p}|\boldsymbol{u})\cdot\overline{\boldsymbol{R}}_u(\boldsymbol{u}|\boldsymbol{p}_0)\cdot\boldsymbol{E}^i(\boldsymbol{p}_0)+\overline{\boldsymbol{M}}_h^{1-,0-}(\boldsymbol{p}|\boldsymbol{p}_0)\cdot\boldsymbol{E}^i(\boldsymbol{p}_0)$$
$$=-\frac{2(\epsilon_0\epsilon_1)^{\frac{1}{2}}\alpha_1(\boldsymbol{p})}{(\epsilon_1-\epsilon_0)}\boldsymbol{E}^{1-}(\boldsymbol{p}), \tag{71}$$

where the matrices $\overline{\boldsymbol{M}}_h$ are given by:

$$\overline{\boldsymbol{M}}_h^{1b,0a}(\boldsymbol{u}|\boldsymbol{p})\equiv\frac{I(b\alpha_1(\boldsymbol{u})-a\alpha_0(\boldsymbol{p})|\boldsymbol{u}-\boldsymbol{p})}{b\alpha_1(\boldsymbol{u})-a\alpha_0(\boldsymbol{p})}\overline{\boldsymbol{M}}^{1b,0a}(\boldsymbol{u}|\boldsymbol{p}), \tag{72}$$

$$\overline{\boldsymbol{M}}_h^{0b,1a}(\boldsymbol{u}|\boldsymbol{p})\equiv\frac{I(b\alpha_0(\boldsymbol{u})-a\alpha_1(\boldsymbol{p})|\boldsymbol{u}-\boldsymbol{p})}{b\alpha_0(\boldsymbol{u})-a\alpha_1(\boldsymbol{p})}\overline{\boldsymbol{M}}^{0b,1a}(\boldsymbol{u}|\boldsymbol{p}), \tag{73}$$

with

$$\overline{\boldsymbol{M}}^{1b,0a}(\boldsymbol{u}|\boldsymbol{p}) = \begin{pmatrix} ||\boldsymbol{u}||||\boldsymbol{p}|| + ab\alpha_1(\boldsymbol{u})\alpha_0(\boldsymbol{p})\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}} & -b\epsilon_0^{\frac{1}{2}} K_0\alpha_1(\boldsymbol{u})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}})_z \\ a\epsilon_1^{\frac{1}{2}} K_0\alpha_0(\boldsymbol{p})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}})_z & (\epsilon_0\epsilon_1)^{\frac{1}{2}} K_0^2\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}} \end{pmatrix}, \tag{74}$$

$$\overline{\boldsymbol{M}}^{0b,1a}(\boldsymbol{u}|\boldsymbol{p}= \begin{pmatrix} ||\boldsymbol{u}||||\boldsymbol{p}|| + ab\alpha_0(\boldsymbol{u})\alpha_1(\boldsymbol{p})\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}} & -b\epsilon_1^{\frac{1}{2}} K_0\alpha_0(\boldsymbol{u})\,(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}})_z \\ a\epsilon_0^{\frac{1}{2}} K_0\alpha_1(\boldsymbol{p})\,(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}})_z & (\epsilon_0\epsilon_1)^{\frac{1}{2}} K_0^2\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}} \end{pmatrix}, \tag{75}$$

and

$$I(\alpha|\boldsymbol{p}) \equiv \int \mathrm{d}^2\boldsymbol{x}\exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} - i\alpha h(\boldsymbol{x})). \tag{76}$$

Inside the slab the scattered field by the planar surface is related to the incident field by the relation

$$\boldsymbol{E}^{1+}(\boldsymbol{u}) = \overline{\boldsymbol{r}}^{H21}(\boldsymbol{u})\cdot\boldsymbol{E}^{1-}(\boldsymbol{u}), \tag{77}$$

where $\overline{\boldsymbol{r}}^{H\,21}$ is a diagonal matrix

$$\overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{p}) = \exp^{2\,\mathrm{i}\alpha_1(\boldsymbol{p})H}\,\overline{\boldsymbol{r}}^{21}(\boldsymbol{p}), \tag{78}$$

$$\overline{\boldsymbol{r}}^{21}(\boldsymbol{p}) \equiv \begin{pmatrix} \frac{\epsilon_2\alpha_1(\boldsymbol{p})-\epsilon_1\alpha_2(\boldsymbol{p})}{\epsilon_2\alpha_1(\boldsymbol{p})+\epsilon_1\alpha_2(\boldsymbol{p})} & 0 \\ 0 & \frac{\alpha_1(\boldsymbol{p})-\alpha_2(\boldsymbol{p})}{\alpha_1(\boldsymbol{p})+\alpha_2(\boldsymbol{p})} \end{pmatrix}, \tag{79}$$

this matrix contains the reflection coefficients for a planar surface located at $z = -H$ which separates two media of permittivity ϵ_1 and ϵ_2. The phase factor $\exp(2\mathrm{i}\alpha_1(\boldsymbol{p})H)$ describes the extra path of the scattered wave due to the planar surface. Collecting the integral Eqs. (70) and (71), the matrix $\overline{\boldsymbol{R}}_u$ is a solution of the equation

$$\int \frac{\mathrm{d}^2\boldsymbol{u}}{(2\pi)^2}\left[\overline{\boldsymbol{M}}_h^{1+,0+}(\boldsymbol{p}|\boldsymbol{u}) + \overline{\boldsymbol{r}}^{\,H\,21}(\boldsymbol{p})\cdot\overline{\boldsymbol{M}}_h^{1-,0+}(\boldsymbol{p}|\boldsymbol{u})\right]\cdot\overline{\boldsymbol{R}}_u(\boldsymbol{u}|\boldsymbol{p}_0) = -\left[\overline{\boldsymbol{M}}_h^{1+,0-}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\boldsymbol{r}}^{\,H\,21}(\boldsymbol{p})\cdot\overline{\boldsymbol{M}}_h^{1-,0-}(\boldsymbol{p}|\boldsymbol{p}_0)\right]. \tag{80}$$

In order to construct a perturbative development, the method consists to expand in Taylor series $I(\alpha|\boldsymbol{p})$ with respect to h. We obtain for the matrix $\overline{\boldsymbol{R}}_u$ an expansion analogous to Eq. (24)

$$\begin{aligned}\overline{\boldsymbol{R}}_u(\boldsymbol{p}|\boldsymbol{p}_0) \;=\; & (2\pi)^2\delta(\boldsymbol{p}-\boldsymbol{p}_0)\overline{\boldsymbol{X}}_u^{(0)}(\boldsymbol{p}_0) + \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_0) \\ & +\alpha_0(\boldsymbol{p}_0)\int\frac{\mathrm{d}^2\boldsymbol{p}_1}{(2\pi)^2}\overline{\boldsymbol{X}}_u^{(2)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0)h(\boldsymbol{p}-\boldsymbol{p}_1)h(\boldsymbol{p}_1-\boldsymbol{p}_0),\end{aligned} \tag{81}$$

with the following expressions for the matrices $\overline{\boldsymbol{X}}_u$

$$\overline{\boldsymbol{X}}_u^{(0)}(\boldsymbol{p}_0) = -\left[\frac{\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{p}_0|\boldsymbol{p}_0)}{\alpha_1(\boldsymbol{p}_0)-\alpha_0(\boldsymbol{p}_0)} - \overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{p}_0)\cdot\frac{\overline{\boldsymbol{M}}^{1-,0+}(\boldsymbol{p}_0|\boldsymbol{p}_0)}{\alpha_1(\boldsymbol{p}_0)+\alpha_0(\boldsymbol{p}_0)}\right]^{-1} \cdot\left[\frac{\overline{\boldsymbol{M}}^{1+,0-}(\boldsymbol{p}_0|\boldsymbol{p}_0)}{\alpha_1(\boldsymbol{p}_0)+\alpha_0(\boldsymbol{p}_0)} + \overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{p}_0)\cdot\frac{\overline{\boldsymbol{M}}^{1-,0-}(\boldsymbol{p}_0|\boldsymbol{p}_0)}{-\alpha_1(\boldsymbol{p}_0)+\alpha_0(\boldsymbol{p}_0)}\right], = \left(\overline{\boldsymbol{r}}^{10}(\boldsymbol{p}_0)+\overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{p}_0)\right)\cdot\left[\overline{1}+\overline{\boldsymbol{r}}^{10}(\boldsymbol{p}_0)\cdot\overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{p}_0)\right]^{-1}, \quad (82)$$

$\overline{\boldsymbol{r}}^{10}(\boldsymbol{p}_0)$ is given by (C13).

$$\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{u}|\boldsymbol{p}_0) \equiv 2i\ \overline{\boldsymbol{Q}}^{++}(\boldsymbol{u}|\boldsymbol{p}_0), \quad (83)$$

$$\overline{\boldsymbol{X}}_u^{(2)}(\boldsymbol{u}|\boldsymbol{p}_1|\boldsymbol{p}_0) = \alpha_1(\boldsymbol{u})\overline{\boldsymbol{Q}}^{-+}(\boldsymbol{u}|\boldsymbol{p}_0)+\alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{Q}}^{+-}(\boldsymbol{u}|\boldsymbol{p}_0) -2\overline{\boldsymbol{P}}^{+}(\boldsymbol{u}|\boldsymbol{p}_1)\cdot\overline{\boldsymbol{Q}}^{++}(\boldsymbol{p}_1|\boldsymbol{p}_0), \quad (84)$$

$\overline{\boldsymbol{Q}}$ and $\overline{\boldsymbol{P}}$ are given in Appendix C.

In order to obtain the scattering matrix for a slab with a rough surface at the upper boundary in the SSA approximation, we follow the same method of identification between the SSA and SPM described in Section 7, and we get

$$\overline{\boldsymbol{R}}_u^{10}(\boldsymbol{p}|\boldsymbol{p}_0) = \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}\cdot\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\mathrm{d}^2\,\boldsymbol{r} \times\exp^{-\mathrm{i}\,(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\,h(\boldsymbol{r})}\left\{(2\pi)^2\delta(\boldsymbol{\xi})\,\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\right. +\frac{\mathrm{i}}{2}\left[\overline{\boldsymbol{X}}_u^{(2)}(\boldsymbol{p}|\boldsymbol{p}-\boldsymbol{\xi}|\boldsymbol{p}_0)+\overline{\boldsymbol{X}}_u^{(2)}(\boldsymbol{p}|\boldsymbol{p}_0+\boldsymbol{\xi}|\boldsymbol{p}_0)\right. \left.\left.+\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\right]h(\boldsymbol{\xi})\right\}. \quad (85)$$

The last step is to introduce in Eq. (85) the $\overline{\boldsymbol{X}}_u$ reciprocal matrices to complete the expression of the scattering matrix $\overline{\boldsymbol{R}}_u^{10}(\boldsymbol{p}|\boldsymbol{p}_0)$.

8.1. Applications

We take as a first example a slab of thickness $H = 500\,\mathrm{nm}$, with an upper rough surface $\sigma = 15\,\mathrm{nm}$, $l = 100\,\mathrm{nm}$, and a lower planar surface made of a perfect conductor (structure no 8). The successive media have a permittivity: $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \mathrm{i}0.0075$. The incident field is normal to the slab, and the wavelength $\lambda = 632.8\,\mathrm{nm}$. The scattered intensities for the polarizations VV and HH are presented in Fig. 26, a

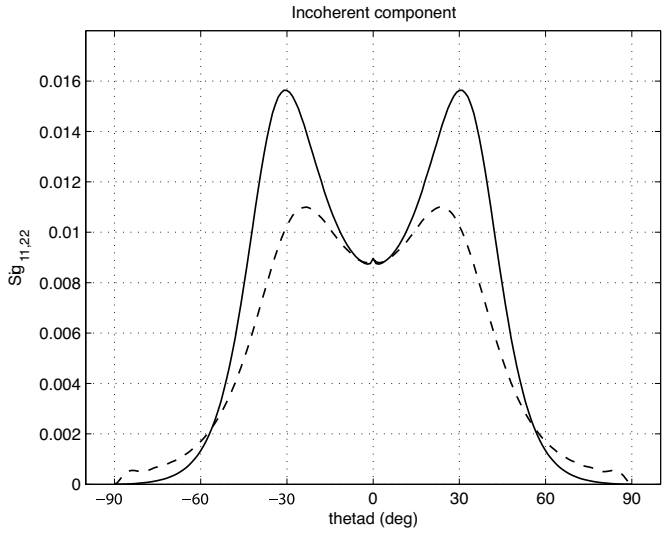


Figure 26. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2, VV (dashed line), HH (solid line), Surface height $\sigma = 15\,\text{nm}$, correlation length $l = 100\,\text{nm}$, slab thickness 500 nm. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \text{i}\,0.0075$, $\epsilon_2 = +\text{i}\infty$. Incident angles: $\theta_i = 0°$, $\phi_i = 0°$, wavelength $\lambda = 632.8\,\text{nm}$.

comparison with the SPM (see [26] Fig. 10) shows that the magnitude are the same, but the difference between the maxima for $\theta_d = \pm 30°$ and the minimum for $\theta_d = 0°$ is more pronounced in the SPM case. For the crossed polarizations VH and HV shown in Fig. 27 the shape of the intensities is identical but the magnitudes are half of the SPM case. Taking the same structure, with an angle of incidence $\theta_i = 20°$, the results are shown in Figs. 29–31. The intensities are concentrated in the backscattering region for the polarizations VV and HH, while for the VH and VH the intensities are maximum in a region opposite the incident scattering angle.

An other structure (no 10) is obtained from structure no 8 where the infinite conducting planar surface is replaced by a silver planar surface of permittivity $\epsilon_2 = -18.3 + \text{i}0.55$. In Fig. 35 is shown the intensities for the polarizations VV and HH, the HH component has the same maxima for $\theta_d = \pm 40°$ as in the SPM case (see Fig. 13 in [26]), but the difference between the maxima and the minimum ($\theta_d = 0°$) is less important. For the polarizations HV and VH, the intensities behavior with the scattering angle are similar but reduced by approximately a half compared to the SPM case.

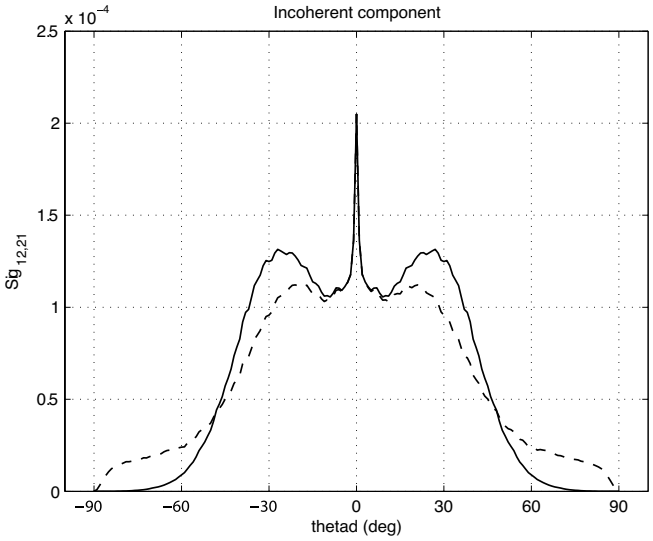


Figure 27. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

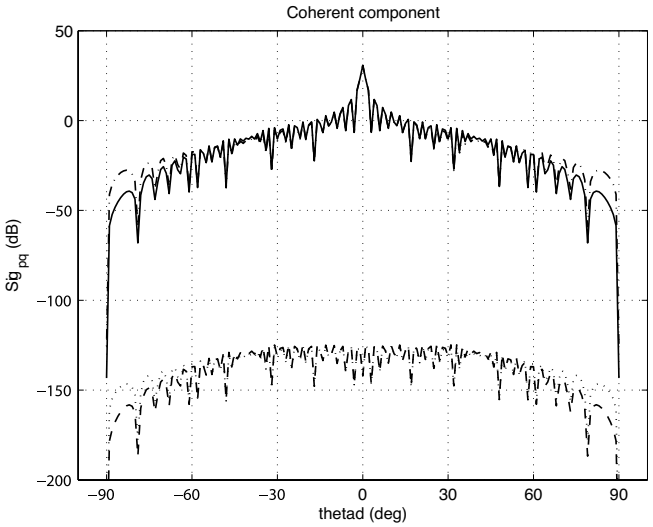


Figure 28. Characteristics of Fig. 26. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

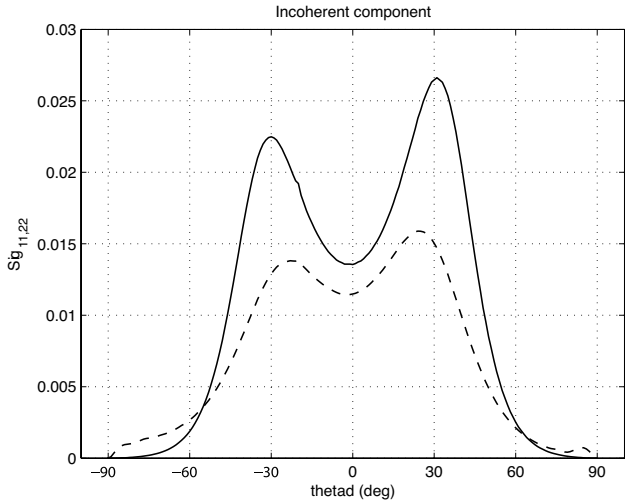


Figure 29. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2, VV (dashed line), HH (solid line), Surface height $\sigma = 15\,\mathrm{nm}$, correlation length $l = 100\,\mathrm{nm}$, slab thickness $500\,\mathrm{nm}$. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \mathrm{i}\,0.0075$, $\epsilon_2 = +\mathrm{i}\infty$. Incident angles: $\theta_i = 20°$, $\phi_i = 0°$, wavelength $\lambda = 632.8\,\mathrm{nm}$.

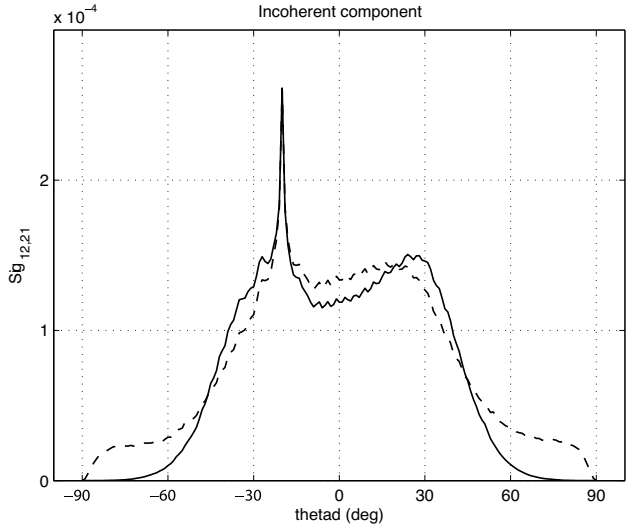


Figure 30. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

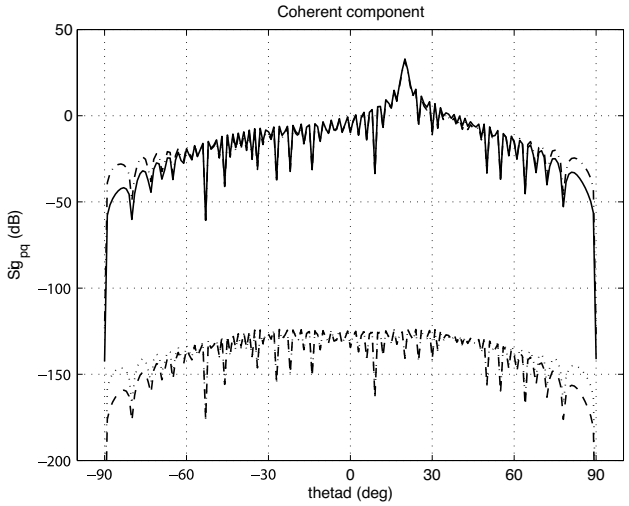


Figure 31. Characteristics of Fig. 29. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

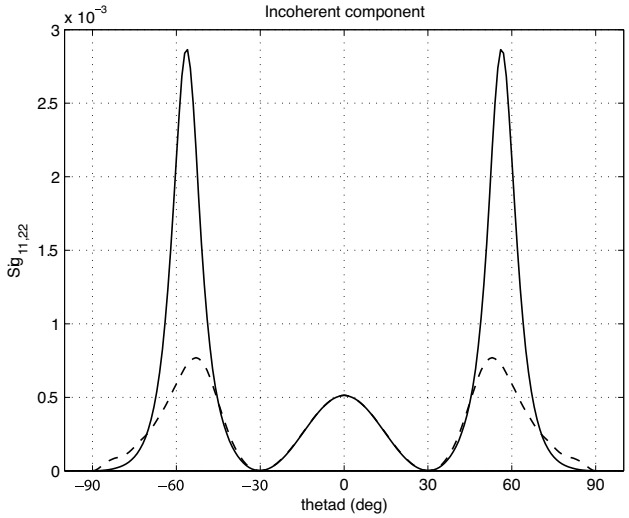


Figure 32. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2, VV (dashed line), HH (solid line), Surface height $\sigma = 15\,\text{nm}$, correlation length $l = 100\,\text{nm}$, slab thickness 1000 nm. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \text{i}\,0.0075$, $\epsilon_2 = +\text{i}\infty$. Incident angles: $\theta_i = 0°$, $\phi_i = 0°$, wavelength $\lambda = 632.8\,\text{nm}$.

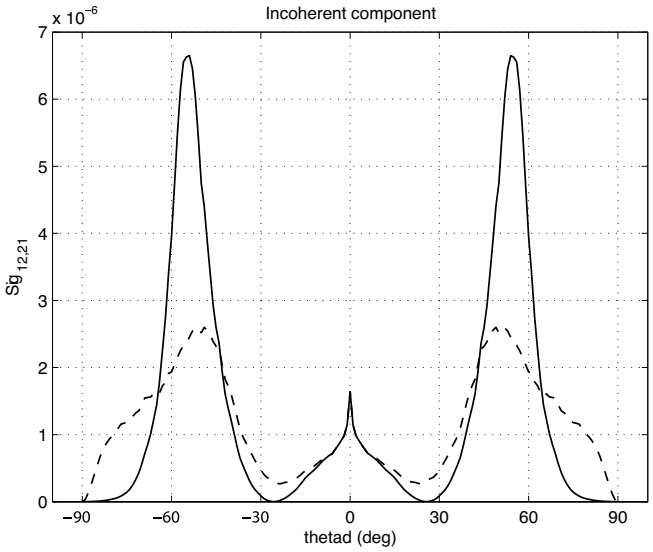


Figure 33. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

The fact to add a slab under a rough surface has a significant influence on the scattered intensity. To illustrate this point we take the structure no 1 (a rough surface between two semi-infinite media) and introduce a slab of thickness $H = 500\,\text{nm}$ with an infinite conducting lower planar surface (structure no 11).

The results are presented in Figs. 38–40, we observe the same maximum around the backscattering direction for the polarizations VV and HH but an increase of the scattered intensity by a factor 100. We notice for the polarizations HV and VH the presence of small oscillations for $\theta_d > 60^\circ$ due to the integration method.

9. A SLAB WITH TWO ROUGH BOUNDARIES

In the previous sections we have examined the cases where only on rough surface participated to the scattering process, in the present section our purpose is to show how light can interact with a slab delimited by two rough surfaces. This configuration is shown in Fig. 1, where three regions are characterized by different permittivities homogenous and isotropic, ϵ_0, ϵ_1 and ϵ_2. A slab is delimited by two rough surfaces located at $z = h_1(\boldsymbol{x})$ and $z = -H + h_2(\boldsymbol{x})$, $\boldsymbol{x} = (x, y)$.

Since the SSA method involves a knowledge of the scattering

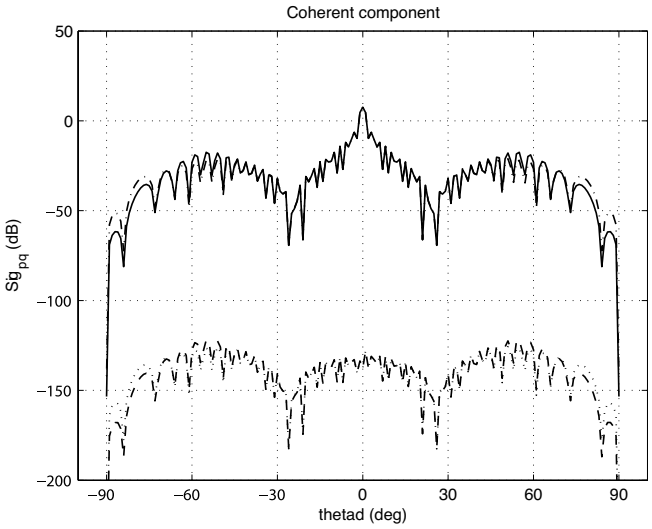


Figure 34. Characteristics of Fig. 32. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

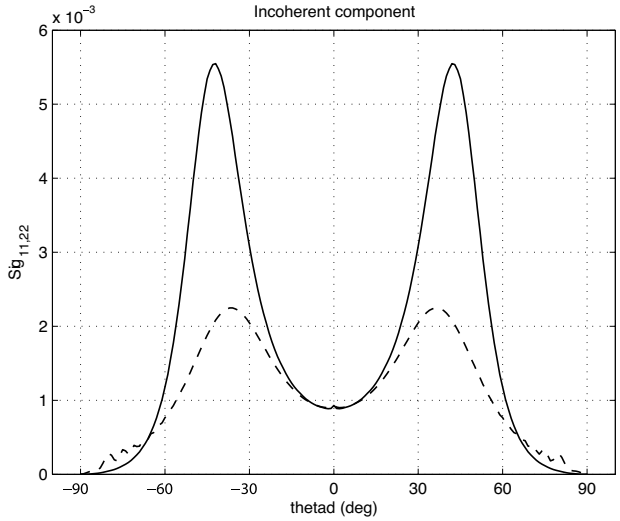


Figure 35. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2, VV (dashed line), HH (solid line), Surface height $\sigma = 15\,\text{nm}$, correlation length $l = 100\,\text{nm}$, slab thickness 500 nm. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + \text{i}\,0.0075$, $\epsilon_2 = -18.3 + \text{i}0.55$. Incident angles: $\theta_i = 0°$, $\phi_i = 0°$, wavelength $\lambda = 632.8\,\text{nm}$.

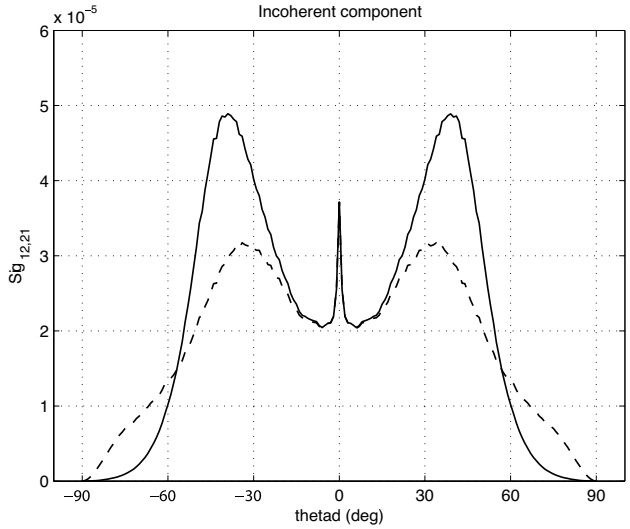


Figure 36. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

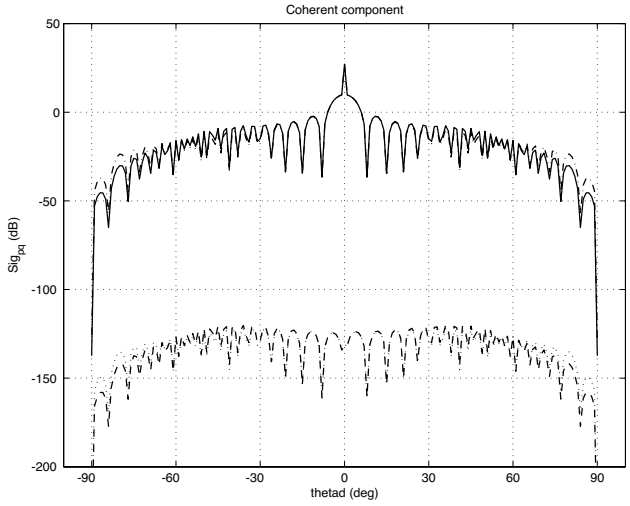


Figure 37. Characteristics of Fig. 35. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

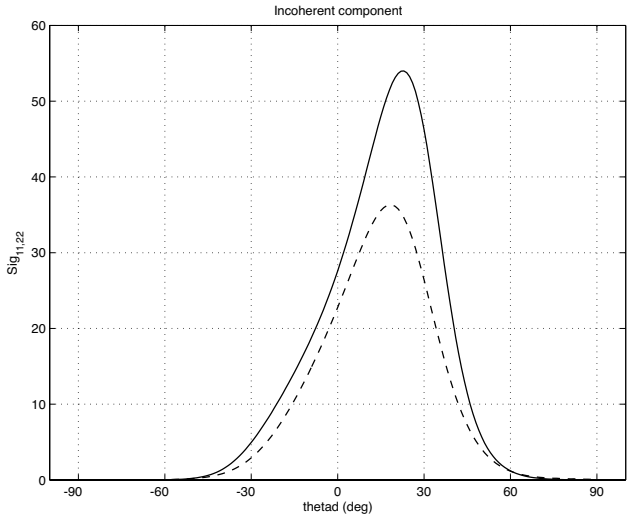


Figure 38. Incoherent components $\gamma^{incoh}(\theta_d)$ to the order 2. VV (dashed line), HH (solid line), Surface height $\sigma = 0.223\,\mu$m, correlation length $l = 1.42\,\mu$m, slab thickness 500 nm. Permittivities $\epsilon_0 = 1$, $\epsilon_1 = 1.62 + \mathrm{i}\,0.001$, $\epsilon_2 = +\mathrm{i}\infty$. Incident angles: $\theta_i = 20°$, $\phi_i = 0°$, wavelength $\lambda = 632.8$ nm.

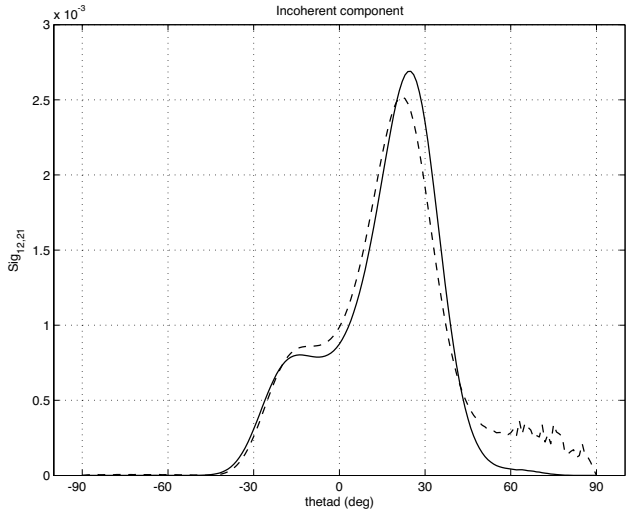


Figure 39. Same characteristics as the previous figure. Incoherent component VH (dashed line), HV (solid line).

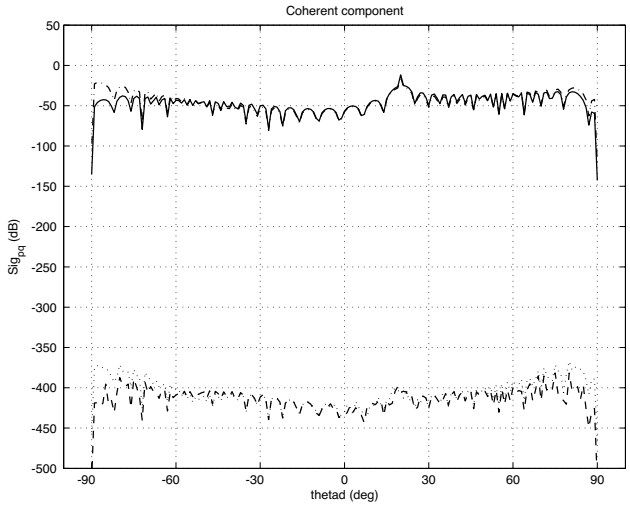


Figure 40. Characteristics of Fig. 38. Coherent components $\gamma^{coh}(\theta_d)$, VV (dashed-dotted line), HH (solid line), VH (dotted line) and HV (dashed line).

matrices calculated in the small-perturbation method, we summarize the results already obtained in [27] and needed in the following. For a system with two rough surfaces the perturbative development of the scattering matrix $\overline{\boldsymbol{R}}$ can be expanded as:

$$\begin{aligned}\overline{\boldsymbol{R}} \;=\; & \overline{\boldsymbol{R}}^{(00)} + \overline{\boldsymbol{R}}^{(10)} + \overline{\boldsymbol{R}}^{(01)} + \overline{\boldsymbol{R}}^{(11)} + \overline{\boldsymbol{R}}^{(20)} + \overline{\boldsymbol{R}}^{(21)} + \overline{\boldsymbol{R}}^{(12)} \\ & + \overline{\boldsymbol{R}}^{(22)} + \overline{\boldsymbol{R}}^{(30)} + \overline{\boldsymbol{R}}^{(03)} + \dots ,\end{aligned} \tag{86}$$

where the terms associated with the products of the heights of the two surfaces $h_1^n\, h_2^m$ are labelled $\overline{\boldsymbol{R}}^{(nm)}$.

Concerning the bistatic incoherent cross-sections we decompose their expressions into three terms corresponding to the contributions of the upper and lower surfaces alone plus a contribution due to the interference

$$\overline{\gamma}^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0) = \overline{\gamma}_u^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\gamma}_d^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\gamma}_{ud}^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0), \tag{87}$$

as an example

$$\begin{aligned}\overline{\gamma}_u^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0) \;=\; & \frac{K_0^2 \cos^2\theta}{A\,(2\pi)^2 \cos\theta_0} \Big[< \overline{\boldsymbol{R}}^{(10)} \odot \overline{\boldsymbol{R}}^{(10)} > + < \overline{\boldsymbol{R}}^{(20)} \odot \overline{\boldsymbol{R}}^{(20)} > \\ & + < \overline{\boldsymbol{R}}^{(30)} \odot \overline{\boldsymbol{R}}^{(10)} > \Big] ,\end{aligned} \tag{88}$$

corresponds to the contribution of the upper surface ($h_2(\boldsymbol{x}) = 0$), where the perturbative expansion is limited to the order 3 as a function of mean height σ_1. In a similar way the contribution due to the lower surface can be written by permuting the upper indices. The interference term $\overline{\gamma}_{up}^{incoh}$, contains the contributions of the field interacting with the two rough surfaces, and the dominant parts are given by

$$\begin{aligned}\overline{\gamma}_{ud}^{incoh}(\boldsymbol{p}|\boldsymbol{p}_0) \;=\; & \frac{K_0^2\cos^2\theta}{A(2\pi)^2\cos\theta_0}\Big[<\overline{\boldsymbol{R}}^{(10)}\odot\overline{\boldsymbol{R}}^{(12)}>+<\overline{\boldsymbol{R}}^{(12)}\odot\overline{\boldsymbol{R}}^{(10)}> \\ & +<\overline{\boldsymbol{R}}^{(01)}\odot\overline{\boldsymbol{R}}^{(21)}>+<\overline{\boldsymbol{R}}^{(21)}\odot\overline{\boldsymbol{R}}^{(01)}> \\ & +<\overline{\boldsymbol{R}}^{(11)}\odot\overline{\boldsymbol{R}}^{(11)}>+\ldots\Big], \end{aligned} \tag{89}$$

these contributions contain all the terms with $\sigma_1^i\,\sigma_2^j$ $(1 \le i+j \le 4)$. If the values σ_1 and σ_2 are close their contributions will be equivalent to fourth order terms in (88), (89). So we have supposed in Eq. (89) that the terms corresponding to $\sigma_1^4\,\sigma_2^2$, $\sigma_1^2\,\sigma_2^4$, $\sigma_1^4\,\sigma_2^4$ are negligible compared to the terms kept in Eq. (89), moreover, due to their complexity these terms of sixth order are not calculated.

In the case of the small-slope method we will study a perturbative development of the scattered field which depends on the slope of the surfaces h_1, h_2. The scattering matrix we have used in Sections 4, 7 and 8, must be generalized to the case with two surfaces. It is clear that several generalizations can be proposed, we choose the simplest one by making an ansatz similar to the functional form proposed by Voronovich, namely

$$\begin{aligned}\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0)=\int d^2\boldsymbol{r}d^2\boldsymbol{r}'\exp\big[&-i(\boldsymbol{p}-\boldsymbol{p}_0)\cdot(\boldsymbol{r}+\boldsymbol{r}') \\ &-i(\alpha(\boldsymbol{p})+\alpha(\boldsymbol{p}_0))(h_1(\boldsymbol{r})+h_2(\boldsymbol{r}'))\big] \\ &\times\overline{\boldsymbol{\Phi}}\left[\boldsymbol{p},\boldsymbol{p}_0;\ \boldsymbol{r};\ \boldsymbol{r}';\ [h_1(\boldsymbol{r})];\ [h_2(\boldsymbol{r}')]\right]. \end{aligned} \tag{90}$$

Introducing the Fourier transform of the functional $\overline{\boldsymbol{\Phi}}$, we have

$$\begin{aligned}\overline{\boldsymbol{R}}(\boldsymbol{p},\boldsymbol{p}_0) \;=\; & \int d^2\boldsymbol{r}d^2\boldsymbol{r}'\frac{d^2\boldsymbol{\xi}}{(2\pi)^2}\frac{d^2\boldsymbol{\xi}'}{(2\pi)^2}\exp\left[-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}\right. \\ & \left.-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}')\cdot\boldsymbol{r}'-i(\alpha(\boldsymbol{p})+\alpha(\boldsymbol{p}_0))(h_1(\boldsymbol{r})+h_2(\boldsymbol{r}'))\right] \\ & \times\tilde{\boldsymbol{\Phi}}\left[\boldsymbol{p},\boldsymbol{p}_0;\ \boldsymbol{\xi};\ \boldsymbol{\xi}';\ [h_1(\boldsymbol{\xi})];\ [h_2(\boldsymbol{\xi}')]\right]. \end{aligned} \tag{91}$$

In this expression the functional $\tilde{\boldsymbol{\Phi}}$ is expanded in a Taylor series in powers of h_1 and h_2 taking into account the translational invariance.

In order to simplify the formulas in the following

i) we omit the dependance on h_1, h_2 in the $\tilde{\mathbf{\Phi}}$ argument
ii) We introduce the notations $\tilde{\mathbf{\Phi}}^{(n\,m)ijk}$ where n refers to the dependance on the number of heights of the upper surface, m the number for the lower surface
iii) i, j, k represent the order according to which the field interacts successively with the surfaces h_1 and h_2
iv) the differential elements $d^2\boldsymbol{\xi}$ have to be divided by $(2\pi)^2$, and each function $\delta()$ multiplied by $(2\pi)^2$.

We then obtain the following expansion:

$$\tilde{\mathbf{\Phi}}(\boldsymbol{p}, \boldsymbol{p}_0,\ \boldsymbol{\xi},\ \boldsymbol{\xi}') = \tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi})\delta(\boldsymbol{\xi}) + \tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}')\delta(\boldsymbol{\xi}') \quad (92)$$

$$+ \int d^2\boldsymbol{\xi}_1 \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_1)\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_1) \quad (93)$$

$$+ \int d^2\boldsymbol{\xi}_2 \delta(\boldsymbol{\xi}' - \boldsymbol{\xi}_2)\tilde{\mathbf{\Phi}}^{(01)}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_2) \quad (94)$$

$$+ \int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 \delta(\boldsymbol{\xi} + \boldsymbol{\xi}' - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)$$

$$\Big[\tilde{\mathbf{\Phi}}^{(11)12}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2) \quad (95)$$

$$+\tilde{\mathbf{\Phi}}^{(11)21}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)\Big] \quad (96)$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\tilde{\mathbf{\Phi}}^{(20)}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi})h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2) \quad (97)$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 \delta(\boldsymbol{\xi}' - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\tilde{\mathbf{\Phi}}^{(02)}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}', \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2) \quad (98)$$

$$+ \int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 \delta(\boldsymbol{\xi} + \boldsymbol{\xi}' - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_3)\Big[$$

$$\tilde{\mathbf{\Phi}}^{(21)112}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \quad (99)$$

$$+\tilde{\mathbf{\Phi}}^{(21)121}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_3, \boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_2) \quad (100)$$

$$+\tilde{\mathbf{\Phi}}^{(21)211}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_3, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)\Big] \quad (101)$$

$$+ \int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 \delta(\boldsymbol{\xi} + \boldsymbol{\xi}' - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_3)\Big[$$

$$\tilde{\mathbf{\Phi}}^{(12)221}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \quad (102)$$

$$+\tilde{\mathbf{\Phi}}^{(12)212}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_3, \boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_2) \quad (103)$$

$$+\tilde{\mathbf{\Phi}}^{(12)122}(\boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{\xi}_3, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)\Big] \quad (104)$$

$$+ \int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_3)$$

$$\times\tilde{\mathbf{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \tag{105}$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3\delta(\boldsymbol{\xi}'-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3)$$

$$\times\tilde{\mathbf{\Phi}}^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \tag{106}$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 d^2\boldsymbol{\xi}_4\delta(\boldsymbol{\xi}+\boldsymbol{\xi}'-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3-\boldsymbol{\xi}_4)\Big[$$

$$\tilde{\mathbf{\Phi}}^{(22)1122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \tag{107}$$

$$+\tilde{\mathbf{\Phi}}^{(22)1212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \tag{108}$$

$$+\tilde{\mathbf{\Phi}}^{(22)1221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \tag{109}$$

$$+\tilde{\mathbf{\Phi}}^{(22)2112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \tag{110}$$

$$+\tilde{\mathbf{\Phi}}^{(22)2121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \tag{111}$$

$$+\tilde{\mathbf{\Phi}}^{(22)2211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\Big] \tag{112}$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 d^2\boldsymbol{\xi}_4\delta(\boldsymbol{\xi}+\boldsymbol{\xi}'-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3-\boldsymbol{\xi}_4)\Big[$$

$$\tilde{\mathbf{\Phi}}^{(31)1112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \tag{113}$$

$$+\tilde{\mathbf{\Phi}}^{(31)1121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_3) \tag{114}$$

$$+\tilde{\mathbf{\Phi}}^{(31)1211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \tag{115}$$

$$+\tilde{\mathbf{\Phi}}^{(31)2111}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)\Big] \tag{116}$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 d^2\boldsymbol{\xi}_4\delta(\boldsymbol{\xi}+\boldsymbol{\xi}'-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3-\boldsymbol{\xi}_4)\Big[$$

$$\tilde{\mathbf{\Phi}}^{(13)1222}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \tag{117}$$

$$+\tilde{\mathbf{\Phi}}^{(13)2122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \tag{118}$$

$$+\tilde{\mathbf{\Phi}}^{(13)2212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_3) \tag{119}$$

$$+\tilde{\mathbf{\Phi}}^{(13)2221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\Big] \tag{120}$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 d^2\boldsymbol{\xi}_4\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3-\boldsymbol{\xi}_4)$$

$$\times\tilde{\mathbf{\Phi}}^{(40)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \tag{121}$$

$$+\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 d^2\boldsymbol{\xi}_4\delta(\boldsymbol{\xi}'-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3-\boldsymbol{\xi}_4)$$

$$\times\tilde{\mathbf{\Phi}}^{(04)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4). \tag{122}$$

The computation of $\tilde{\mathbf{\Phi}}^{(n\,m)}$ follows the method proposed by

Voronovich, we consider successively the terms of order $n+m = 1,2,3,4$ in the previous expansion, and identify them with the equivalent order of the small-perturbation method. The expansion to the 4th order is required to take into account the interactions between the two surfaces.

9.1. Expansion of the Scattering Matrix according to the Order

9.1.1. Order $\mathbf{n+m=1}$

We get for this order the expression:

$$\int d^2\boldsymbol{r}\frac{d^2\boldsymbol{\xi}}{(2\pi)^2}\left[1-i(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h_1(\boldsymbol{r})\right]\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi})\delta(\boldsymbol{\xi})$$
$$+\int d^2\boldsymbol{r}'\frac{d^2\boldsymbol{\xi}'}{(2\pi)^2}\left[1-i(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))h_2(\boldsymbol{r}')\right]\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}')\delta(\boldsymbol{\xi}'), \quad (123)$$

after some calculations we obtain:

$$(2\pi)^2\delta(\boldsymbol{p}-\boldsymbol{p}_0)\left[\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right]$$
$$-i(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)h_1(\boldsymbol{p}-\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)h_2(\boldsymbol{p}-\boldsymbol{p}_0)\right], \quad (124)$$

a comparison with the SPM leads to the expressions $\tilde{\boldsymbol{\Phi}}_u^{(0)}$ and $\tilde{\boldsymbol{\Phi}}_d^{(0)}$ in term of the known operators $\overline{\boldsymbol{X}}^{(1)}$

$$\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0) = \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0), \quad (125)$$

$$\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0) = \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0). \quad (126)$$

Making use of the relation (42) we also obtain the following contributions to the order $n+m=2$

$$\tilde{\boldsymbol{\Phi}}^{(20)} = -(\alpha_0(\boldsymbol{p}_0)\alpha_0(\boldsymbol{p}))^2\tilde{\boldsymbol{\Phi}}_u^{(0)}h_1\,h_1, \quad (127)$$

$$\tilde{\boldsymbol{\Phi}}^{(02)} = -(\alpha_0(\boldsymbol{p}_0)\alpha_0(\boldsymbol{p}))^2\tilde{\boldsymbol{\Phi}}_d^{(0)}h_2\,h_2, \quad (128)$$

$$\tilde{\boldsymbol{\Phi}}^{(11)12} = -(\alpha_0(\boldsymbol{p}_0)\alpha_0(\boldsymbol{p}))^2\left(\tilde{\boldsymbol{\Phi}}_u^{(0)}+\tilde{\boldsymbol{\Phi}}_d^{(0)}\right)h_1\,h_2, \quad (129)$$

$$\tilde{\boldsymbol{\Phi}}^{(11)21} = -(\alpha_0(\boldsymbol{p}_0)\alpha_0(\boldsymbol{p}))^2\left(\tilde{\boldsymbol{\Phi}}_u^{(0)}+\tilde{\boldsymbol{\Phi}}_d^{(0)}\right)h_2\,h_1. \quad (130)$$

9.1.2. Order $\mathbf{n+m=2}$

The computation of the order 2 involves a power of h_1 and h_2 such that $n+m=2$, moreover, the terms of order 1 in h Eqs. (93), (94) must be replaced by terms of order 0 and 2, we then obtain

$$\int d^2\boldsymbol{r}d^2\boldsymbol{r}'\frac{d^2\boldsymbol{\xi}}{(2\pi)^2}\frac{d^2\boldsymbol{\xi}'}{(2\pi)^2}\exp\left[-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}')\cdot\boldsymbol{r}'\right].$$
$$\exp\left[-i(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))(h_1(\boldsymbol{r})+h_2(\boldsymbol{r}'))\right]$$
$$\times\left[\text{Eq. }(92)+\text{Eq. }(95-98)\right]. \tag{131}$$

After integration and introducing Eqs. (127)–(130), we obtain the result

$$\int\frac{d^2\boldsymbol{p}_1}{(2\pi)^2}\left[-\frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}{2}(\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0))\right.$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_1,\boldsymbol{p}_1-\boldsymbol{p}_0)\right]h_1(\boldsymbol{p}-\boldsymbol{p}_1)h_2(\boldsymbol{p}_1-\boldsymbol{p}_0) \tag{132}$$
$$+\int\frac{d^2\boldsymbol{p}_1}{(2\pi)^2}\left[-\frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}{2}(\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0))\right.$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_1,\boldsymbol{p}_1-\boldsymbol{p}_0)\right]h_2(\boldsymbol{p}-\boldsymbol{p}_1)h_1(\boldsymbol{p}_1-\boldsymbol{p}_0) \tag{133}$$
$$+\int\frac{d^2\boldsymbol{p}_1}{(2\pi)^2}\left[-\frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}{2}\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right.$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_1,\boldsymbol{p}_1-\boldsymbol{p}_0)\right]h_1(\boldsymbol{p}-\boldsymbol{p}_1)h_1(\boldsymbol{p}_1-\boldsymbol{p}_0) \tag{134}$$
$$+\int\frac{d^2\boldsymbol{p}_1}{(2\pi)^2}\left[-\frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}{2}\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right.$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_1,\boldsymbol{p}_1-\boldsymbol{p}_0)\right]h_2(\boldsymbol{p}-\boldsymbol{p}_1)h_2(\boldsymbol{p}_1-\boldsymbol{p}_0). \tag{135}$$

From this expansion we can derive the expressions of $\tilde{\boldsymbol{\Phi}}^{(n\,m)}$ in terms of the matrices $\tilde{\boldsymbol{X}}$ obtained in SPM. In the above expression the last two terms Eqs. (134), (135) must be identified with $\overline{\boldsymbol{R}}^{(20)}$ and $\overline{\boldsymbol{R}}^{(02)}$, see Eqs. (35), (36) in [27], we deduce

$$\tilde{\boldsymbol{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1) = \frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}{2}\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)$$
$$+\alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(20)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0), \tag{136}$$
$$\tilde{\boldsymbol{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1) = \frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}{2}\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)$$
$$+\alpha(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(02)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0). \tag{137}$$

We know with the reduction formula (42) that a term of order 20 can be decomposed into a term of order 1 and a term of order 3, for example

$$\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1)=\frac{\mathrm{i}}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\tilde{\mathbf{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_1,\boldsymbol{p}_1-\boldsymbol{p}_0), \tag{138}$$

and from Eqs. (125), (126) the first order terms give for $\tilde{\mathbf{\Phi}}^{(10)}$ and $\tilde{\mathbf{\Phi}}^{(01)}$ an expression in terms of the known operators $\overline{\boldsymbol{X}}_u^{(1)}$, $\overline{\boldsymbol{X}}_d^{(1)}$, $\overline{\boldsymbol{X}}^{(20)}$, $\overline{\boldsymbol{X}}^{(02)}$

$$\begin{aligned}\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1) &= \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big[\overline{\boldsymbol{X}}^{(20)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0)\\ &\quad+\mathrm{i}\,\frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\Big], \qquad (139)\\ \tilde{\mathbf{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1) &= \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big[\overline{\boldsymbol{X}}^{(02)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0)\\ &\quad+\mathrm{i}\,\frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\Big]. \qquad (140)\end{aligned}$$

Extra terms of order 3 can be deduced (We introduce in $\tilde{\mathbf{\Phi}}$ a lower index to make reference to the origin of their order when a confusion is possible.)

$$\begin{aligned}\tilde{\mathbf{\Phi}}_2^{(30)} &= -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(20)}h_1\,h_1\,h_1, \qquad (141)\\ \tilde{\mathbf{\Phi}}_2^{(03)} &= -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(02)}h_2\,h_2\,h_2, \qquad (142)\\ \tilde{\mathbf{\Phi}}_2^{(21)112} &= -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(20)}h_1\,h_1\,h_2, \qquad (143)\\ \tilde{\mathbf{\Phi}}_2^{(21)211} &= -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(20)}h_2\,h_1\,h_1, \qquad (144)\\ \tilde{\mathbf{\Phi}}_2^{(12)122} &= -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(02)}h_1\,h_2\,h_2, \qquad (145)\\ \tilde{\mathbf{\Phi}}_2^{(12)221} &= -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(02)}h_2\,h_2\,h_1. \qquad (146)\end{aligned}$$

For the first two terms in Eqs. (132), (133) we can make an identification with $\overline{\boldsymbol{R}}^{(11)}$, (see Eq. (34) in [27]), they give

$$\begin{aligned}\tilde{\mathbf{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1) &= \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(11)12}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0)\\ &\quad+\frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2(\tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p}|\boldsymbol{p}_0)+\tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p}|\boldsymbol{p}_0)),\end{aligned} \tag{147}$$

$$\tilde{\boldsymbol{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1) = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(11)21}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0) + \frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2(\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p}|\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p}|\boldsymbol{p}_0)). \tag{148}$$

Here, we notice that the relation (42) linking the orders $n-1, n, n+1$, and the formula (138) for one surface can be extended to the case of 2 surfaces, for example

$$\tilde{\boldsymbol{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}_1) = \frac{\mathrm{i}}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\tilde{\boldsymbol{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_1,\boldsymbol{p}_1-\boldsymbol{p}_0), \tag{149}$$

where in the calculations we keep all the terms of $\tilde{\boldsymbol{\Phi}}^{(11)}$ giving a contribution to $\tilde{\boldsymbol{\Phi}}^{(10)}$ and $\tilde{\boldsymbol{\Phi}}^{(01)}$.

We see that Eqs. (147), (148) give new contributions to the order $n+m=3$, they have to be included in the next approximation otherwise these contributions will be missing in the calculations of the coupling between the two surfaces at higher order.

$$\tilde{\boldsymbol{\Phi}}_2^{(21)112} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\boldsymbol{\Phi}}^{(11)12}h_1\,h_1\,h_2, \tag{150}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(21)121} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(11)12}+\tilde{\boldsymbol{\Phi}}^{(11)21}\right]h_1\,h_2\,h_1, \tag{151}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(21)211} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\boldsymbol{\Phi}}^{(11)21}h_2\,h_1\,h_1, \tag{152}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(12)221} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\boldsymbol{\Phi}}^{(11)21}h_2\,h_2\,h_1, \tag{153}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(12)212} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(11)12}+\tilde{\boldsymbol{\Phi}}^{(11)21}\right]h_2\,h_1\,h_2, \tag{154}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(12)122} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\boldsymbol{\Phi}}^{(11)12}h_1\,h_1\,h_2, \tag{155}$$

next we add Eqs. (143)–(146), and we get the following terms to be included in the next order.

$$\tilde{\boldsymbol{\Phi}}_2^{(21)112} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(20)}+\tilde{\boldsymbol{\Phi}}^{(11)12}\right]h_1\,h_1\,h_2, \tag{156}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(21)121} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(11)12}+\tilde{\boldsymbol{\Phi}}^{(11)21}\right]h_1\,h_2\,h_1 \tag{157}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(21)211} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(20)}+\tilde{\boldsymbol{\Phi}}^{(11)21}\right]h_2\,h_1\,h_1, \tag{158}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(12)221} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(02)}+\tilde{\boldsymbol{\Phi}}^{(11)21}\right]h_2\,h_2\,h_1, \tag{159}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(12)212} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(11)12}+\tilde{\boldsymbol{\Phi}}^{(11)21}\right]h_2\,h_1\,h_2, \tag{160}$$

$$\tilde{\boldsymbol{\Phi}}_2^{(12)122} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\boldsymbol{\Phi}}^{(02)}+\tilde{\boldsymbol{\Phi}}^{(11)12}\right]h_1\,h_1\,h_2. \tag{161}$$

9.1.3. Order $\mathbf{n+m=3}$

We have to collect in Eqs. (92)–(122) all the terms up to the order 3 in h, excepted those of power 2, and add Eqs. (141), (142), Eqs. (156)–(161) obtained from the order 2, we get the contributions

$$\int d^2\boldsymbol{r}d^2\boldsymbol{r}'\frac{d^2\boldsymbol{\xi}}{(2\pi)^2}\frac{d^2\boldsymbol{\xi}'}{(2\pi)^2}\exp\left[-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}')\cdot\boldsymbol{r}'\right]$$
$$\times\exp\left[-i(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))(h_1(\boldsymbol{r})+h_2(\boldsymbol{r}'))\right]$$
$$\times[\text{Eqs. (92–94)}+\text{Eqs. (99–106)}]\,, \tag{162}$$

after some calculations (162) gives:

$$\int d^2\boldsymbol{\xi}_1d^2\boldsymbol{\xi}_2d^2\boldsymbol{\xi}_3\,\delta(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3)\Bigg[$$
$$-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\Big[\Big[\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\Big]$$
$$\{h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)$$
$$+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)\}$$
$$+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\Big] \tag{163}$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1)\,\{h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)$$
$$+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)+h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)$$
$$+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)+h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)\} \tag{164}$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1)\,\{h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)$$
$$+h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_2)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)$$
$$+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)+h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)\} \tag{165}$$
$$+\left[\tilde{\boldsymbol{\Phi}}^{(21)112}+\tilde{\boldsymbol{\Phi}}_2^{(21)112}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \tag{166}$$
$$+\left[\tilde{\boldsymbol{\Phi}}^{(21)121}+\tilde{\boldsymbol{\Phi}}_2^{(21)121}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_2) \tag{167}$$
$$+\left[\tilde{\boldsymbol{\Phi}}^{(21)211}+\tilde{\boldsymbol{\Phi}}_2^{(21)211}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2) \tag{168}$$
$$+\left[\tilde{\boldsymbol{\Phi}}^{(12)221}+\tilde{\boldsymbol{\Phi}}_2^{(12)221}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \tag{169}$$
$$+\left[\tilde{\boldsymbol{\Phi}}^{(12)212}+\tilde{\boldsymbol{\Phi}}_2^{(12)212}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_2) \tag{170}$$
$$+\left[\tilde{\boldsymbol{\Phi}}^{(12)122}+\tilde{\boldsymbol{\Phi}}_2^{(12)122}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2) \tag{171}$$

$$+\left[\tilde{\mathbf{\Phi}}^{(30)}+\tilde{\mathbf{\Phi}}_2^{(30)}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \quad (172)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(03)}+\tilde{\mathbf{\Phi}}_2^{(03)}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)\Big], \quad (173)$$

where the operators $\tilde{\mathbf{\Phi}}_2^{(03)}$ and $\tilde{\mathbf{\Phi}}_2^{(30)}$ are given by Eqs. (141), (142), $\tilde{\mathbf{\Phi}}_2^{(12)ijk}$ and $\tilde{\mathbf{\Phi}}_2^{(21)ijk}$ by Eqs. (156)–(161), $\tilde{\mathbf{\Phi}}^{(10)}$ and $\tilde{\mathbf{\Phi}}^{(01)}$ by Eqs. (139), (140), $\tilde{\mathbf{\Phi}}_u^{(0)}$ and $\tilde{\mathbf{\Phi}}_d^{(0)}$ by Eqs. (125), (126).

Reordering the previous expression with respect to the h_i products leads to:

$$\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3\,\delta(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3)\Big[$$
$$\Big\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)$$
$$+\tilde{\mathbf{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\mathbf{\Phi}}_2^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\Big\}h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \quad (174)$$

$$+\Big\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\mathbf{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)$$
$$+\tilde{\mathbf{\Phi}}^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\mathbf{\Phi}}_2^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\Big\}h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \quad (175)$$

$$+\Big\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right]$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)+\tilde{\mathbf{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)\right]$$
$$+\tilde{\mathbf{\Phi}}^{(21)112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\mathbf{\Phi}}_2^{(21)112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\Big\}$$
$$\times h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \quad (176)$$

$$+\Big\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right]$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)$$
$$+\tilde{\mathbf{\Phi}}^{(21)121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\mathbf{\Phi}}_2^{(21)121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\Big\}$$
$$\times h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_2) \quad (177)$$

$$+\left\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right]\right.$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[\tilde{\boldsymbol{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)+\tilde{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)\right]$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(21)211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\boldsymbol{\Phi}}_2^{(21)211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\right\}$$
$$\times h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2) \tag{178}$$

$$+\left\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right]\right.$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(12)212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\boldsymbol{\Phi}}_2^{(12)212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\right\}$$
$$\times h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_2) \tag{179}$$
$$+\left\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right]\right.$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[\tilde{\boldsymbol{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)+\tilde{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)\right]$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(12)221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\boldsymbol{\Phi}}_2^{(12)221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\right\}$$
$$h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \tag{180}$$
$$+\left\{-\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\boldsymbol{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\right]\right.$$
$$-\frac{1}{2!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[\tilde{\boldsymbol{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)+\tilde{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3)\right]$$
$$\left.+\tilde{\boldsymbol{\Phi}}^{(12)122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)+\tilde{\boldsymbol{\Phi}}_2^{(12)122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\right\}$$
$$\times\ h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)]\,. \tag{181}$$

In order to identify the different terms with the operators $\overline{\boldsymbol{R}}^{(n\,m)}$ (see Eqs. (34)–(40) in [27]) of the perturbative development we make the variable substitutions $\boldsymbol{\xi}_1 = \boldsymbol{p} - \boldsymbol{p}_1$, $\boldsymbol{\xi}_2 = \boldsymbol{p}_1 - \boldsymbol{p}_2$ and integrate the delta functions. Next, following the Voronovich method, we identify $\tilde{\boldsymbol{\Phi}}^{(30)}$ with $\overline{\boldsymbol{R}}^{(30)}$ obtained in the SPM method and then deduce $\tilde{\boldsymbol{\Phi}}^{(20)}$, similarly for $\tilde{\boldsymbol{\Phi}}^{(03)}$. The terms of $\tilde{\boldsymbol{\Phi}}^{(21)}$ have to be identified to $\overline{\boldsymbol{R}}^{(21)}$ and according to their heights values contribute to $\tilde{\boldsymbol{\Phi}}^{(20)}$ or $\tilde{\boldsymbol{\Phi}}^{(11)}$, similarly for $\tilde{\boldsymbol{\Phi}}^{(12)}$.

So in a first step we obtain:

$$\tilde{\boldsymbol{\Phi}}^{(30)} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(30)} - \tilde{\boldsymbol{\Phi}}_2^{(30)} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(10)} + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3\tilde{\boldsymbol{\Phi}}_u^{(0)}, \tag{182}$$

$$\tilde{\boldsymbol{\Phi}}^{(03)} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(03)} - \tilde{\boldsymbol{\Phi}}_2^{(03)} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(01)} + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3\tilde{\boldsymbol{\Phi}}_d^{(0)}, \tag{183}$$

$$\tilde{\boldsymbol{\Phi}}^{(21)112} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(21)112} - \tilde{\boldsymbol{\Phi}}_2^{(21)112} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2\left[\tilde{\boldsymbol{\Phi}}^{(10)} + \tilde{\boldsymbol{\Phi}}^{(01)}\right] + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)} + \tilde{\boldsymbol{\Phi}}_d^{(0)}\right], \tag{184}$$

$$\tilde{\boldsymbol{\Phi}}^{(21)121} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(21)121} - \tilde{\boldsymbol{\Phi}}_2^{(21)121} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(10)} + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)} + \tilde{\boldsymbol{\Phi}}_d^{(0)}\right], \tag{185}$$

$$\tilde{\boldsymbol{\Phi}}^{(21)211} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(21)211} - \tilde{\boldsymbol{\Phi}}_2^{(21)211} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2\left[\tilde{\boldsymbol{\Phi}}^{(10)} + \tilde{\boldsymbol{\Phi}}^{(01)}\right] + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)} + \tilde{\boldsymbol{\Phi}}_d^{(0)}\right], \tag{186}$$

$$\tilde{\boldsymbol{\Phi}}^{(12)212} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(12)212} - \tilde{\boldsymbol{\Phi}}_2^{(12)212} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(01)} + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)} + \tilde{\boldsymbol{\Phi}}_d^{(0)}\right], \tag{187}$$

$$\tilde{\boldsymbol{\Phi}}^{(12)221} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(12)221} - \tilde{\boldsymbol{\Phi}}_2^{(12)221} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2\left[\tilde{\boldsymbol{\Phi}}^{(10)} + \tilde{\boldsymbol{\Phi}}^{(01)}\right] + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3\left[\tilde{\boldsymbol{\Phi}}_u^{(0)} + \tilde{\boldsymbol{\Phi}}_d^{(0)}\right], \tag{188}$$

$$\tilde{\mathbf{\Phi}}^{(12)122} = \alpha_0(\boldsymbol{p}_0)\overline{\boldsymbol{X}}^{(12)122} - \tilde{\mathbf{\Phi}}_2^{(12)122} + \frac{1}{2!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2 \left[\tilde{\mathbf{\Phi}}^{(10)} + \tilde{\mathbf{\Phi}}^{(01)}\right] + \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^3 \left[\tilde{\mathbf{\Phi}}_u^{(0)} + \tilde{\mathbf{\Phi}}_d^{(0)}\right], \tag{189}$$

A second step consists to solve the above equations, giving the order 2 terms:

$$\begin{aligned}\tilde{\mathbf{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = {} & \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0)} \left\{ \frac{1}{4} \left[\overline{\boldsymbol{X}}^{(21)112}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) \right.\right. \\ & \left. + \overline{\boldsymbol{X}}^{(21)121}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}^{(12)212}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}^{(12)122}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) \right] \\ & + \mathrm{i}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0)) \left[\frac{5}{8} \left(\overline{\boldsymbol{X}}^{(20)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}^{(02)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) \right) \right. \\ & \left. + \overline{\boldsymbol{X}}^{(11)12}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) + \frac{1}{2}\overline{\boldsymbol{X}}^{(11)21}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) \right] \\ & \left. - \frac{59}{48}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2 \left(\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) \right) \right\},\end{aligned} \tag{190}$$

$$\begin{aligned}\tilde{\mathbf{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = {} & \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0)} \left\{ \frac{1}{4} \left[\overline{\boldsymbol{X}}^{(21)121}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) \right.\right. \\ & \left. + \overline{\boldsymbol{X}}^{(21)211}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}^{(12)221}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}^{(12)212}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) \right] \\ & + \mathrm{i}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0)) \left[\frac{5}{8} \left(\overline{\boldsymbol{X}}^{(20)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}^{(02)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) \right) \right. \\ & \left. + \frac{1}{2}\overline{\boldsymbol{X}}^{(11)12}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}^{(11)21}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) \right] \\ & \left. - \frac{59}{48}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2 \left(\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) + \overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) \right) \right\},\end{aligned} \tag{191}$$

$$\begin{aligned}\tilde{\mathbf{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = {} & \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0)} \left\{ \overline{\boldsymbol{X}}^{(30)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) \right. \\ & + \mathrm{i}\frac{3}{2}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}^{(20)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) \\ & \left. - \frac{11}{12}(\alpha_0(\boldsymbol{p}) + \alpha_0(\boldsymbol{p}_0))^2 \overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0) \right\},\end{aligned} \tag{192}$$

$$\tilde{\mathbf{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{\overline{\boldsymbol{X}}^{(03)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0) + \mathrm{i}\frac{3}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\overline{\boldsymbol{X}}^{(02)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0) - \frac{11}{12}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0)\Big\}. \quad (193)$$

The expressions (190)–(193) contain the operators $\overline{\boldsymbol{X}}$ already calculated.

In addition, Eqs. (182)–(189) will contribute also to the 4th order through the terms:

$$\tilde{\mathbf{\Phi}}_3^{(22)1122} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)112}+\tilde{\mathbf{\Phi}}^{(12)122}\right], \quad (194)$$

$$\tilde{\mathbf{\Phi}}_3^{(22)1221} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(12)221}+\tilde{\mathbf{\Phi}}^{(12)122}\right], \quad (195)$$

$$\tilde{\mathbf{\Phi}}_3^{(22)1212} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)121}+\tilde{\mathbf{\Phi}}^{(12)212}\right], \quad (196)$$

$$\tilde{\mathbf{\Phi}}_3^{(22)2112} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)211}+\tilde{\mathbf{\Phi}}^{(21)112}\right], \quad (197)$$

$$\tilde{\mathbf{\Phi}}_3^{(22)2211} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)211}+\tilde{\mathbf{\Phi}}^{(12)221}\right], \quad (198)$$

$$\tilde{\mathbf{\Phi}}_3^{(22)2121} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)121}+\tilde{\mathbf{\Phi}}^{(12)212}\right], \quad (199)$$

$$\tilde{\mathbf{\Phi}}_3^{(13)1222} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(12)122}+\tilde{\mathbf{\Phi}}^{(03)}\right], \quad (200)$$

$$\tilde{\mathbf{\Phi}}_3^{(13)2122} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(12)212}+\tilde{\mathbf{\Phi}}^{(12)122}\right], \quad (201)$$

$$\tilde{\mathbf{\Phi}}_3^{(13)2212} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(12)212}+\tilde{\mathbf{\Phi}}^{(12)221}\right], \quad (202)$$

$$\tilde{\mathbf{\Phi}}_3^{(13)2221} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(12)221}+\tilde{\mathbf{\Phi}}^{(03)}\right], \quad (203)$$

$$\tilde{\mathbf{\Phi}}_3^{(31)1112} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)112}+\tilde{\mathbf{\Phi}}^{(30)}\right], \quad (204)$$

$$\tilde{\mathbf{\Phi}}_3^{(31)1121} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)121}+\tilde{\mathbf{\Phi}}^{(21)112}\right], \quad (205)$$

$$\tilde{\mathbf{\Phi}}_3^{(31)1211} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)121}+\tilde{\mathbf{\Phi}}^{(21)211}\right], \quad (206)$$

$$\tilde{\mathbf{\Phi}}_3^{(31)2111} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\left[\tilde{\mathbf{\Phi}}^{(21)211}+\tilde{\mathbf{\Phi}}^{(30)}\right], \quad (207)$$

$$\tilde{\mathbf{\Phi}}_3^{(40)} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(30)}, \quad (208)$$

$$\tilde{\mathbf{\Phi}}_3^{(04)} = -\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))\tilde{\mathbf{\Phi}}^{(03)}, \quad (209)$$

we recall that the lower index 3 refers to the original order of the terms.

9.1.4. Order $\mathbf{n+m=4}$

This order will produce the expression of the operators to the order 3, namely, $\tilde{\mathbf{\Phi}}^{(21)}$, $\tilde{\mathbf{\Phi}}^{(12)}$, $\tilde{\mathbf{\Phi}}^{(30)}$, $\tilde{\mathbf{\Phi}}^{(03)}$. In the $\tilde{\mathbf{\Phi}}$ expansion Eqs. (92)–(122) we have to retain the terms with $n+m=0,1,2,4$, and in Eq. (91) make a development in powers $h_1\,h_2$ up to 4, and also take into account contributions Eqs. (194)–(209) obtained from the previous order.

All together, we derive the expression:

$$
\begin{aligned}
&\int d^2\boldsymbol{\xi}_1 d^2\boldsymbol{\xi}_2 d^2\boldsymbol{\xi}_3 d^2\boldsymbol{\xi}_4\,\delta(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3-\boldsymbol{\xi}_4)\Big\{\\
&\frac{1}{4!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^4\Big[\Big(\tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\Big)\\
&\Big[h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\Big]\\
&+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\\
&+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\Big] \qquad (210)\\
&+\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1)\\
&\Big[h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\Big] \qquad (211)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\tilde{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1)\\
&\Big[h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\Big] \qquad (212)\\
&-\frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\\
&\Big[h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\Big] \qquad (213)\\
&-\frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \qquad (214)
\end{aligned}
$$

$$
\begin{aligned}
&\Big[h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\Big]\\
&-\frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\\
&\Big[h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)\\
&+h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)+h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)\\
&+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)\\
&+h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\Big] \qquad (215)
\end{aligned}
$$

$$-\frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$$
$$\Big[h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)+h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)$$
$$+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)$$
$$+h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)+h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)$$
$$+h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)\Big] \tag{216}$$

$$+\left[\tilde{\boldsymbol{\Phi}}^{(22)1122}+\tilde{\boldsymbol{\Phi}}_3^{(22)1122}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \tag{217}$$

$$+\left[\tilde{\boldsymbol{\Phi}}^{(22)1221}+\tilde{\boldsymbol{\Phi}}_3^{(22)1221}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \tag{218}$$
$$+\left[\tilde{\boldsymbol{\Phi}}^{(22)1212}+\tilde{\boldsymbol{\Phi}}_3^{(22)1212}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \tag{219}$$

$$+\left[\tilde{\boldsymbol{\Phi}}^{(22)2112}+\tilde{\boldsymbol{\Phi}}_3^{(22)2112}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \tag{220}$$

$$+\left[\tilde{\boldsymbol{\Phi}}^{(22)2121}+\tilde{\boldsymbol{\Phi}}_3^{(22)2121}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \tag{221}$$

$$+\left[\tilde{\boldsymbol{\Phi}}^{(22)2211}+\tilde{\boldsymbol{\Phi}}_3^{(22)2211}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \tag{222}$$

$$+\left[\tilde{\boldsymbol{\Phi}}^{(13)1222}+\tilde{\boldsymbol{\Phi}}_3^{(13)1222}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \tag{223}$$

$$+\left[\tilde{\boldsymbol{\Phi}}^{(13)2122}+\tilde{\boldsymbol{\Phi}}_3^{(13)2122}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \tag{224}$$

$$+\left[\tilde{\mathbf{\Phi}}^{(13)2212}+\tilde{\mathbf{\Phi}}_3^{(13)2212}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_3) \quad (225)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(13)2221}+\tilde{\mathbf{\Phi}}_3^{(13)2221}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \quad (226)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(31)1112}+\tilde{\mathbf{\Phi}}_3^{(31)1112}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4) \quad (227)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(31)1121}+\tilde{\mathbf{\Phi}}_3^{(31)1121}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_3) \quad (228)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(31)1211}+\tilde{\mathbf{\Phi}}_3^{(31)1211}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \quad (229)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(31)2111}+\tilde{\mathbf{\Phi}}_3^{(31)2111}\right]$$
$$(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)h_2(\boldsymbol{\xi}_4)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \quad (230)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(40)}+\tilde{\mathbf{\Phi}}_3^{(40)}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_4) \quad (231)$$

$$+\left[\tilde{\mathbf{\Phi}}^{(04)}+\tilde{\mathbf{\Phi}}_3^{(04)}\right](\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)$$
$$\times h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_4)\Big\}. \quad (232)$$

In the previous formula we collect the terms according to the ordered appearance of h_1 and h_2, they are 16 such combinations. For the purpose to make the formulas shorter, we introduce the notations:

$$\begin{aligned}\Omega^0(\boldsymbol{p},\boldsymbol{p}_0) &= \Omega_u^0(\boldsymbol{p},\boldsymbol{p}_0)+\Omega_d^0(\boldsymbol{p},\boldsymbol{p}_0)\\ &= \frac{1}{4!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^4\Big[\tilde{\mathbf{\Phi}}_u^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)+\tilde{\mathbf{\Phi}}_d^{(0)}(\boldsymbol{p},\boldsymbol{p}_0)\Big], \quad (233)\end{aligned}$$

$$\Omega^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1) = \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\tilde{\boldsymbol{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1), \tag{234}$$

$$\Omega^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1) = \frac{\mathrm{i}}{3!}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\tilde{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1), \tag{235}$$

$$\Omega^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2), \tag{236}$$

$$\Omega^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2), \tag{237}$$

$$\Omega^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2), \tag{238}$$

$$\Omega^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \frac{1}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\tilde{\boldsymbol{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2), \tag{239}$$

and we omit the dependance upon the variables in Eqs. (210)–(232)

$$\int \frac{d^2\boldsymbol{\xi}_1}{(2\pi)^2}\frac{d^2\boldsymbol{\xi}_2}{(2\pi)^2}\frac{d^2\boldsymbol{\xi}_3}{(2\pi)^2}\frac{d^2\boldsymbol{\xi}_4}{(2\pi)^2}\,\delta(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_3-\boldsymbol{\xi}_4)\Big[$$

$$\Big\{\tilde{\boldsymbol{\Phi}}^{(22)1122}+\tilde{\boldsymbol{\Phi}}_3^{(22)1122}-\Omega^{(20)}-\Omega^{(02)}-\Omega^{(11)12}-\Omega^{(01)}$$

$$-\Omega^{(10)}+\Omega^{(0)}\Big\}\,h_1h_1h_2h_2 \tag{240}$$

$$+\Big\{\tilde{\boldsymbol{\Phi}}^{(22)1221}+\tilde{\boldsymbol{\Phi}}_3^{(22)1221}-\Omega^{(11)21}-\Omega^{(11)12}$$

$$-\Omega^{(10)}+\Omega^{(0)}\Big\}\,h_1h_2h_2h_1 \tag{241}$$

$$+\Big\{\tilde{\boldsymbol{\Phi}}^{(22)1212}+\tilde{\boldsymbol{\Phi}}_3^{(22)1212}-\Omega^{(11)21}-\Omega^{(11)12}-\Omega^{(01)}$$

$$-\Omega^{(10)}+\Omega^{(0)}\Big\}\,h_1h_2h_1h_2 \tag{242}$$

$$+\Big\{\tilde{\boldsymbol{\Phi}}^{(22)2112}+\tilde{\boldsymbol{\Phi}}_3^{(22)2112}-\Omega^{(11)21}-\Omega^{(11)12}$$

$$-\Omega^{(01)}+\Omega^{(0)}\Big\}\,h_2h_1h_1h_2 \tag{243}$$

$$+\Big\{\tilde{\boldsymbol{\Phi}}^{(22)2121}+\tilde{\boldsymbol{\Phi}}_3^{(22)2121}-\Omega^{(11)21}-\Omega^{(11)12}-\Omega^{(01)}$$

$$-\Omega^{(10)}+\Omega^{(0)}\Big\}\,h_2h_1h_2h_1 \tag{244}$$

$$+\Big\{\tilde{\Phi}^{(22)2211}+\tilde{\boldsymbol{\Phi}}_3^{(22)2211}-\Omega^{(20)}-\Omega^{(02)}-\Omega^{(11)21}-\Omega^{(01)}$$

$$-\Omega^{(10)}+\Omega^{(0)}\Big\}\,h_2h_2h_1h_1 \tag{245}$$

$$+\left\{\tilde{\mathbf{\Phi}}^{(13)1222}+\tilde{\mathbf{\Phi}}_3^{(13)1222}-\Omega^{(02)}-\Omega^{(11)12}-\Omega^{(01)}\right.$$
$$\left.-\Omega^{(10)}+\Omega^{(0)}\right\}h_1h_2h_2h_2 \tag{246}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(13)2122}+\tilde{\mathbf{\Phi}}_3^{(13)2122}-\Omega^{(02)}-\Omega^{(11)21}-\Omega^{(11)12}\right.$$
$$\left.-\Omega^{(01)}+\Omega^{(0)}\right\}h_2h_1h_2h_2 \tag{247}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(13)2212}+\tilde{\mathbf{\Phi}}_3^{(13)2212}-\Omega^{(02)}-\Omega^{(11)12}\right.$$
$$\left.-\Omega^{(01)}+\Omega^{(0)}\right\}h_2h_2h_1h_2 \tag{248}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(13)2221}+\tilde{\mathbf{\Phi}}_3^{(13)2221}-\Omega^{(02)}-\Omega^{(11)21}-\Omega^{(01)}\right.$$
$$\left.-\Omega^{(10)}+\Omega^{(0)}\right\}h_2h_2h_2h_1 \tag{249}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(31)1112}+\tilde{\mathbf{\Phi}}_3^{(31)1112}-\Omega^{(20)}-\Omega^{(11)12}-\Omega^{(01)}\right.$$
$$\left.-\Omega^{(10)}+\Omega^{(0)}\right\}h_1h_1h_1h_2 \tag{250}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(31)1121}+\tilde{\mathbf{\Phi}}_3^{(31)1121}-\Omega^{(20)}-\Omega^{(11)21}\right.$$
$$\left.-\Omega^{(10)}+\Omega^{(0)}\right\}h_1h_1\,h_2h_1 \tag{251}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(31)1211}+\tilde{\mathbf{\Phi}}_3^{(31)1211}-\Omega^{(20)}-\Omega^{(11)21}-\Omega^{(11)12}\right.$$
$$\left.-\Omega^{(10)}+\Omega^{(0)}\right\}h_1h_2h_1h_1 \tag{252}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(31)2111}+\tilde{\mathbf{\Phi}}_3^{(31)2111}-\Omega^{(20)}-\Omega^{(11)21}-\Omega^{(01)}\right.$$
$$\left.-\Omega^{(10)}+\Omega^{(0)}\right\}h_2h_1h_1h_1 \tag{253}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(40)}+\tilde{\mathbf{\Phi}}_3^{(40)}-\Omega^{(20)}-\Omega^{(10)}+\Omega_u^{(0)}\right\}h_1h_1h_1h_1 \tag{254}$$
$$+\left\{\tilde{\mathbf{\Phi}}^{(04)}+\tilde{\mathbf{\Phi}}_3^{(04)}-\Omega^{(02)}-\Omega^{(01)}+\Omega_d^{(0)}\right\}h_2h_2h_2h_2\Big]. \tag{255}$$

In this formula we have to identify the terms $\tilde{\mathbf{\Phi}}^{(22)ijkl}$ with the corresponding terms $\overline{\boldsymbol{X}}^{(22)ijkl}$ in SPM. In the same way $\tilde{\mathbf{\Phi}}^{(13)ijkl}$ and $\tilde{\mathbf{\Phi}}^{(31)ijkl}$ are identified with $\overline{\boldsymbol{X}}^{(13)ijkl}$ and $\overline{\boldsymbol{X}}^{(31)ijkl}$ respectively, also, $\tilde{\mathbf{\Phi}}^{(40)}$, $\tilde{\mathbf{\Phi}}^{(04)}$ with $\overline{\boldsymbol{X}}^{(40)}$, $\overline{\boldsymbol{X}}^{(04)}$ (The calculation of the operators $\overline{\boldsymbol{X}}$ to the fourth order is in progress). The terms $\tilde{\mathbf{\Phi}}_3$ are given by Eqs. (194)–

(209). We introduce new definitions

$$\begin{aligned}
\overline{\boldsymbol{X}}_u^{(1)} &= \overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}_d^{(1)} &= \overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p}|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(11)ij} &= \overline{\boldsymbol{X}}^{(11)ij}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(02),(20)} &= \overline{\boldsymbol{X}}^{(02),(20)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(21)ijk} &= \overline{\boldsymbol{X}}^{(21)ijk}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(12)ijk} &= \overline{\boldsymbol{X}}^{(12)ijk}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(03),(30)} &= \overline{\boldsymbol{X}}^{(03),(30)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(21)ijkl} &= \overline{\boldsymbol{X}}^{(22)ijkl}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{\xi}_3|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(13),(31)} &= \overline{\boldsymbol{X}}^{(13),(31)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{\xi}_3|\boldsymbol{p}_0),\\
\overline{\boldsymbol{X}}^{(04),(40)} &= \overline{\boldsymbol{X}}^{(04),(40)}(\boldsymbol{p}|\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{\xi}_3|\boldsymbol{p}_0).
\end{aligned}$$

After some calculations, (For all these calculations we have used the MAPLE software, Waterloo Maple Inc.) we find for the operators of order 3, $\tilde{\boldsymbol{\Phi}}^{(12)}$, $\tilde{\boldsymbol{\Phi}}^{(21)}$, $\tilde{\boldsymbol{\Phi}}^{(30)}$ and $\tilde{\boldsymbol{\Phi}}^{(03)}$:

$$\begin{aligned}
&\tilde{\boldsymbol{\Phi}}^{(12)221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3) = \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{\\
&\frac{1}{5}\left[\overline{\boldsymbol{X}}^{(22)1221}+\overline{\boldsymbol{X}}^{(22)2121}+\overline{\boldsymbol{X}}^{(22)2211}+\overline{\boldsymbol{X}}^{(13)2221}+\overline{\boldsymbol{X}}^{(13)2212}\right]\\
&+\mathrm{i}\frac{1}{240}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\Big[72\overline{\boldsymbol{X}}^{(21)211}+90\overline{\boldsymbol{X}}^{(21)121}+18\overline{\boldsymbol{X}}^{(21)112}\\
&+138\overline{\boldsymbol{X}}^{(12)212}+66\overline{\boldsymbol{X}}^{(12)122}+216\overline{\boldsymbol{X}}^{(12)221}+24\overline{\boldsymbol{X}}^{(30)}+120\overline{\boldsymbol{X}}^{(03)}\Big]\\
&-\frac{1}{240}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\Big[312\overline{\boldsymbol{X}}^{(11)12}+516\overline{\boldsymbol{X}}^{(11)21}+749\overline{\boldsymbol{X}}^{(02)}\\
&+389\overline{\boldsymbol{X}}^{(20)}\Big]\\
&-\mathrm{i}\frac{1}{480}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left(1445\overline{\boldsymbol{X}}_u^{(1)}+1837\overline{\boldsymbol{X}}_d^{(1)}\right)\Big\}, \qquad (256)
\end{aligned}$$

$$\begin{aligned}
&\tilde{\boldsymbol{\Phi}}^{(12)212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3) = \frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{\\
&\frac{1}{5}\left[\overline{\boldsymbol{X}}^{(22)2121}+\overline{\boldsymbol{X}}^{(22)1212}+\overline{\boldsymbol{X}}^{(22)2112}+\overline{\boldsymbol{X}}^{(13)2122}+\overline{\boldsymbol{X}}^{(13)2212}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{i}\frac{1}{240}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\Big[72\overline{\boldsymbol{X}}^{(21)211}+78\overline{\boldsymbol{X}}^{(21)112}+150\overline{\boldsymbol{X}}^{(21)121}\\
&+246\overline{\boldsymbol{X}}^{(12)212}+72\overline{\boldsymbol{X}}^{(12)221}+78\overline{\boldsymbol{X}}^{(12)122}+48\overline{\boldsymbol{X}}^{(03)}\Big]\\
&-\frac{1}{240}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[535\overline{\boldsymbol{X}}^{(02)}+391\overline{\boldsymbol{X}}^{(20)}+552\overline{\boldsymbol{X}}^{(11)12}\right.\\
&\left.+540\overline{\boldsymbol{X}}^{(11)21}\right]\\
&-\mathrm{i}\frac{1}{480}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left(1735\overline{\boldsymbol{X}}_u^{(1)}+1895\overline{\boldsymbol{X}}_d^{(1)}\right)\Big\},
\end{aligned}
\tag{257}
$$

$$
\begin{aligned}
&\tilde{\boldsymbol{\Phi}}^{(12)122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)=\frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{\\
&\frac{1}{5}\left[\overline{\boldsymbol{X}}^{(22)1221}+\overline{\boldsymbol{X}}^{(22)1212}+\overline{\boldsymbol{X}}^{(22)1122}+\overline{\boldsymbol{X}}^{(13)2122}+\overline{\boldsymbol{X}}^{(13)1222}\right]\\
&+\mathrm{i}\frac{1}{240}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\Big[78\overline{\boldsymbol{X}}^{(21)112}+18\overline{\boldsymbol{X}}^{(21)211}+96\overline{\boldsymbol{X}}^{(21)121}\\
&+144\overline{\boldsymbol{X}}^{(12)212}+222\overline{\boldsymbol{X}}^{(12)122}+66\overline{\boldsymbol{X}}^{(12)221}+24\overline{\boldsymbol{X}}^{(30)}+120\overline{\boldsymbol{X}}^{(03)}\Big]\\
&-\frac{1}{240}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[540\overline{\boldsymbol{X}}^{(11)12}+324\overline{\boldsymbol{X}}^{(11)21}+764\overline{\boldsymbol{X}}^{(02)}\right.\\
&\left.+404\overline{\boldsymbol{X}}^{(20)}\right]\\
&-\mathrm{i}\frac{1}{480}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\Big(1504\overline{\boldsymbol{X}}_u^{(1)}+1896\overline{\boldsymbol{X}}_d^{(1)}\Big)\Big\},
\end{aligned}
\tag{258}
$$

$$
\begin{aligned}
&\tilde{\boldsymbol{\Phi}}^{(21)112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)=\frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{\\
&\frac{1}{5}\left[\overline{\boldsymbol{X}}^{(22)2112}+\overline{\boldsymbol{X}}^{(22)1212}+\overline{\boldsymbol{X}}^{(22)1122}+\overline{\boldsymbol{X}}^{(31)1121}+\overline{\boldsymbol{X}}^{(31)1112}\right]\\
&+\mathrm{i}\frac{1}{240}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\Big[216\overline{\boldsymbol{X}}^{(21)112}+138\overline{\boldsymbol{X}}^{(21)121}+66\overline{\boldsymbol{X}}^{(21)211}\\
&+90\overline{\boldsymbol{X}}^{(12)212}+18\overline{\boldsymbol{X}}^{(12)221}+72\overline{\boldsymbol{X}}^{(12)122}+120\overline{\boldsymbol{X}}^{(30)}+24\overline{\boldsymbol{X}}^{(03)}\Big]\\
&-\frac{1}{240}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[516\overline{\boldsymbol{X}}^{(11)12}+312\overline{\boldsymbol{X}}^{(11)21}+389\overline{\boldsymbol{X}}^{(02)}\right.\\
&\left.+749\overline{\boldsymbol{X}}^{(20)}\right]\\
&-\mathrm{i}\frac{1}{480}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left(1445\overline{\boldsymbol{X}}_u^{(1)}+1837\overline{\boldsymbol{X}}_d^{(1)}\right)\Big\},
\end{aligned}
\tag{259}
$$

$$\tilde{\mathbf{\Phi}}^{(21)121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)=\frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{$$
$$\frac{1}{5}\left[\overline{\boldsymbol{X}}^{(22)1221}+\overline{\boldsymbol{X}}^{(22)1212}+\overline{\boldsymbol{X}}^{(22)2121}+\overline{\boldsymbol{X}}^{(31)1121}+\overline{\boldsymbol{X}}^{(31)1211}\right]$$
$$+\mathrm{i}\frac{1}{240}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\Big[78\overline{\boldsymbol{X}}^{(21)211}+72\overline{\boldsymbol{X}}^{(21)112}+246\overline{\boldsymbol{X}}^{(21)121}$$
$$+72\overline{\boldsymbol{X}}^{(12)122}+150\overline{\boldsymbol{X}}^{(12)212}+78\overline{\boldsymbol{X}}^{(12)221}+48\overline{\boldsymbol{X}}^{(30)}\Big]$$
$$-\frac{1}{240}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[540\overline{\boldsymbol{X}}^{(11)12}+552\overline{\boldsymbol{X}}^{(11)21}+391\overline{\boldsymbol{X}}^{(02)}\right.$$
$$\left.+535\overline{\boldsymbol{X}}^{(20)}\right]$$
$$-\mathrm{i}\frac{1}{480}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left(1895\overline{\boldsymbol{X}}_u^{(1)}+1735\overline{\boldsymbol{X}}_d^{(1)}\right)\Big\},\qquad(260)$$

$$\tilde{\mathbf{\Phi}}^{(21)211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)=\frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{$$
$$\frac{1}{5}\left[\overline{\boldsymbol{X}}^{(22)2112}+\overline{\boldsymbol{X}}^{(22)2211}+\overline{\boldsymbol{X}}^{(22)2121}+\overline{\boldsymbol{X}}^{(31)2111}+\overline{\boldsymbol{X}}^{(31)1211}\right]$$
$$+\mathrm{i}\frac{1}{240}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\Big[222\overline{\boldsymbol{X}}^{(21)211}+66\overline{\boldsymbol{X}}^{(21)112}+144\overline{\boldsymbol{X}}^{(21)121}$$
$$+96\overline{\boldsymbol{X}}^{(12)212}+18\overline{\boldsymbol{X}}^{(12)122}+78\overline{\boldsymbol{X}}^{(12)221}+120\overline{\boldsymbol{X}}^{(30)}+24\overline{\boldsymbol{X}}^{(03)}\Big]$$
$$-\frac{1}{240}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\left[324\overline{\boldsymbol{X}}^{(11)12}+540\overline{\boldsymbol{X}}^{(11)21}+404\overline{\boldsymbol{X}}^{(02)}\right.$$
$$\left.+764\overline{\boldsymbol{X}}^{(20)}\right]$$
$$-\mathrm{i}\frac{1}{480}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\left(1896\overline{\boldsymbol{X}}_u^{(1)}+1504\overline{\boldsymbol{X}}_d^{(1)}\right)\Big\},\qquad(261)$$

$$\tilde{\mathbf{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)=\frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{\overline{\boldsymbol{X}}^{(40)}$$
$$+\mathrm{i}\frac{3}{2}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\overline{\boldsymbol{X}}^{(30)}-\frac{29}{12}(\alpha_0(\boldsymbol{p})$$
$$+\alpha_0(\boldsymbol{p}_0)^2\overline{\boldsymbol{X}}^{(20)}-\mathrm{i}\frac{3}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\overline{\boldsymbol{X}}_u^{(1)}\Big\},\qquad(262)$$

$$\tilde{\mathbf{\Phi}}^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)=\frac{\mathrm{i}\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\Big\{\overline{\boldsymbol{X}}^{(04)}$$
$$+\mathrm{i}\frac{3}{2}\left(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)\right)\overline{\boldsymbol{X}}^{(03)}-\frac{29}{12}(\alpha_0(\boldsymbol{p})$$
$$+\alpha_0(\boldsymbol{p}_0)^2\overline{\boldsymbol{X}}^{(02)}-\mathrm{i}\frac{3}{2}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^3\overline{\boldsymbol{X}}_d^{(1)}\Big\}. \tag{263}$$

We have checked that the numerical coefficients in front of the operators $\overline{\boldsymbol{X}}$ are equal when the symmetry $h_1 \leftrightarrow h_2$ is applied, so we verify through this symmetry the correspondence: $\tilde{\mathbf{\Phi}}^{(12)122} \leftrightarrow \tilde{\mathbf{\Phi}}^{(21)211}$, $\tilde{\mathbf{\Phi}}^{(12)221} \leftrightarrow \tilde{\mathbf{\Phi}}^{(21)112}$, $\tilde{\mathbf{\Phi}}^{(12)212} \leftrightarrow \tilde{\mathbf{\Phi}}^{(21)121}$, $\tilde{\mathbf{\Phi}}^{(30)} \leftrightarrow \tilde{\mathbf{\Phi}}^{(03)}$.

9.2. Expressions of the Scattering Matrices

Once we have calculated the functionals $\tilde{\mathbf{\Phi}}^{(ij)}$ up to the order 3, we are in a position to deduce the expressions of the scattering matrices which are defined in Section 3. We define a new integration operator $\mathcal{J}^{(n)}$

$$\begin{aligned}\mathcal{J}^{(n)} \;=\; &\int d^2\boldsymbol{r}d^2\boldsymbol{r}'\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{\xi}'}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2\boldsymbol{\xi}_n}{(2\pi)^2} \\ &\exp\left[-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi})\cdot\boldsymbol{r}-i(\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\xi}')\cdot\boldsymbol{r}'\right. \\ &\left.-i(\alpha(\boldsymbol{p})+\alpha(\boldsymbol{p}_0))(h_1(\boldsymbol{r})+h_2(\boldsymbol{r}'))\right].\end{aligned} \tag{264}$$

With this operator $\overline{\boldsymbol{R}}^{(ij)}$ can be written (we give inside brackets the reference equation of the formulas obtained for $\tilde{\mathbf{\Phi}}$)

$$\overline{\boldsymbol{R}}^{(10)}(\boldsymbol{p}|\boldsymbol{p}_0)=\mathcal{J}^{(1)}\tilde{\mathbf{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_1)$$
$$[\text{Eq. (139)}], \tag{265}$$
$$\overline{\boldsymbol{R}}^{(01)}(\boldsymbol{p}|\boldsymbol{p}_0)=\mathcal{J}^{(1)}\tilde{\mathbf{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_1)$$
$$[\text{Eq. (140)}], \tag{266}$$
$$\overline{\boldsymbol{R}}^{(11)}(\boldsymbol{p}|\boldsymbol{p}_0)=\mathcal{J}^{(2)}\Big[\tilde{\mathbf{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)$$
$$+\tilde{\mathbf{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_1)\Big]$$
$$[\text{Eq. (190, 191)}], \tag{267}$$

$$\overline{\boldsymbol{R}}^{(20)}(\boldsymbol{p}|\boldsymbol{p}_0)=\mathcal{J}^{(2)}\tilde{\mathbf{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi})h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)$$
$$[\text{Eq. (192)}], \tag{268}$$
$$\overline{\boldsymbol{R}}^{(02)}(\boldsymbol{p}|\boldsymbol{p}_0)=\mathcal{J}^{(2)}\tilde{\mathbf{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)$$
$$[\text{Eq. (193)}], \tag{269}$$

$$\overline{\boldsymbol{R}}^{(21)}(\boldsymbol{p}|\boldsymbol{p}_0) = \mathcal{J}^{(3)}\Big[\tilde{\boldsymbol{\Phi}}^{(21)112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) + \tilde{\boldsymbol{\Phi}}^{(21)121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_2) + \tilde{\boldsymbol{\Phi}}^{(21)211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)\Big] \quad [\text{Eqs. } (259-261)] \tag{270}$$

$$\overline{\boldsymbol{R}}^{(12)}(\boldsymbol{p}|\boldsymbol{p}_0) = \mathcal{J}^{(3)}\Big[\tilde{\boldsymbol{\Phi}}^{(12)221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) + \tilde{\boldsymbol{\Phi}}^{(12)212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_2) + \tilde{\boldsymbol{\Phi}}^{(12)122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)\Big] \quad [\text{Eq. } (256-258)] \tag{271}$$

$$\overline{\boldsymbol{R}}^{(30)}(\boldsymbol{p}|\boldsymbol{p}_0) = \mathcal{J}^{(3)}\tilde{\boldsymbol{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_1(\boldsymbol{\xi}_1)h_1(\boldsymbol{\xi}_2)h_1(\boldsymbol{\xi}_3) \quad [\text{Eq. } (262)]\,, \tag{272}$$

$$\overline{\boldsymbol{R}}^{(03)}(\boldsymbol{p}|\boldsymbol{p}_0) = \mathcal{J}^{(3)}\tilde{\boldsymbol{\Phi}}^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)h_2(\boldsymbol{\xi}_1)h_2(\boldsymbol{\xi}_2)h_2(\boldsymbol{\xi}_3) \quad [\text{Eq. } (263)]\,. \tag{273}$$

Using the same notations as in the SPM case, the average [32] over the surface realizations are given by:

$$\begin{aligned}
&< \overline{\boldsymbol{R}}^{(10)}(\boldsymbol{p}|\boldsymbol{p}_0) \odot \overline{\boldsymbol{R}}^{(10)}(\boldsymbol{p}|\boldsymbol{p}_0) >= \\
&\frac{-\alpha_0(\boldsymbol{p}_0)\alpha_0(\boldsymbol{p})}{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\sigma_1^2/2\right]} \\
&\times \int \mathrm{d}^2\boldsymbol{r}\int \mathrm{d}^2\boldsymbol{r}'\exp^{[-\mathrm{i}\,(\boldsymbol{p}-\boldsymbol{p}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')]}\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\,W_{11}(\boldsymbol{r}-\boldsymbol{r}')\right]} \\
&\Big\{\Big[\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p},\boldsymbol{p}_0) \\
&-\frac{\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}W_{11}(\boldsymbol{\xi})(\exp^{\mathrm{i}\,\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}')}-1)\overline{\boldsymbol{\Sigma}}_u(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\Big] \\
&\odot\Big[\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p},\boldsymbol{p}_0) \\
&-\frac{\mathrm{i}\,(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}W_{11}(\boldsymbol{\xi})(\exp^{\mathrm{i}\,\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}')}-1)\overline{\boldsymbol{\Sigma}}_u(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\Big]^* \\
&+\frac{1}{4}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}W_{11}(\boldsymbol{\xi})\overline{\boldsymbol{\Sigma}}_u(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\odot\overline{\boldsymbol{\Sigma}}_u(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\Big\}\,,
\end{aligned} \tag{274}$$

$$\begin{aligned}
&< \overline{\boldsymbol{R}}^{(01)}(\boldsymbol{p}|\boldsymbol{p}_0) \odot \overline{\boldsymbol{R}}^{(01)}(\boldsymbol{p}|\boldsymbol{p}_0) >= \\
&\frac{-\alpha_0(\boldsymbol{p}_0)\alpha_0(\boldsymbol{p})}{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2}\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\,\sigma_2^2/2\right]}
\end{aligned}$$

$$\times \int \mathrm{d}^2\boldsymbol{r} \int \mathrm{d}^2\boldsymbol{r}' \exp^{[-\mathrm{i}\,(\boldsymbol{p}-\boldsymbol{p}_0)\cdot(\boldsymbol{r}-\boldsymbol{r}')]} \exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\, W_{22}(\boldsymbol{r}-\boldsymbol{r}')\right]}$$
$$\Big\{\Big[\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p},\boldsymbol{p}_0$$
$$) - \frac{\mathrm{i}\,(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int \frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2} W_{22}(\boldsymbol{\xi})(\exp^{\mathrm{i}\,\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}')}-1)\overline{\boldsymbol{\Sigma}}_d(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\Big]$$
$$\odot \Big[\overline{\boldsymbol{X}}_d^{(1)}(\boldsymbol{p},\boldsymbol{p}_0)$$
$$-\frac{\mathrm{i}\,(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int \frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2} W_{22}(\boldsymbol{\xi})(\exp^{\mathrm{i}\,\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}')}-1)\overline{\boldsymbol{\Sigma}}_d(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\Big]^*$$
$$+\frac{1}{4}\int \frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2} W_{22}(\boldsymbol{\xi})\overline{\boldsymbol{\Sigma}}_d(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\odot\overline{\boldsymbol{\Sigma}}_d(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})\Big\}, \tag{275}$$

$$<\overline{\boldsymbol{R}}^{(11)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(11)}(\boldsymbol{p}|\boldsymbol{p}_0)>=$$
$$\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2(W_{11}(0)+W_{22}(0))\right]}$$
$$\times\int d^2\boldsymbol{x}_1\, d^2\boldsymbol{x}_2\, d^2\boldsymbol{x}_3\, \exp^{-\mathrm{i}((\boldsymbol{p}-\boldsymbol{p}_0)(\boldsymbol{x}_1-\boldsymbol{x}_3)}\exp^{[-\mathrm{i}((\boldsymbol{p}-\boldsymbol{p}_0)(\boldsymbol{x}_2-\boldsymbol{x}_3)]}\times$$
$$\times\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2(W_{11}(\boldsymbol{x}_1-\boldsymbol{x}_3)+W_{22}(\boldsymbol{x}_2-\boldsymbol{x}_3))\right]}$$
$$\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}_1(\boldsymbol{x}_1-\boldsymbol{x}_3)]}\exp^{[\mathrm{i}\boldsymbol{\xi}_2(\boldsymbol{x}_2-\boldsymbol{x}_3)]}\times$$
$$\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}_3}{(2\pi)^2}\Big[\overline{\boldsymbol{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3,\boldsymbol{\xi}_1+\boldsymbol{\xi}_2-\boldsymbol{\xi}_3)$$
$$+\overline{\boldsymbol{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1+\boldsymbol{\xi}_2-\boldsymbol{\xi}_3,\boldsymbol{\xi}_3)\Big]$$
$$\odot\Big[\overline{\boldsymbol{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_3,\boldsymbol{\xi}_1+\boldsymbol{\xi}_2-\boldsymbol{\xi}_3)$$
$$+\overline{\boldsymbol{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1+\boldsymbol{\xi}_2-\boldsymbol{\xi}_3,\boldsymbol{\xi}_3)\Big]^* W_{11}(\boldsymbol{\xi}_3)W_{22}(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2-\boldsymbol{\xi}_3), \tag{276}$$

$$<\overline{\boldsymbol{R}}^{(20)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(20)}(\boldsymbol{p}|\boldsymbol{p}_0)>=\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2\,W_{11}(0)\right]}$$
$$\times\int d^2\boldsymbol{x}_1 d^2\boldsymbol{x}_2\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)(\boldsymbol{x}_1-\boldsymbol{x}_2)]}\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\,W_{11}(\boldsymbol{x}_1-\boldsymbol{x}_2)\right]}$$
$$\Big\{\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\overline{\boldsymbol{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1)W_{11}(\boldsymbol{\xi}_1)$$
$$\odot\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}\overline{\boldsymbol{\Phi}}^{(20)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2)W_{11}(\boldsymbol{\xi}_2)$$

$$+2\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}_1(\boldsymbol{x}_1-\boldsymbol{x}_2)]}\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}\overline{\boldsymbol{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)$$
$$\odot\overline{\boldsymbol{\Phi}}^{(20)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)W_{11}(\boldsymbol{\xi}_2)W_{11}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\Big\}, \tag{277}$$

$$<\overline{\boldsymbol{R}}^{(02)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(02)}(\boldsymbol{p}|\boldsymbol{p}_0)>=\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2\,W_{22}(0)\right]}$$
$$\times\int d^2\boldsymbol{x}_1 d^2\boldsymbol{x}_2\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)(\boldsymbol{x}_1-\boldsymbol{x}_2)]}\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2\,W_{22}(\boldsymbol{x}_1-\boldsymbol{x}_2)\right]}$$
$$\times\Big\{\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\overline{\boldsymbol{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1)W_{22}(\boldsymbol{\xi}_1)$$
$$\odot\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}\overline{\boldsymbol{\Phi}}^{(02)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2)W_{22}(\boldsymbol{\xi}_2)$$
$$+2\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}_1(\boldsymbol{x}_1-\boldsymbol{x}_2)]}\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}\overline{\boldsymbol{\Phi}}^{(02)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)$$
$$\odot\overline{\boldsymbol{\Phi}}^{(02)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)W_{22}(\boldsymbol{\xi}_2)W_{22}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\Big\}, \tag{278}$$

$$<\overline{\boldsymbol{R}}^{(30)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(10)}(\boldsymbol{p}|\boldsymbol{p}_0)>=\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2W_{11}(0)\right]}$$
$$\times\int d^2\boldsymbol{x}_1\,d^2\boldsymbol{x}_2\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)(\boldsymbol{x}_1-\boldsymbol{x}_2)]}\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2W_{11}(\boldsymbol{x}_1-\boldsymbol{x}_2)\right]}$$
$$\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}(\boldsymbol{x}_1-\boldsymbol{x}_2)]}W_{11}(\boldsymbol{\xi})\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}W_{11}(\boldsymbol{\xi}_2)$$
$$\Big[\overline{\boldsymbol{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi},\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2)\odot\overline{\boldsymbol{\Phi}}^{(10)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi})$$
$$+\overline{\boldsymbol{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,\boldsymbol{\xi},-\boldsymbol{\xi}_2)\odot\overline{\boldsymbol{\Phi}}^{(10)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi})$$
$$+\overline{\boldsymbol{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2,\boldsymbol{\xi})\odot\overline{\boldsymbol{\Phi}}^{(10)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi})\Big], \tag{279}$$

$$<\overline{\boldsymbol{R}}^{(03)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(01)}(\boldsymbol{p}|\boldsymbol{p}_0)>=\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2W_{22}(0)\right]}$$
$$\times\int d^2\boldsymbol{x}_1\,d^2\boldsymbol{x}_2\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)(\boldsymbol{x}_1-\boldsymbol{x}_2)]}\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2W_{22}(\boldsymbol{x}_1-\boldsymbol{x}_2)\right]}$$
$$\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}(\boldsymbol{x}_1-\boldsymbol{x}_2)]}W_{22}(\boldsymbol{\xi})\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}W_{22}(\boldsymbol{\xi}_2)$$
$$\Big[\overline{\boldsymbol{\Phi}}^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi},\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2)\odot\overline{\boldsymbol{\Phi}}^{(01)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi})$$
$$+\overline{\boldsymbol{\Phi}}^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,\boldsymbol{\xi},-\boldsymbol{\xi}_2)\odot\overline{\boldsymbol{\Phi}}^{(01)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi})$$

$$+\overline{\boldsymbol{\Phi}}^{(03)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2,\boldsymbol{\xi})\odot\overline{\boldsymbol{\Phi}}^{(01)\,*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi})\Big], \tag{280}$$

$$\begin{aligned}
&<\overline{\boldsymbol{R}}^{(12)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(10)}(\boldsymbol{p}|\boldsymbol{p}_0)>= \\
&\exp\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2(W_{11}(0)+W_{22}(0))\right] \\
&\times\int d^2\boldsymbol{x}\,\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{x}]}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}]}W_{11}(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}) \\
&\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}W_{22}(\boldsymbol{\xi}_1)\Big[\overline{\boldsymbol{\Phi}}^{(12)122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1) \\
&+\overline{\boldsymbol{\Phi}}^{(12)212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},-\boldsymbol{\xi}_1) \\
&+\overline{\boldsymbol{\Phi}}^{(12)221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\Big]\odot\overline{\boldsymbol{\Phi}}^{(10)*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}) \\
&\times\int d^2\boldsymbol{x}_1\exp^{[-\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}_1]}\exp\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2W_{11}(\boldsymbol{x}-\boldsymbol{x}_1)\right],
\end{aligned} \tag{281}$$

$$\begin{aligned}
&<\overline{\boldsymbol{R}}^{(10)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(12)}(\boldsymbol{p}|\boldsymbol{p}_0)>= \\
&\exp\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2(W_{11}(0)+W_{22}(0))\right]\int d^2\boldsymbol{x}\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{x}]} \\
&\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}]}W_{11}(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\overline{\boldsymbol{\Phi}}^{(10)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\odot \\
&\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}W_{22}(\boldsymbol{\xi}_1)\Big[\overline{\boldsymbol{\Phi}}^{(12)122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1) \\
&+\overline{\boldsymbol{\Phi}}^{(12)212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},-\boldsymbol{\xi}_1) \\
&+\overline{\boldsymbol{\Phi}}^{(12)221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\Big]^* \\
&\int d^2\boldsymbol{x}_1\exp^{[-\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}_1]}\exp\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2W_{11}(\boldsymbol{x}-\boldsymbol{x}_1)\right],
\end{aligned} \tag{282}$$

$$\begin{aligned}
&<\overline{\boldsymbol{R}}^{(21)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(01)}(\boldsymbol{p}|\boldsymbol{p}_0)>= \\
&\exp\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2(W_{11}(0)+W_{22}(0))\right] \\
&\times\int d^2\boldsymbol{x}\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{x}]}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}]}W_{22}(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}) \\
&\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}W_{11}(\boldsymbol{\xi}_1)\Big[\overline{\boldsymbol{\Phi}}^{(21)112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1)
\end{aligned}$$

$$+\overline{\boldsymbol{\Phi}}^{(21)121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},-\boldsymbol{\xi}_1)$$
$$+\overline{\boldsymbol{\Phi}}^{(21)211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\Big]\odot\overline{\boldsymbol{\Phi}}^{(01)*}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})$$
$$\times\int d^2\boldsymbol{x}_1\exp^{[-\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}_1]}\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2W_{22}(\boldsymbol{x}-\boldsymbol{x}_1)\right]}, \tag{283}$$

$$<\overline{\boldsymbol{R}}^{(01)}(\boldsymbol{p}|\boldsymbol{p}_0)\odot\overline{\boldsymbol{R}}^{(21)}(\boldsymbol{p}|\boldsymbol{p}_0)>=$$
$$\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2(W_{11}(0)+W_{22}(0))\right]}\int d^2\boldsymbol{x}\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{x}]}$$
$$\times\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\exp^{[\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}]}W_{22}(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\overline{\boldsymbol{\Phi}}^{(01)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\odot$$
$$\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}W_{11}(\boldsymbol{\xi}_1)\Big[\overline{\boldsymbol{\Phi}}^{(21)112}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1)$$
$$+\overline{\boldsymbol{\Phi}}^{(21)121}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi},-\boldsymbol{\xi}_1)$$
$$+\overline{\boldsymbol{\Phi}}^{(21)211}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi})\Big]^*$$
$$\times\int d^2\boldsymbol{x}_1\exp^{[-\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}_1]}\exp^{\left[(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2W_{22}(\boldsymbol{x}-\boldsymbol{x}_1)\right]}\,. \tag{284}$$

In these formulas W is given by Eqs. (4), (9).

To compute the coherent cross-sections we need the following averages:

$$<\overline{\boldsymbol{R}}^{(10)}(\boldsymbol{p}|\boldsymbol{p}_0)>=$$
$$\mathrm{i}\frac{\alpha_0(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)}\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2W_{11}(0)\right]}\int d^2\boldsymbol{r}\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{r}]}$$
$$\left\{\overline{\boldsymbol{X}}_u^{(1)}(\boldsymbol{p},\boldsymbol{p}_0)+\frac{(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))}{2}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\overline{\boldsymbol{\Sigma}}_u(\boldsymbol{p}|\boldsymbol{p}_0|\boldsymbol{\xi})W_{11}(\boldsymbol{\xi})\right\}, \tag{285}$$

$$<\overline{\boldsymbol{R}}^{(20)}(\boldsymbol{p}|\boldsymbol{p}_0)>=\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2W_{11}(0)\right]}$$
$$\times\int d^2\boldsymbol{r}\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{r}]}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^2}\overline{\boldsymbol{\Phi}}^{(20)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi},-\boldsymbol{\xi})W_{11}(\boldsymbol{\xi})\,, \tag{286}$$

$$<\overline{\boldsymbol{R}}^{(30)}(\boldsymbol{p}|\boldsymbol{p}_0)>=\exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2W_{11}(0)\right]}$$
$$\times\int d^2\boldsymbol{r}\exp^{[-\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{r}]}\int\frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2}W_{11}(\boldsymbol{\xi}_1)\int\frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}W_{11}(\boldsymbol{\xi}_2)$$

$$\left[\overline{\boldsymbol{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2)+\overline{\boldsymbol{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_2)\right.$$
$$\left.\overline{\boldsymbol{\Phi}}^{(30)}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_2,-\boldsymbol{\xi}_2,\boldsymbol{\xi}_1)\right], \tag{287}$$

$$< \overline{\boldsymbol{R}}^{(11)}(\boldsymbol{p}|\boldsymbol{p}_0) >=$$
$$-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2 \exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2(W_{11}(0)+W_{22}(0))\right]}$$
$$\times \int d^2\boldsymbol{r} \exp^{[-2\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{r}]} \int \frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2} \int \frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2}$$
$$\times \left[\overline{\boldsymbol{\Phi}}^{(11)12}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_1,-(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_2))\right.$$
$$\left.+\overline{\boldsymbol{\Phi}}^{(11)21}(\boldsymbol{p},\boldsymbol{p}_0,-(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_2),\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_1)\right]$$
$$\times W_{11}(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_1)W_{22}(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_2), \tag{288}$$

$$< \overline{\boldsymbol{R}}^{(12)}(\boldsymbol{p}|\boldsymbol{p}_0) >=$$
$$-\mathrm{i}(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0)) \exp^{\left[-(\alpha_0(\boldsymbol{p})+\alpha_0(\boldsymbol{p}_0))^2/2(W_{11}(0)+W_{22}(0))\right]}$$
$$\times \int d^2\boldsymbol{r} \exp^{[-2\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}_0)\boldsymbol{r}]} \int \frac{\mathrm{d}^2\boldsymbol{\xi}_1}{(2\pi)^2} \int \frac{\mathrm{d}^2\boldsymbol{\xi}_2}{(2\pi)^2} W_{11}(\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_1)W_{22}(\boldsymbol{\xi}_2)$$
$$\times \left[\overline{\boldsymbol{\Phi}}^{(12)221}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_2)\right.$$
$$+\overline{\boldsymbol{\Phi}}^{(12)212}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{\xi}_1,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_2,-\boldsymbol{\xi}_1)$$
$$\left.+\overline{\boldsymbol{\Phi}}^{(12)122}(\boldsymbol{p},\boldsymbol{p}_0,\boldsymbol{p}-\boldsymbol{p}_0+\boldsymbol{\xi}_2,\boldsymbol{\xi}_1,-\boldsymbol{\xi}_1)\right]. \tag{289}$$

9.3. Applications

As an example of application we take a slab of thickness $H = 500\,\mathrm{nm}$, with an upper rough surface characterized by the parameters: rms height $\sigma_1 = 15\,\mathrm{nm}$, correlation length $l_1 = 100\,\mathrm{nm}$, and a lower rough surface: $\sigma_2 = 5\,\mathrm{nm}$, $l_2 = 100\,\mathrm{nm}$. The permittivity of the successive media is: $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + i\,0.0075$, and $\epsilon_2 = -18.3 + i\,0.55$. Incident angles: $\theta_i = 0^\circ$, $\phi_i = 0^\circ$, wavelength $\lambda = 632.8\,\mathrm{nm}$.

The incoherent bistatic cross-sections for the 4 polarization states as a function of the scattering angle are shown in Fig. 41. The calculations are performed with 16 Fourier modes. The results are qualitatively similar to those obtained in the SPM case (see [27] Fig. 4), we notice for the polarization $H-H$ that the maximum and minimum are larger. The Fig. 42 shows the enhancement of the backscattering

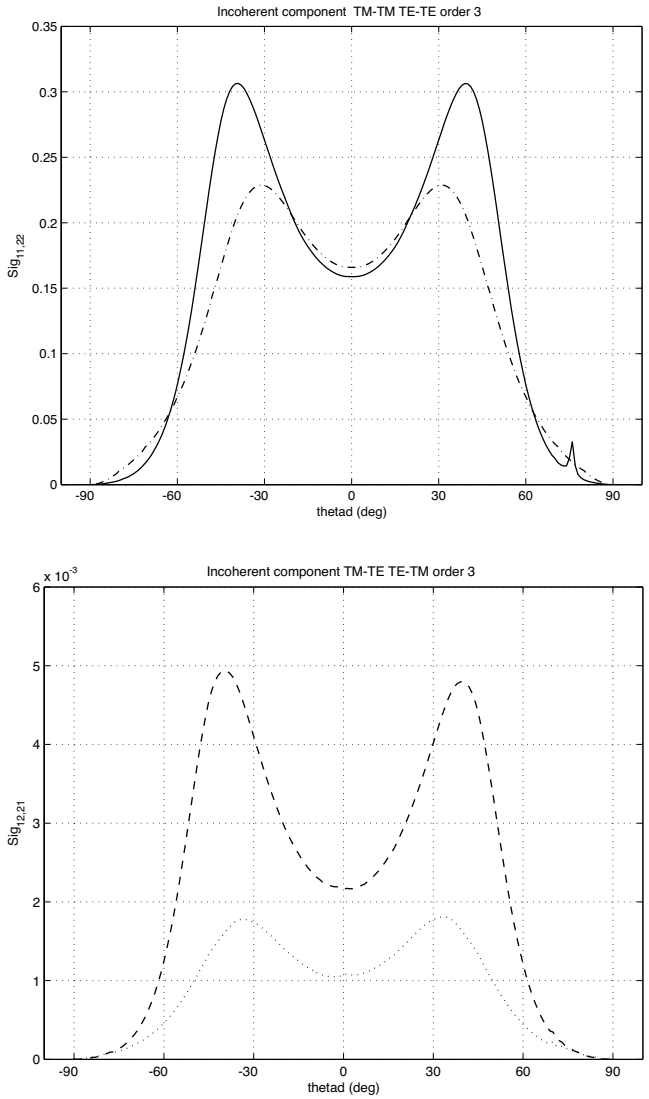


Figure 41. Incoherent cross-sections to the order 3 for an incident polarized wave $\lambda = 632.8\,\text{nm}$. Permittivity of the media: $\epsilon_0 = 1$, $\epsilon_1 = 2.6896 + i0.0075$, $\epsilon_2 = -18.3 + 0.55i$. Slab thickness $H = 500\,\text{nm}$. Upper rough surace; height $\sigma_1 = 15\,\text{nm}$, correlation length $l_1 = 100\,\text{nm}$, lower rough surface: $\sigma_2 = 5\,\text{nm}$, $l_2 = 100\,\text{nm}$. Angles: $\theta_i = 0°$, $\phi_i = 0°$. Polarizations: VV (dashed-dotted line), HH (solid line), HV (dashed line), VH (dotted line). Calculations are done with 16 Fourier modes.

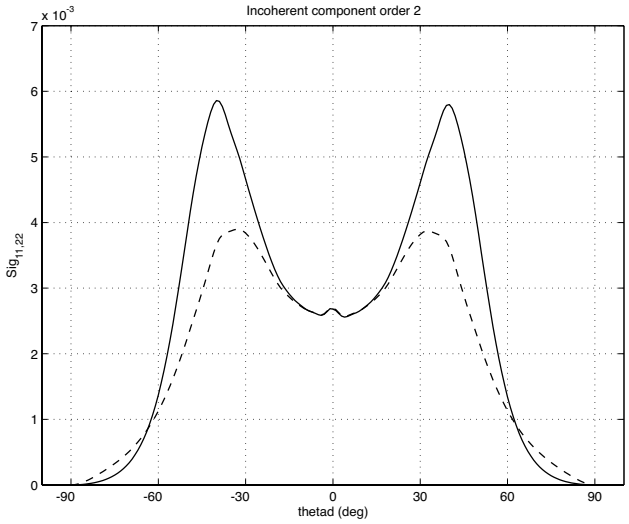
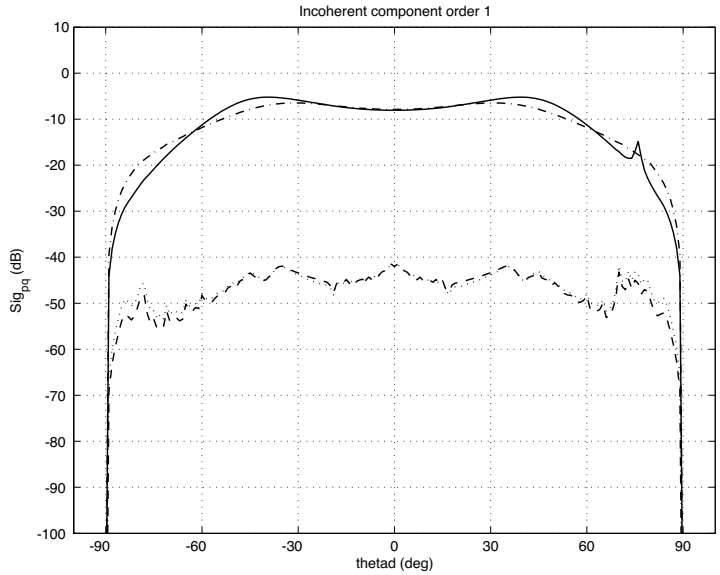
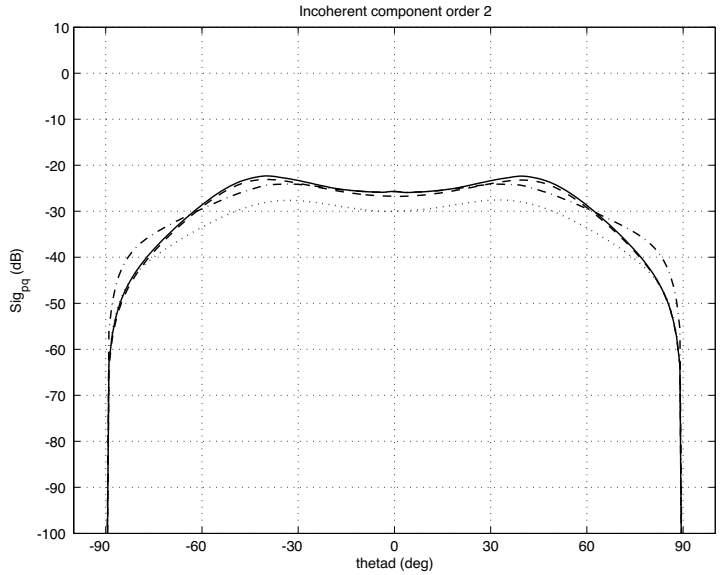


Figure 42. Incoherent cross-sections contribution to order 2. Polarization TE-TE (solid line), TM-TM (dashed line).

for $\theta = 0^\circ$ due to the order 2 contribution, this phenomena was also observed in the SPM case [27]. In order to get an estimate of the magnitude of the different order contributions, we show in Fig. 43 the cross-sections for the different polarizations states according to the order. We notice that the cross-section values decrease with increasing order, giving a justification of a perturbative development, although, no proof of convergence exists. The order 1 polarizations TE-TE, TM-TM are dominant, the polarizations TE-TM, TM-TE give smaller contributions and the order 1 and 2 are close. The order 3 polarizations are 40 dB lower compared to order 1 or 2.

In the calculations of the cross-sections the number of Fourier modes plays a significant role on their magnitude. A calculation with 256 modes at the order 1, is given in Fig. 44, the results show a better agreement with the SPM case. We have studied the contributions given by the upper and lower surfaces separately. In Fig. 45 are drawn for the 4 states of polarization the corresponding cross-sections limited to order 1, the lower surface contributes less than the upper one for the polarizations TE-TE, TM-TM, while for TE-TM, TM-TE we observe the opposite effect.


Incoherent component order 1
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
Sig$_{pq}$ (dB)
-90
-60
-30
0
30
60
90
thetad (deg)


Incoherent component order 2
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
Sig$_{pq}$ (dB)
-90
-60
-30
0
30
60
90
thetad (deg)

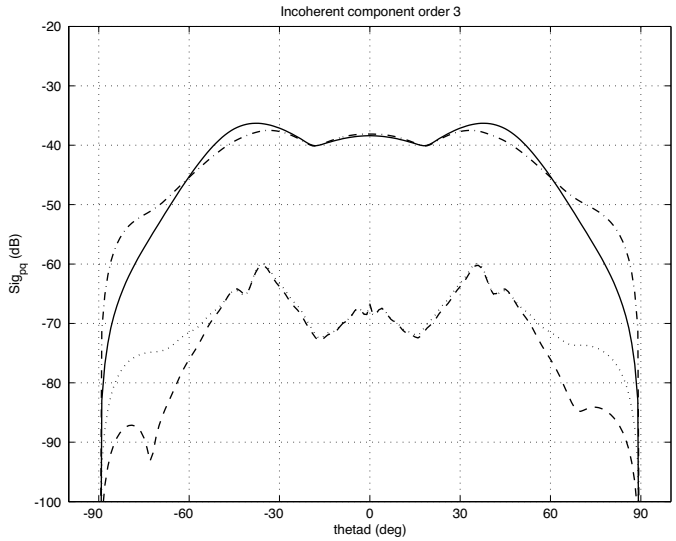
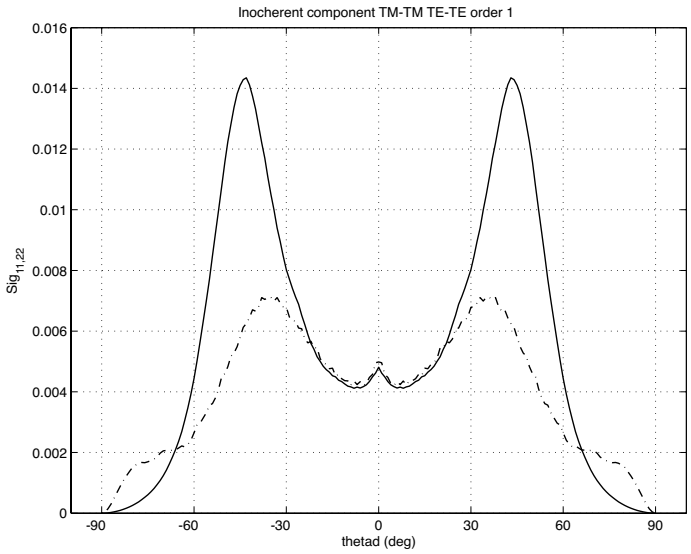


Figure 43. Contributions to the polarizations for different orders, 1, 2, 3. Parameters are the same as in Fig. 41. VV (dashed-dotted line), HH (solid line), HV (dashed line), VH (dashed line).



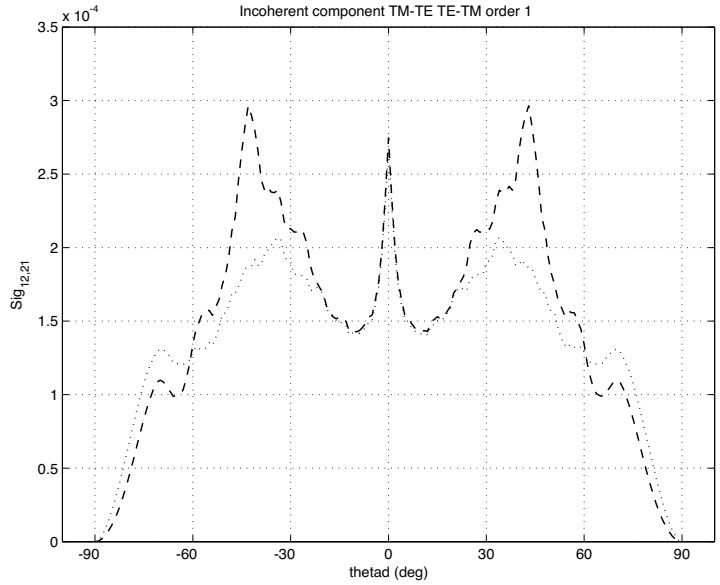
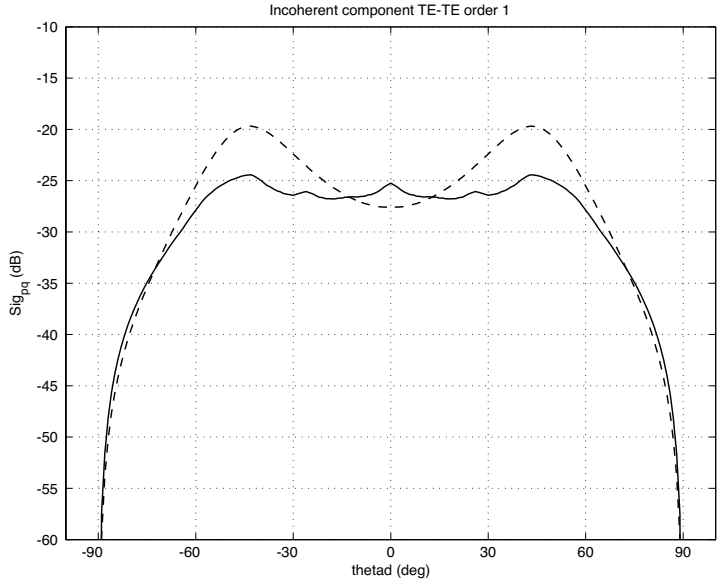
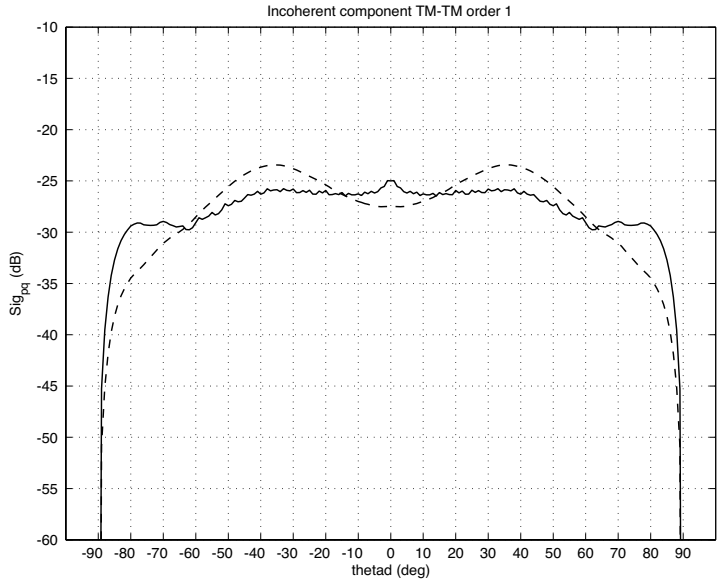
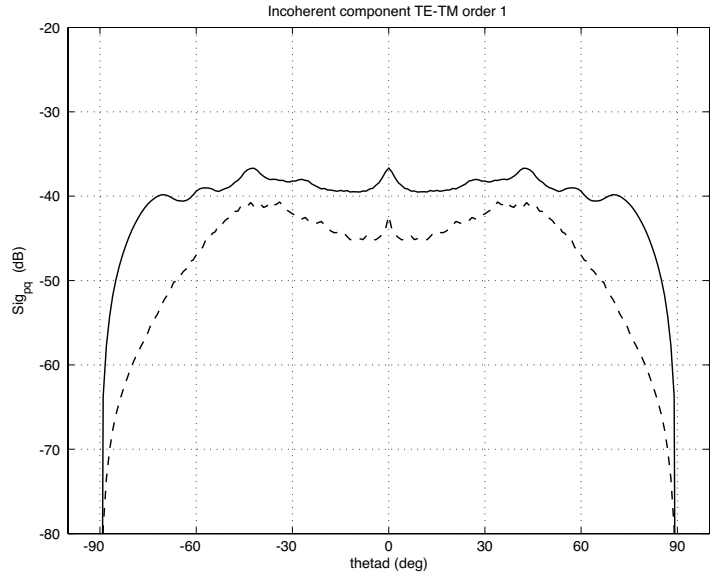


Figure 44. Contribution of the order 1 with 256 Fourier modes. Parameters are the same as in Fig. 41. VV (dashed-dotted line), HH (solid line), HV (dashed line), VH (dashed line).




Incoherent component TM-TM order 1
Sig_pq (dB)
thetad (deg)
-10
-15
-20
-25
-30
-35
-40
-45
-50
-55
-60
-90 -80 -70 -60 -50 -40 -30 -20 -10 0 10 20 30 40 50 60 70 80 90


Incoherent component TE-TM order 1
Sig_pq (dB)
thetad (deg)
-20
-30
-40
-50
-60
-70
-80
-90 -60 -30 0 30 60 90

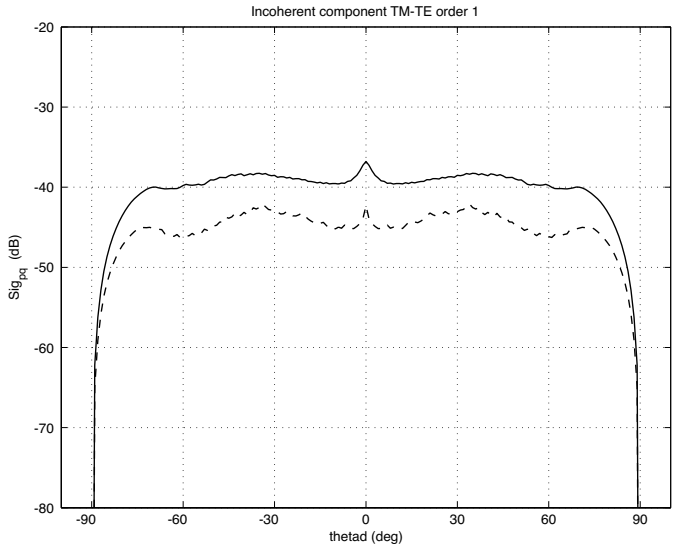


Figure 45. Contribution of the order 1 to different polarizations states due to the upper rough surface (dashed line), the lower surface (solid line). Same parameters as in Fig. 44.

10. CONCLUSION

In this paper, we have presented very new results on the small-slope approximation. In the development of the SSA series, we have taken into account the third-order SPM (small-perturbation method) kernel. We have generalized the Voronovich ansatz to a layer bounded with two randomly rough surfaces. The functional introduced by Voronovich is expanded in a Taylor series in powers of the different heights h_1 and h_2 of the rough surfaces taking into account the translational invariance. We consider successively the terms of order $n + m = 1, 2, 3, 4$ where n and m are the powers of h_1 and respectively h_2. We have introduced new terms in the SSA development to consider the coupling between the two rough surfaces. We have given the complete expressions of the scattering matrices and the expression of the needed cross-section for the different polarization states by introducing the Muller matrices. With this new formulation of the SSA, we have observed the backscattering enhancement for a slightly rough layer. We also have performed a comparison between our formulation of the small-slope approximation (SSA) and the formulation of the small-perturbation method (SPM) we have developed for different dielectric and metallic structures. Four types of structure are studied: a rough surface

separating two infinite media, a slab with upper rough surface and a lower rough surface, and finally the general case where a slab is delimited by two rough surfaces. The calculation of the scattering amplitudes involves a knowledge of the SPM scattering matrices, we have used those obtained in [26, 27].

We have calculated the scattered intensity up to the order 2 for the first 3 structures, and up to the order 3 for the last one. The global form of the intensity spectra for the 4 polarizations states are similar for both methods, however, some differences exist concerning the maxima and minima obtained, the SSA has a tendency to increase their values. In the case of a slab delimited by two rough surfaces, it was difficult to put in evidence the satellite peaks we observed in the SPM [27]. In fact, the SSA method combines different orders of the SPM, so the resulting contributions can hidden this effect. Further studies with more appropriate integration methods are required to address this issue. The numerical calculation of the intensities is performed by a FFT method and we have noticed a sensitivity of the results on the number of Fourier modes which are used.

This type of simulation computation can give some experimental conditions and specifications to realize highly integrated optical devices that use metallic or metallo-dielectric nano-scale structures.

APPENDIX

In order to make the paper self-contained we give in the appendices a summary of the formulas derived in [26] in the case of the small-perturbation method. Appendix A contains the scattering matrices for a rough surface separating two semi-infinite media, Appendix B, for a rough surface on the bottom side of a slab, and Appendix C for a rough surface on the upper side of a slab.

APPENDIX A. DEFINITION OF THE SCATTERING MATRICES FOR A SINGLE ROUGH SURFACE

$$\overline{\boldsymbol{X}}^{(0)}_{s\,\epsilon_0,\epsilon_1}(\boldsymbol{p}_0) = \overline{\boldsymbol{D}}^{-}_{10}(\boldsymbol{p}_0)\cdot\left[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{p}_0)\right]^{-1}, \tag{A1}$$

$$\overline{\boldsymbol{X}}^{(1)}_{s\,\epsilon_0,\epsilon_1}(\boldsymbol{u}|\boldsymbol{p}_0) = 2i\,\overline{\boldsymbol{Q}}^{+}(\boldsymbol{u}|\boldsymbol{p}_0), \tag{A2}$$

$$\begin{aligned}\overline{\boldsymbol{X}}^{(2)}_{s\,\epsilon_0,\epsilon_1}(\boldsymbol{u}|\boldsymbol{p}_1|\boldsymbol{p}_0) =& \alpha_1(\boldsymbol{u})\,\overline{\boldsymbol{Q}}^{+}(\boldsymbol{u}|\boldsymbol{p}_0) + \alpha_0(\boldsymbol{p}_0)\,\overline{\boldsymbol{Q}}^{-}(\boldsymbol{u}|\boldsymbol{p}_0)\\ &- 2\overline{\boldsymbol{P}}(\boldsymbol{u}|\boldsymbol{p}_1)\cdot\overline{\boldsymbol{Q}}^{+}(\boldsymbol{p}_1|\boldsymbol{p}_0),\end{aligned} \tag{A3}$$

where

$$\overline{\boldsymbol{Q}}^{\pm}(\boldsymbol{u}|\boldsymbol{p}_0) \equiv \frac{\alpha_1(\boldsymbol{u})-\alpha_0(\boldsymbol{u})}{2\alpha_0(\boldsymbol{p}_0)}[\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{u})]^{-1} \cdot [\overline{\boldsymbol{M}}^{1+,0-}(\boldsymbol{u}|\boldsymbol{p}_0) \pm \overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{p}_0)\cdot\overline{\boldsymbol{X}}^{(0)}(\boldsymbol{p}_0)], \quad \text{(A4)}$$

or explicitly:

$$\overline{\boldsymbol{Q}}^{+}(\boldsymbol{u}|\boldsymbol{p}_0) = (\epsilon_1-\epsilon_0)\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{u})]^{-1}\cdot \begin{pmatrix} \epsilon_1\,||\boldsymbol{u}||||\boldsymbol{p}_0|| - \epsilon_0\,\alpha_1(\boldsymbol{u})\,\alpha_1(\boldsymbol{p})\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_0 & -\epsilon_0^{\frac{1}{2}}\,K_0\,\alpha_1(\boldsymbol{u})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_0)_z \\ -\epsilon_0^{\frac{1}{2}}\,K_0\,\alpha_1(\boldsymbol{p})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_0)_z & \epsilon_0\,K_0^2\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_0 \end{pmatrix} \cdot[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{p}_0)]^{-1}, \quad \text{(A5)}$$

$$\overline{\boldsymbol{Q}}^{-}(\boldsymbol{u}|\boldsymbol{p}_0) = \frac{(\epsilon_1-\epsilon_0)}{\alpha_0(\boldsymbol{p}_0)}\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{u})]^{-1}\cdot \begin{pmatrix} \epsilon_0\,\alpha_1(\boldsymbol{p})||\boldsymbol{u}||||\boldsymbol{p}_0|| - \epsilon_1\,\alpha_1(\boldsymbol{u})\alpha_0^2(\boldsymbol{p}_0)\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_0 & -\epsilon_0^{\frac{1}{2}}\,K_0\alpha_1(\boldsymbol{u})\alpha_1(\boldsymbol{p})\,(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_0)_z \\ -\epsilon_0^{\frac{1}{2}}\,K_0\epsilon_1\alpha_0^2(\boldsymbol{p}_0)(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_0)_z & \epsilon_0\,K_0^2\,\alpha_1(\boldsymbol{p})\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_0 \end{pmatrix} \cdot[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{p}_0)]^{-1}, \quad \text{(A6)}$$

$$\overline{\boldsymbol{P}}(\boldsymbol{u}|\boldsymbol{p}_1) \equiv (\alpha_1(\boldsymbol{u})-\alpha_0(\boldsymbol{u}))\,[\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{u})]^{-1}\,\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{p}_1) \quad \text{(A7)}$$

$$= (\epsilon_1-\epsilon_0)\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{u})]^{-1}\cdot \begin{pmatrix} ||\boldsymbol{u}||||\boldsymbol{p}|| + \alpha_1(\boldsymbol{u})\,\alpha_0(\boldsymbol{p})\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_1 & -\epsilon_0^{\frac{1}{2}}\,K_0\,\alpha_1(\boldsymbol{u})\,(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_1)_z \\ \epsilon_0^{-\frac{1}{2}}\,K_0\,\alpha_0(\boldsymbol{p})\,(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_1)_z & K_0^2\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_1 \end{pmatrix}, \quad \text{(A8)}$$

where

$$\overline{\boldsymbol{D}}^{\pm}_{10}(\boldsymbol{p}_0) \equiv \begin{pmatrix} \epsilon_1\,\alpha_0(\boldsymbol{p}_0) \pm \epsilon_0\,\alpha_1(\boldsymbol{p}) & 0 \\ 0 & \alpha_0(\boldsymbol{p}_0) \pm \alpha_1(\boldsymbol{p}) \end{pmatrix}. \quad \text{(A9)}$$

APPENDIX B. DEFINITION OF THE SCATTERING MATRICES FOR A SLAB WITH A ROUGH SURFACE ON THE BOTTOM SIDE

$$\overline{\boldsymbol{X}}^{(0)}_{d}(\boldsymbol{p}_0) = \left(\overline{\boldsymbol{V}}^{10}(\boldsymbol{p}_0) + \overline{\boldsymbol{V}}^{H\,21}(\boldsymbol{p}_0)\right)\left[\overline{1} + \overline{\boldsymbol{V}}^{10}(\boldsymbol{p}_0)\cdot\overline{\boldsymbol{V}}^{H21}(\boldsymbol{p}_0)\right]^{-1}, \quad \text{(B1)}$$

$$\overline{\boldsymbol{X}}^{(1)}_{d}(\boldsymbol{p}|\boldsymbol{p}_0) = \overline{\boldsymbol{T}}^{10}(\boldsymbol{p})\cdot\overline{\boldsymbol{U}}^{(0)}(\boldsymbol{p})\cdot\overline{\boldsymbol{X}}^{H(1)}_{s\epsilon_1,\epsilon_2}(\boldsymbol{p}|\boldsymbol{p}_0)\cdot\overline{\boldsymbol{U}}^{(0)}(\boldsymbol{p}_0)\cdot\overline{\boldsymbol{T}}^{10}(\boldsymbol{p}_0), \quad \text{(B2)}$$

$$\overline{\boldsymbol{X}}_d^{(2)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0) = \overline{\boldsymbol{T}}^{10}(\boldsymbol{p}) \cdot \overline{\boldsymbol{U}}^{(0)}(\boldsymbol{p}) \cdot \Big[\overline{\boldsymbol{X}}_{s\,\epsilon_1,\epsilon_2}^{H\,(2)}(\boldsymbol{p}|\boldsymbol{p}_1|\boldsymbol{p}_0) - \alpha_1(\boldsymbol{p}_1)\, \overline{\boldsymbol{X}}_{s\,\epsilon_1,\epsilon_2}^{H\,(1)}(\boldsymbol{p}|\boldsymbol{p}_1) \cdot \overline{\boldsymbol{U}}^{(0)}(\boldsymbol{p}_1) \cdot \overline{\boldsymbol{V}}^{10}(\boldsymbol{p}_1) \cdot \overline{\boldsymbol{X}}_{s\,\epsilon_1,\epsilon_2}^{H\,(1)}(\boldsymbol{p}_1|\boldsymbol{p}_0)\Big] \cdot \overline{\boldsymbol{U}}^{(0)}(\boldsymbol{p}_0) \cdot \overline{\boldsymbol{T}}^{10}(\boldsymbol{p}_0)\,. \tag{B3}$$

In these formulas $\overline{\boldsymbol{X}}_{s\,\epsilon_1,\epsilon_2}^{(i)}$ are defined in Appendix A and

$$\overline{\boldsymbol{X}}_{s\,\epsilon_1,\epsilon_2}^{H\,(n)}(\boldsymbol{p}|\boldsymbol{p}_0) \equiv \exp(\mathrm{i}(\alpha_1(\boldsymbol{p}) + \alpha_1(\boldsymbol{p}_0))\, H)\, \overline{\boldsymbol{X}}_{s\,\epsilon_1,\epsilon_2}^{(n)}(\boldsymbol{p}|\boldsymbol{p}_0)\ , \tag{B4}$$

where we have replaced the permittivities ϵ_0 by ϵ_1, and ϵ_1 by ϵ_2. The expressions of the other matrices are given by:

$$\overline{\boldsymbol{V}}^{10}(\boldsymbol{p}_0) \equiv \overline{\boldsymbol{D}}_{10}^{-}(\boldsymbol{p}_0)[\overline{\boldsymbol{D}}_{10}^{+}(\boldsymbol{p}_0)]^{-1}\,, \tag{B5}$$

and $\overline{\boldsymbol{D}}_{10}$ is defined by (A9).

$$\overline{\boldsymbol{V}}^{H\,21}(\boldsymbol{p}_0) \equiv \exp(2\,i\,\alpha_1(\boldsymbol{p}_0)\, H)\, \overline{\boldsymbol{D}}_{21}^{-}(\boldsymbol{p}_0)\left[\overline{\boldsymbol{D}}_{21}^{+}(\boldsymbol{p}_0)\right]^{-1}\,, \tag{B6}$$

$$\overline{\boldsymbol{D}}_{21}^{\pm}(\boldsymbol{p}_0) \equiv \begin{pmatrix} \epsilon_2\,\alpha_1(\boldsymbol{p}_0) \pm \epsilon_1\,\alpha_2(\boldsymbol{p}_0) & 0 \\ 0 & \alpha_1(\boldsymbol{p}_0) \pm \alpha_2(\boldsymbol{p}_0) \end{pmatrix}\,, \tag{B7}$$

$$\overline{\boldsymbol{T}}^{10}(\boldsymbol{p}_0) = \alpha_1(\boldsymbol{p}_0) \begin{pmatrix} (\epsilon_0\,\epsilon_1)^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \cdot [\overline{\boldsymbol{D}}_{10}^{+}(\boldsymbol{p}_0)]^{-1}\,, \tag{B8}$$

$$\overline{\boldsymbol{U}}^{(0)}(\boldsymbol{p}_0) = \left[\overline{1} + \overline{\boldsymbol{V}}^{10}(\boldsymbol{p}_0) \cdot \overline{\boldsymbol{V}}^{H\,21}(\boldsymbol{p}_0)\right]^{-1}\,. \tag{B9}$$

APPENDIX C. DEFINITION OF THE SCATTERING MATRICES FOR A SLAB WITH A ROUGH SURFACE ON THE UPPER SIDE

$$\overline{\boldsymbol{Q}}^{ba}(\boldsymbol{u}|\boldsymbol{p}_0) \equiv \frac{i}{2\,\alpha_0(\boldsymbol{p}_0)} \left[\frac{\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{u})}{\alpha_1(\boldsymbol{u}) - \alpha_0(\boldsymbol{u})} - \overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{u}) \cdot \frac{\overline{\boldsymbol{M}}^{1-,0+}(\boldsymbol{u}|\boldsymbol{u})}{\alpha_1(\boldsymbol{u}) + \alpha_0(\boldsymbol{u})}\right]^{-1} \cdot \Big[\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{p}_0) \cdot \overline{\boldsymbol{X}}_{s\,\epsilon_0,\epsilon_1}^{(0)}(\boldsymbol{p}_0) + a\,\overline{\boldsymbol{M}}^{1+,0-}(\boldsymbol{u}|\boldsymbol{p}_0) + b\,\overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{u}) \cdot \Big(\overline{\boldsymbol{M}}^{1-,0+}(\boldsymbol{u}|\boldsymbol{p}_0) \cdot \overline{\boldsymbol{X}}_{s\,\epsilon_0,\epsilon_1}^{(0)}(\boldsymbol{p}_0) + a\,\overline{\boldsymbol{M}}^{1-,0-}(\boldsymbol{u}|\boldsymbol{p}_0)\Big)\Big]\,, \tag{C1}$$

$$\overline{\boldsymbol{P}}^{\pm}(\boldsymbol{u}|\boldsymbol{p}1) \equiv \left[\frac{\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{u})}{\alpha_1(\boldsymbol{u}) - \alpha_0(\boldsymbol{u})} - \overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{u}) \cdot \frac{\overline{\boldsymbol{M}}^{1-,0+}(\boldsymbol{u}|\boldsymbol{u})}{\alpha_1(\boldsymbol{u}) + \alpha_0(\boldsymbol{u})}\right]^{-1} \cdot \left[\overline{\boldsymbol{M}}^{1+,0+}(\boldsymbol{u}|\boldsymbol{p}1) \pm \overline{\boldsymbol{r}}^{H\,21}(\boldsymbol{u}) \cdot \overline{\boldsymbol{M}}^{1-,0+}(\boldsymbol{u}|\boldsymbol{p}1)\right]\,, \tag{C2}$$

where $a = \pm$, $b = \pm$ are the indices related to the direction of propagation of the waves, $-$ downward, $+$ upward with respect to $z > 0$ direction. After some calculations:

$$\overline{\boldsymbol{Q}}^{++}(\boldsymbol{u}|\boldsymbol{p}_0) = (\epsilon_1 - \epsilon_0)\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{u})]^{-1}\cdot$$
$$\begin{pmatrix} \epsilon_1\,\overline{\boldsymbol{A}}^{++} - \epsilon_0\,\alpha_1(\boldsymbol{u})\,\alpha_1(\boldsymbol{p}_0)\,\overline{\boldsymbol{B}}^{--} & -\epsilon_0^{\frac{1}{2}}\,\alpha_1(\boldsymbol{u})\,\overline{\boldsymbol{J}}^{-+} \\ -\epsilon_0^{\frac{1}{2}}\,\alpha_1(\boldsymbol{p}_0)\,\overline{\boldsymbol{G}}^{+-} & \epsilon_0\,K_0^2\,\overline{\boldsymbol{C}}^{++} \end{pmatrix}$$
$$\cdot\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{p}_0)]^{-1}\,, \tag{C3}$$

$$\overline{\boldsymbol{Q}}^{-+}(\boldsymbol{u}|\boldsymbol{p}_0) = (\epsilon_1 - \epsilon_0)\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{u})]^{-1}\cdot$$
$$\begin{pmatrix} \epsilon_1\,\overline{\boldsymbol{A}}^{-+} - \epsilon_0\,\alpha_1(\boldsymbol{u})\,\alpha_1(\boldsymbol{p}_0)\,F_V^{+}(\boldsymbol{u})\,\overline{\boldsymbol{B}}^{+-} & -\epsilon_0^{\frac{1}{2}}\,\alpha_1(\boldsymbol{u})\,\overline{\boldsymbol{J}}^{++} \\ -\epsilon_0^{\frac{1}{2}}\,\alpha_1(\boldsymbol{p}_0)\,\overline{\boldsymbol{G}}^{--} & \epsilon_0\,K_0^2\,\overline{\boldsymbol{C}}^{-+} \end{pmatrix}$$
$$\cdot\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{p}_0)]^{-1}\,, \tag{C4}$$

$$\overline{\boldsymbol{Q}}^{+-}(\boldsymbol{u}|\boldsymbol{p}_0) = \frac{(\epsilon_1 - \epsilon_0)}{\alpha_0(\boldsymbol{p}_0)}[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{u})]^{-1}\cdot$$
$$\begin{pmatrix} \epsilon_0\alpha_1(\boldsymbol{p}_0)\overline{\boldsymbol{A}}^{+-} - \epsilon_1\,\alpha_1(\boldsymbol{u})\alpha_0^2(\boldsymbol{p}_0)\overline{\boldsymbol{B}}^{-+} & -\epsilon_0^{\frac{1}{2}}\alpha_1(\boldsymbol{u})\alpha_1(\boldsymbol{p}_0)\overline{\boldsymbol{J}}^{--} \\ -\epsilon_0^{\frac{1}{2}}\epsilon_1\alpha_0^2(\boldsymbol{p}_0)\overline{\boldsymbol{G}}^{++} & \epsilon_0 K_0^2\alpha_1(\boldsymbol{p}_0)\overline{\boldsymbol{C}}^{+-} \end{pmatrix}$$
$$\cdot\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{p}_0)]^{-1}, \tag{C5}$$

$$\overline{\boldsymbol{Q}}^{--}(\boldsymbol{u}|\boldsymbol{p}_0) = \frac{(\epsilon_1 - \epsilon_0)}{\alpha_0(\boldsymbol{p}_0)}[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{u})]^{-1}\cdot$$
$$\begin{pmatrix} \epsilon_0\alpha_1(\boldsymbol{p}_0)\overline{\boldsymbol{A}}^{--} - \epsilon_1\alpha_1(\boldsymbol{u})\alpha_0^2(\boldsymbol{p}_0)\overline{\boldsymbol{B}}^{++} & -\epsilon_0^{\frac{1}{2}}\alpha_1(\boldsymbol{u})\alpha_1(\boldsymbol{p}_0)\overline{\boldsymbol{J}}^{+-} \\ -\epsilon_0^{\frac{1}{2}}\epsilon_1\alpha_0^2(\boldsymbol{p}_0)\overline{\boldsymbol{G}}^{-+} & \epsilon_0 K_0^2\,\alpha_1(\boldsymbol{p}_0)\overline{\boldsymbol{C}}^{--} \end{pmatrix}$$
$$\cdot\,[\overline{\boldsymbol{D}}^{+}_{10}(\boldsymbol{p}_0)]^{-1}, \tag{C6}$$

where

$$\overline{\boldsymbol{A}}^{a,b} = ||\boldsymbol{u}||||\boldsymbol{p}_0||\,F_V^a(\boldsymbol{u})F_V^b(\boldsymbol{p}_0)\,, \tag{C7}$$
$$\overline{\boldsymbol{B}}^{a,b} = F_V^a(\boldsymbol{u})\,F_V^b(\boldsymbol{p}_0)\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_0\,, \tag{C8}$$
$$\overline{\boldsymbol{C}}^{a.b} = F_H^a(\boldsymbol{u})F_H^b(\boldsymbol{p}_0)\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_0\,, \tag{C9}$$
$$\overline{\boldsymbol{J}}^{a,b} = F_V^a(\boldsymbol{u})F_H^b(\boldsymbol{p}_0)\,(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_0)_z\,, \tag{C10}$$
$$\overline{\boldsymbol{G}}^{a,b} = F_H^a(\boldsymbol{u})F_V^b(\boldsymbol{p}_0)\,(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_0)_z\,, \tag{C11}$$

$$\begin{pmatrix} F_V^{\pm}(\boldsymbol{p}_0) & 0 \\ 0 & F_H^{\pm}(\boldsymbol{p}_0) \end{pmatrix} = \left(\overline{1} \pm \overline{\boldsymbol{V}}^{H\,21}(\boldsymbol{p}_0)\right) \left(\overline{1} + \overline{\boldsymbol{r}}^{10}(\boldsymbol{p}_0) \cdot \overline{\boldsymbol{V}}^{H\,21}(\boldsymbol{p}_0)\right)^{-1}, \tag{C12}$$

the matrix $\overline{\boldsymbol{r}}^{\,10}$ represents the reflection coefficient of a planar surface located at $z = 0$ and separating two media of permittivity ϵ_0 et ϵ_1: (In [26] Eqs. (150), (151) have a misprint.)

$$\overline{\boldsymbol{r}}^{10}(\boldsymbol{p}_0) = \begin{pmatrix} \frac{\epsilon_1\,\alpha_0(\boldsymbol{p}_0) - \epsilon_0\,\alpha_1(\boldsymbol{p}_0)}{\epsilon_1\,\alpha_0(\boldsymbol{p}_0) + \epsilon_0\,\alpha_1(\boldsymbol{p}_0)} & 0 \\ 0 & \frac{\alpha_0(\boldsymbol{p}_0) - \alpha_1(\boldsymbol{p}_0)}{\alpha_0(\boldsymbol{p}_0) + \alpha_1(\boldsymbol{p}_0)} \end{pmatrix}, \tag{C13}$$

$\overline{\boldsymbol{V}}^{H\,21}$ is given by Eq. (B6).

The explicit form of the matrices $\overline{\boldsymbol{P}}^{\pm}$ is the following:

$$\overline{\boldsymbol{P}}^{+}(\boldsymbol{u}|\boldsymbol{p}_0) = (\epsilon_1 - \epsilon_0)\,[\overline{\boldsymbol{D}}_{10}^{+}(\boldsymbol{u})]^{-1} \cdot \begin{pmatrix} ||\boldsymbol{u}||||\boldsymbol{p}||F_V^{+}(\boldsymbol{u}) + \alpha_1(\boldsymbol{u})\alpha_0(\boldsymbol{p})F_V^{-}(\boldsymbol{u})\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_1 & -\epsilon_0^{\frac{1}{2}}\,K_0\alpha_1(\boldsymbol{u})F_V^{-}(\boldsymbol{u})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_1)_z \\ \epsilon_0^{-\frac{1}{2}}\,K_0\alpha_0(\boldsymbol{p})F_H^{+}(\boldsymbol{u})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_1)_z & K_0^2 F_H^{+}(\boldsymbol{u})\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_1 \end{pmatrix}, \tag{C14}$$

$$\overline{\boldsymbol{P}}^{-}(\boldsymbol{u}|\boldsymbol{p}_0) = (\epsilon_1 - \epsilon_0)[\overline{\boldsymbol{D}}_{10}^{+}(\boldsymbol{u})]^{-1} \cdot \begin{pmatrix} ||\boldsymbol{u}||||\boldsymbol{p}||F_V^{-}(\boldsymbol{u}) + \alpha_1(\boldsymbol{u})\alpha_0(\boldsymbol{p})F_V^{+}(\boldsymbol{u})\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_1 & -\epsilon_0^{\frac{1}{2}}\,K_0\alpha_1(\boldsymbol{u})F_V^{+}(\boldsymbol{u})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_1)_z \\ \epsilon_0^{-\frac{1}{2}}\,K_0\alpha_0(\boldsymbol{p})F_H^{-}(\boldsymbol{u})(\hat{\boldsymbol{u}}\times\hat{\boldsymbol{p}}_1)_z & K_0^2\,F_H^{-}(\boldsymbol{u})\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{p}}_1 \end{pmatrix}. \tag{C15}$$

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