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TIME-DOMAIN THEORY OF METAL CAVITY RESONATOR

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Abstract—This paper presents a thorough study of the time-domain theory of metal cavity resonators. The completeness of the vector modal functions of a perfectly conducting metal cavity is first proved by symmetric operator theory, and analytic solution for the field distribution inside the cavity excited by an arbitrary source is then obtained in terms of the vector modal functions. The main focus of the present paper is the time-domain theory of a waveguide cavity, for which the excitation problem may be reduced to the solution of a number of modified Klein-Gordon equations. These modified Klein-Gordon equation are then solved by the method of retarded Green's function in order that the causality condition is satisfied. Numerical examples are also presented to demonstrate the time-domain theory. The analysis indicates that the time-domain theory is capable of providing an exact picture for the physical process inside a closed cavity and can overcome some serious problems that may arise in traditional time-harmonic theory due to the lack of causality.

1. INTRODUCTION

The rapid progress in ultra-wideband technologies has prompted the study of time-domain electromagnetics. Compared to the voluminous literature on time-harmonic theory of electromagnetics, the time-domain electromagnetics is still a virgin land to be cultivated. In the time-domain theory, the fields are assumed to start at a finite instant of time and Maxwell equations are solved subject to initial conditions, boundary conditions, excitation conditions and causality. A metal cavity resonator constitutes a typical eigenvalue problem in electromagnetic theory and has been investigated by a number of authors [e.g., 1–5], and the study of the transient process in a metal cavity may be carried out by the field expansions in terms of the modal vector functions. When these expansions are introduced into the time-domain Maxwell equations one may find that the expansion coefficients satisfy the ordinary differential equations of second order [3], which can be easily solved once the initial conditions and the excitations are known. Recently this approach has been used to investigate the responses of the metal cavity to digital signals [4].

Although the study of metal cavity resonator has a long history, there are still some open questions which need to be investigated. This paper attempts to answer these questions, and presents a thorough discussion on the time-domain theory of metal cavities. The paper is organized as follows. Section 2 studies the eigenvalue theory of a perfectly conducting metal cavity filled with lossy medium. A fundamental problem in metal cavity theory is to prove the completeness of its vector modal functions, which has been tried by Kurokawa [1]. But there is a loophole in Kurokawa's approach in which he fails to show the existence of the vector modal functions. The main purpose of Section 2 is to provide a rigorous proof of the completeness of the vector modal functions on the basis of the theory of symmetric operators. Section 3 summarizes the field expansions inside an arbitrary metal cavity filled with lossy medium in terms of the vector modal functions, and discusses the limitations of the time-harmonic theory when it is applied to a closed metal cavity. Section 4 is dedicated to a metal cavity formed by a section of uniform waveguide, i.e., the waveguide cavity. The time-domain theory of the waveguide developed previously [6] is applicable to this case, and the fields inside the waveguide cavity can be expanded in terms of the transverse vector modal functions of the corresponding waveguide. The expansion coefficients of the fields are shown to satisfy the modified Klein-Gordon equation subject to homogeneous boundary conditions, which are then solved by the method of retarded Green's function to satisfy the causality requirement. In order to validate the theory, some examples are expounded in Section 5, and the field responses to typical excitation waveforms are demonstrated.

An important observation in this paper is that the causality plays an important role in determining the field response of a closed cavity. In other words, one must take the initial conditions into consideration in order to obtain a correct field response (transient or steady-state). The time-domain theory shows that a sinusoidal response can be built up if and only if the cavity is excited by sinusoidal source whose frequency coincides with one of the resonant frequencies of the cavity. The traditional time-harmonic theory, however, predicts that the field response is always sinusoidal if the excitation source is sinusoidal.

This discrepancy comes from the lack of causality in traditional time-harmonic theory, which assumes that the source is turned on at $t = -\infty$ instead of a finite instant of time and has ignored the initial conditions.

Another interesting observation is that the field responses in a lossless cavity predicted by the time-harmonic theory are singular everywhere inside the cavity if the frequency of the sinusoidal excitation source coincides with one of the resonant frequencies of the cavity, while the time-domain theory always gives finite field responses. The singularities in time-harmonic theory can be removed by introducing losses inside the cavity, which is essentially required by the uniqueness theorem for a time-harmonic field in a bounded region.

2. EIGENVALUE THEORY FOR METAL CAVITY RESONATOR

The metal cavity resonator constitutes a typical eigenvalue problem, where the eigenvalues correspond to resonant frequencies of the cavity and eigenfunctions correspond to the natural field distributions. Given the eigenvalue problem, one must show the existence and completeness of the eigenfunctions, and the latter implies that the set of eigenfunctions can be used to expand an arbitrary function.

2.1. Eigenvalue Problems for a Metal Cavity

Let us consider a metal cavity with a perfectly conducting wall. It will be assumed that the medium in the cavity is homogeneous and isotropic with medium parameters σ, μ and ε. The volume occupied by the cavity is denoted by V and the boundary of V by S (Figure 1). Since the metallic wall is a perfect conductor, the transient fields in

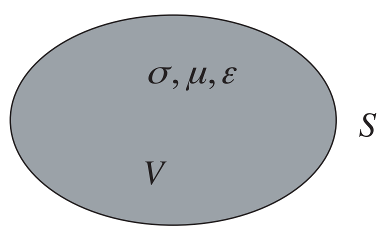

Figure 1. An arbitrary metal cavity.

the cavity satisfy Maxwell equations

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\mu \dfrac{\partial}{\partial t} \boldsymbol{H}(\boldsymbol{r}, t) \\ \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \varepsilon \dfrac{\partial}{\partial t} \boldsymbol{E}(\boldsymbol{r}, t) + \sigma \boldsymbol{E}, \ \boldsymbol{r} \in V \\ \nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = 0 \\ \nabla \cdot \boldsymbol{H}(\boldsymbol{r}, t) = 0 \end{cases} \tag{1}$$

with boundary conditions $\boldsymbol{u}_n \times \boldsymbol{E} = 0$ and $\boldsymbol{u}_n \cdot \boldsymbol{H} = 0$, where $\boldsymbol{u}_n$ is the unit outward normal to the boundary S and all other notations have their usual meaning. From (1), one may obtain

$$\begin{cases} \nabla \times \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) + \mu\varepsilon \dfrac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t^2} + \mu\sigma \dfrac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} = 0, & \boldsymbol{r} \in V \\ \boldsymbol{u}_n \times \boldsymbol{E}(\boldsymbol{r}, t) = 0, & \boldsymbol{r} \in S \end{cases} \tag{2}$$

$$\begin{cases} \nabla \times \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) + \mu\varepsilon \dfrac{\partial^2 \boldsymbol{H}(\boldsymbol{r}, t)}{\partial t^2} + \mu\sigma \dfrac{\partial \boldsymbol{H}(\boldsymbol{r}, t)}{\partial t} = 0, & \boldsymbol{r} \in V \\ \boldsymbol{u}_n \cdot \boldsymbol{H}(\boldsymbol{r}, t) = 0, \ \boldsymbol{u}_n \times \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = 0, & \boldsymbol{r} \in S \end{cases} \tag{3}$$

If the solutions of (2) and (3) can be expressed as a separable function of space and time

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{e}(\boldsymbol{r}) u(t), \quad \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{h}(\boldsymbol{r}) v(t)$$

it follows from (2) and (3) that

$$\begin{cases} \nabla \times \nabla \times \boldsymbol{e} - k_e^2 \boldsymbol{e} = 0, \ \nabla \cdot \boldsymbol{e} = 0, & \boldsymbol{r} \in V \\ \boldsymbol{u}_n \times \boldsymbol{e} = 0, & \boldsymbol{r} \in S \end{cases} \tag{4}$$

$$\begin{cases} \nabla \times \nabla \times \boldsymbol{h} - k_h^2 \boldsymbol{h} = 0, \ \nabla \cdot \boldsymbol{h} = 0, & \boldsymbol{r} \in V \\ \boldsymbol{u}_n \cdot \boldsymbol{h} = 0, \ \boldsymbol{u}_n \times \nabla \times \boldsymbol{h} = 0, & \boldsymbol{r} \in S \end{cases} \tag{5}$$

The functions $u(t)$ and $v(t)$ satisfy

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\eta}{c} \frac{\partial u}{\partial t} + k_e^2 u = 0 \tag{6}$$

$$\frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} + \sigma \frac{\eta}{c} \frac{\partial v}{\partial t} + k_h^2 v = 0 \tag{7}$$

where k_e^2 and k_h^2 are separation constants, $\eta = \sqrt{\mu/\varepsilon}$ and $c = 1/\sqrt{\mu\varepsilon}$. Both (4) and (5) form an eigenvalue problem. However their eigenfunctions do not form a complete set. To overcome this difficulty, (4) and (5) can be modified as [1]

$$\begin{cases} \nabla\times\nabla\times\boldsymbol{e}-\nabla\nabla\cdot\boldsymbol{e}-k_e^2\boldsymbol{e}=0, & \boldsymbol{r}\in V \\ \boldsymbol{u}_n\times\boldsymbol{e}=0,\ \nabla\cdot\boldsymbol{e}=0, & \boldsymbol{r}\in S \end{cases} \tag{8}$$

$$\begin{cases} \nabla\times\nabla\times\boldsymbol{h}-\nabla\nabla\cdot\boldsymbol{h}-k_h^2\boldsymbol{h}=0, & \boldsymbol{r}\in V \\ \boldsymbol{u}_n\cdot\boldsymbol{h}=0,\ \boldsymbol{u}_n\times\nabla\times\boldsymbol{h}=0, & \boldsymbol{r}\in S \end{cases} \tag{9}$$

These are the eigenvalue equations for the metal cavity system.

2.2. Completeness of the Eigenfunctions

The eigenvalue problems (8) and (9) have been discussed by Kurokawa [1]. Kurokawa's approach suffers a drawback that has overlooked the proof of the existence of the eigenfunctions and eigenvalues. In the following, a rigorous approach will be presented, which is similar to what is used in studying the waveguide eigenvalue problem [6]. Let us rewrite the eigenvalue problem (8) as

$$\begin{cases} \boldsymbol{B}(\boldsymbol{e})=\nabla\times\nabla\times\boldsymbol{e}-\nabla\nabla\cdot\boldsymbol{e}=k_e^2\boldsymbol{e}, & \boldsymbol{r}\in V \\ \boldsymbol{u}_n\times\boldsymbol{e}=0,\ \nabla\cdot\boldsymbol{e}=0, & \boldsymbol{r}\in\Gamma \end{cases} \tag{10}$$

where $\boldsymbol{B}=\nabla\times\nabla\times-\nabla\nabla$. The domain of definition of operator $\boldsymbol{B}$ is defined by

$$D(\boldsymbol{B})=\left\{\boldsymbol{e}\,\middle|\,\boldsymbol{e}\in(C^\infty(\Omega))^2,\ \boldsymbol{u}_n\times\boldsymbol{e}=0,\ \nabla\cdot\boldsymbol{e}=0\text{ on }\Gamma\right\}$$

where $C^\infty(V)$ stands for the set of functions that have continuous partial derivatives of any order. Let $L^2(V)$ stand for the space of square-integrable functions defined in V and $H=(L^2(V))^3=L^2(V)\times L^2(V)\times L^2(V)$. For two vector fields $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in $(L^2(V))^3$, the inner product is defined by $(\boldsymbol{e}_1,\boldsymbol{e}_2)=\int_V \boldsymbol{e}_1\cdot\overline{\boldsymbol{e}}_2 dV$ (a bar is used to designate the complex conjugate) and the corresponding norm is denoted by $\|\cdot\|=(\cdot,\cdot)^{1/2}$. (10) can be modified as an equivalent form by adding a term $\xi\boldsymbol{e}$ on both sides

$$\begin{cases} \boldsymbol{A}(\boldsymbol{e})=\nabla\times\nabla\times\boldsymbol{e}-\nabla\nabla\cdot\boldsymbol{e}+\xi\boldsymbol{e}=\left(k_e^2+\xi\right)\boldsymbol{e}, & \boldsymbol{r}\in V \\ \boldsymbol{u}_n\times\boldsymbol{e}=\nabla\cdot\boldsymbol{e}=0, & \boldsymbol{r}\in\Gamma \end{cases} \tag{11}$$

where ξ is an arbitrary positive constant. It is easy to show that the operator $\boldsymbol{A}$ is symmetric, strongly monotone. In fact, for all

$\boldsymbol{e}_1, \boldsymbol{e}_2 \in D(\boldsymbol{A}) = D(\boldsymbol{B})$, the energy product is given by

$$\begin{aligned}(\boldsymbol{e}_1, \boldsymbol{e}_2)_{\boldsymbol{A}} &= (\boldsymbol{A}(\boldsymbol{e}_1), \boldsymbol{e}_2) = \int_V (\nabla\times\nabla\times\boldsymbol{e}_1 - \nabla\nabla\cdot\boldsymbol{e}_1 + \xi\boldsymbol{e}_1)\cdot\overline{\boldsymbol{e}}_2 d\Omega \\ &= \int_V \left[\nabla\times\boldsymbol{e}_1\cdot\nabla\times\overline{\boldsymbol{e}}_2 + (\nabla\cdot\boldsymbol{e}_1)(\nabla\cdot\overline{\boldsymbol{e}}_2) + \xi\boldsymbol{e}_1\cdot\overline{\boldsymbol{e}}_2\right] d\Omega \quad (12)\end{aligned}$$

Therefore the new operator $\boldsymbol{A}$ is symmetric. Thus $\boldsymbol{e}$ can be assumed to be real. $\boldsymbol{A}$ is also strongly monotone since $(\boldsymbol{A}(\boldsymbol{e}), \boldsymbol{e}) \geq \xi\|\boldsymbol{e}\|^2$. Therefore one can introduce the energy inner product of the operator $\boldsymbol{A}$ defined by $(u, v)_{\boldsymbol{A}} = (\boldsymbol{A}u, v)$. The energy space $H_{\boldsymbol{A}}$ is the completion of $D(\boldsymbol{A})$ with respect to the norm $\|\cdot\|_{\boldsymbol{A}} = (\cdot,\cdot)_{\boldsymbol{A}}^{1/2}$ [7]. Let $\boldsymbol{e} \in H_{\boldsymbol{A}}$, and by definition, there exists admissible sequence $\{\boldsymbol{e}_n \in D(\boldsymbol{A})\}$ for $\boldsymbol{e}$ such that $\|\boldsymbol{e}_n - \boldsymbol{e}\| \to 0$ as $n \to \infty$ and $\{\boldsymbol{e}_n\}$ is a Cauchy sequence in $H_{\boldsymbol{A}}$. It follows from (12) that

$$\|\boldsymbol{e}_n-\boldsymbol{e}_m\|_{\boldsymbol{A}}^2 = \|\nabla\times\boldsymbol{e}_n-\nabla\times\boldsymbol{e}_m\|^2+\|\nabla\cdot\boldsymbol{e}_n-\nabla\cdot\boldsymbol{e}_m\|^2+\xi\|\boldsymbol{e}_n-\boldsymbol{e}_m\|^2$$

Consequently $\{\nabla\times\boldsymbol{e}_n\}$ and $\{\nabla\cdot\boldsymbol{e}_n\}$ are Cauchy sequences in H. As a result, there exist $\boldsymbol{h} \in H$, and $\rho \in L^2(V)$ such that $\nabla\times\boldsymbol{e}_n \to \boldsymbol{h}$ and $\nabla\cdot\boldsymbol{e}_n \to \rho$ as $n \to \infty$. From integration by parts, one may write

$$\begin{aligned}\int_V \nabla\times\boldsymbol{e}_n\cdot\boldsymbol{\varphi} dV &= \int_V \boldsymbol{e}_n\cdot\nabla\times\boldsymbol{\varphi} dV, \quad \forall\boldsymbol{\varphi} \in (C_0^\infty(V))^3 \\ \int_V (\nabla\cdot\boldsymbol{e}_n)\varphi dV &= -\int_V \boldsymbol{e}_n\cdot\nabla\varphi dV, \quad \forall\varphi \in C_0^\infty(V)\end{aligned}$$

Letting $n \to \infty$ yields

$$\begin{aligned}\int_V \boldsymbol{h}\cdot\boldsymbol{\varphi} dV &= \int_V \boldsymbol{e}\cdot\nabla\times\boldsymbol{\varphi} dV, \quad \forall\boldsymbol{\varphi} \in (C_0^\infty(V))^3 \\ \int_V \rho\varphi dV &= -\int_V \boldsymbol{e}\cdot\nabla\varphi dV, \quad \forall\varphi \in C_0^\infty(V).\end{aligned}$$

In the above $C_0^\infty(V)$ is the set of all functions in $C^\infty(V)$ that vanish outside a compact subset of V. Therefore $\nabla\times\boldsymbol{e} = \boldsymbol{h}$ and $\nabla\cdot\boldsymbol{e} = \rho$ hold in the generalized sense. For arbitrary $\boldsymbol{e}_1, \boldsymbol{e}_2 \in H_{\boldsymbol{A}}$, there are two admissible functions $\{\boldsymbol{e}_{1n}\}$ and $\{\boldsymbol{e}_{2n}\}$ such that $\|\boldsymbol{e}_{1n} - \boldsymbol{e}_1\| \to 0$ and $\|\boldsymbol{e}_{2n} - \boldsymbol{e}_2\| \to 0$ as $n \to \infty$. Define

$$\begin{aligned}(\boldsymbol{e}_1, \boldsymbol{e}_2)_{\boldsymbol{A}} &= \lim_{n\to\infty}(\boldsymbol{e}_{1n}, \boldsymbol{e}_{2n})_{\boldsymbol{A}} \\ &= \int_V \left[\nabla\times\boldsymbol{e}_1\cdot\nabla\times\overline{\boldsymbol{e}}_2 + (\nabla\cdot\boldsymbol{e}_1)(\nabla\cdot\overline{\boldsymbol{e}}_2) + \xi\boldsymbol{e}_1\cdot\overline{\boldsymbol{e}}_2\right] dV\end{aligned}$$

where the derivatives must be understood in the generalized sense. Now one can show that the embedding $H_{\boldsymbol{A}} \subset H$ is compact. Let $\boldsymbol{J}(\boldsymbol{e}) = \boldsymbol{e}$, $\boldsymbol{e} \in H_{\boldsymbol{A}}$. Then the linear operator $\boldsymbol{J} : H_{\boldsymbol{A}} \to H$ is continuous since

$$\|\boldsymbol{J}(\boldsymbol{e})\|^2 = \|\boldsymbol{e}\|^2 \leq \xi^{-1}\left(\xi\|\boldsymbol{e}\|^2 + \|\nabla \times \boldsymbol{e}\|^2 + \|\nabla \cdot \boldsymbol{e}\|^2\right) = \xi^{-1}\|\boldsymbol{e}\|^2_{H_{\boldsymbol{A}}}.$$

A bounded sequence $\{\boldsymbol{e}_n\} \subset H_{\boldsymbol{A}}$ implies

$$\begin{aligned}\|\boldsymbol{e}_n\|^2_{H_{\boldsymbol{A}}} &= \|\boldsymbol{e}_n\|^2 + \|\nabla \times \boldsymbol{e}_n\|^2 + \|\nabla \cdot \boldsymbol{e}_n\|^2 \\ &= \int\limits_V \left[\xi(e_{nx})^2 + \xi(e_{ny})^2 + (\nabla e_{nx})^2 + (\nabla e_{ny})^2\right] d\Omega \leq c'\end{aligned}$$

where c' is a constant. So the compactness of the operator $\boldsymbol{J}$ follows from the above inequality and the Rellich's theorem [1, 7]. Thus the following theorem may apply to the operator $\boldsymbol{A}$ [7, pp. 284]

Theorem: Let H be a real separable Hilbert space with $\dim H = \infty$ and $\boldsymbol{A} : D(\boldsymbol{A}) \subset H \to H$ be a linear, symmetric operator. Assume that $\boldsymbol{A}$ is strongly monotone, i.e., there exists a constant c_1 such that $(\boldsymbol{A}u, u) > c_1\|u\|^2$ for all $u \in D(\boldsymbol{A})$. Let $\boldsymbol{A}_F$ be the Friedrichs extension of $\boldsymbol{A}$ and $H_{\boldsymbol{A}}$ be the energy space of operator $\boldsymbol{A}$. If the embedding $H_{\boldsymbol{A}} \subset H$ is compact, then the following eigenvalue problem

$$\boldsymbol{A}_F u = \lambda u, \quad u \in D(\boldsymbol{A}_F)$$

has a countable eigenfunctions $\{u_n\}$, which form a complete orthonormal system in the Hilbert space H, with $u_n \in H_{\boldsymbol{A}}$. Each eigenvalue corresponding to u_n has finite multiplicity. Furthermore $\lambda_1 \leq \lambda_2 \leq \cdots$, and $\lambda_n \underset{n\to\infty}{\longrightarrow} \infty$.

It follows from this theorem that there exists a complete set of real eigenfunctions $\{\boldsymbol{e}_n\}$, called electric field modal functions, and the corresponding eigenvalues, denoted by $k^2_{e,n}$, which approach to infinity as $n \to \infty$. The set of modal functions will be assumed to be orthonormal, i.e., $\int\limits_V \boldsymbol{e}_m \cdot \boldsymbol{e}_n dv = \delta_{mn}$. It can be shown that each modal function can be chosen from one of the following three categories [1]:

$$\begin{aligned}\text{I. } \nabla \times \boldsymbol{e}_n &= 0, \quad \nabla \cdot \boldsymbol{e}_n = 0 \\ \text{II. } \nabla \times \boldsymbol{e}_n &\neq 0, \quad \nabla \cdot \boldsymbol{e}_n = 0 \\ \text{III. } \nabla \times \boldsymbol{e}_n &= 0, \quad \nabla \cdot \boldsymbol{e}_n \neq 0.\end{aligned}$$

Similarly one can show that the eigenvalues $k^2_{h,n}$ of (9) are real and positive, and the corresponding magnetic modal functions $\boldsymbol{h}_n$ are real and constitute a complete set, which will be assumed to be orthonormal, i.e., $\int\limits_V \boldsymbol{h}_m \cdot \boldsymbol{h}_n dv = \delta_{mn}$. Also each modal function can be chosen from one of the following three categories:

$$\begin{aligned} \text{I. } \nabla \times \boldsymbol{h}_n &= 0, \quad \nabla \cdot \boldsymbol{h}_n = 0 \\ \text{II. } \nabla \times \boldsymbol{h}_n &\neq 0, \quad \nabla \cdot \boldsymbol{h}_n = 0 \\ \text{III. } \nabla \times \boldsymbol{h}_n &= 0, \quad \nabla \cdot \boldsymbol{h}_n \neq 0. \end{aligned}$$

The modal functions belonging to category II in the two sets of modal functions $\{\boldsymbol{e}_n\}$ and $\{\boldsymbol{h}_n\}$ are related to each other. Actually let $\boldsymbol{e}_n$ belong to category II. Then $k_{e,n} \neq 0$ and one can define a function $\boldsymbol{h}_n$ through

$$\nabla \times \boldsymbol{e}_n = k_{e,n}\boldsymbol{h}_n \tag{13}$$

Therefore $\boldsymbol{h}_n$ belongs to category II. Furthermore

$$\nabla \times \nabla \times \boldsymbol{h} - k_{e,n}^2 \boldsymbol{h} = k_{e,n}^{-1} \nabla \times \left(\nabla \times \nabla \times \boldsymbol{e} - k_{e,n}^2 \boldsymbol{e}\right) = 0, \quad \boldsymbol{r} \in V$$

and

$$\boldsymbol{u}_n \times \nabla \times \boldsymbol{h}_n = k_{e,n}^{-1} \boldsymbol{u}_n \times \nabla \times \nabla \times \boldsymbol{e}_n = k_{e,n}^{-1} \boldsymbol{u}_n \times k_{e,n}^2 \boldsymbol{e}_n = 0, \quad \boldsymbol{r} \in S$$

Consider the integration of $\boldsymbol{u}_n \cdot \boldsymbol{h}_n$ over an arbitrary part of S, denoted ΔS

$$\int\limits_{\Delta S} \boldsymbol{u}_n \cdot \boldsymbol{h}_n ds = k_{e,n}^{-1} \int\limits_{\Delta S} \boldsymbol{u}_n \cdot \nabla \times \boldsymbol{e}_n ds = k_{e,n}^{-1} \int\limits_{\Delta \Gamma} \boldsymbol{e}_n \cdot \boldsymbol{u}_\Gamma d\Gamma$$

where $\Delta\Gamma$ is the closed contour around ΔS and $\boldsymbol{u}_\Gamma$ is the unit tangent vector along the contour. The right-hand side is zero. Thus $\boldsymbol{u}_n \cdot \boldsymbol{h}_n = 0$ since ΔS is arbitrary. Therefore $\boldsymbol{h}_n$ satisfies (9) and the corresponding eigenvalue is $k_{e,n}^2$. If $\boldsymbol{h}_m$ is another eigenfunction corresponding to $\boldsymbol{e}_m$ belonging to category II, then

$$\begin{aligned} \int\limits_V \boldsymbol{h}_m \cdot \boldsymbol{h}_n dv &= (k_{e,m} k_{e,n})^{-1} \int\limits_V \nabla \times \boldsymbol{e}_m \cdot \nabla \times \boldsymbol{e}_n dv \\ &= (k_{e,m} k_{e,n})^{-1} \int\limits_S \boldsymbol{u}_n \times \boldsymbol{e}_m \cdot \nabla \times \boldsymbol{e}_n ds + (k_{e,n}/k_{e,m}) \int\limits_V \boldsymbol{e}_m \cdot \boldsymbol{e}_n dv \\ &= \delta_{mn} \end{aligned}$$

Therefore the eigenfunctions $\boldsymbol{h}_n$ in category II can be derived from the eigenfunction $\boldsymbol{e}_n$ in category II. Conversely if $\boldsymbol{h}_n$ is in category II, one can define $\boldsymbol{e}_n$ by

$$\nabla \times \boldsymbol{h}_n = k_{h,n}\boldsymbol{e}_n \tag{14}$$

and a similar discussion shows that $\boldsymbol{e}_n$ is an eigenfunction of (4) with $k_{h,n}$ being the eigenvalue. So the completeness of the two sets are still guaranteed if the eigenfunctions belonging to category II in $\{\boldsymbol{e}_n\}$ and $\{\boldsymbol{h}_n\}$ are related through either (13) or (14). From now on, (13) and (14) will be assumed to hold, and $k_{e,n} = k_{h,n}$ will be denoted by k_n. Note that the complete set $\{\boldsymbol{e}_n\}$ is most appropriate for the expansion of electric field, and $\{\boldsymbol{h}_n\}$ for the expansion of the magnetic field.

3. TRANSIENT FIELDS IN A METAL CAVITY FILLED WITH LOSSY MEDIUM

If the cavity contains an impressed electric current source $\boldsymbol{J}$ and a magnetic current source $\boldsymbol{J}_m$, the fields excited by these sources satisfy

$$\begin{aligned} \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) &= \varepsilon \frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} + \sigma \boldsymbol{E} + \boldsymbol{J}(\boldsymbol{r}, t), \quad \boldsymbol{r} \in V \\ \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) &= -\mu \frac{\partial \boldsymbol{H}(\boldsymbol{r}, t)}{\partial t} - \boldsymbol{J}_m(\boldsymbol{r}, t), \qquad \boldsymbol{r} \in V \end{aligned} \tag{15}$$

and can be expanded in terms of the vector modal functions as

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}, t) &= \sum_n \boldsymbol{e}_n(\boldsymbol{r}) \int_V \boldsymbol{E}(\boldsymbol{r}, t) \cdot \boldsymbol{e}_n(\boldsymbol{r}) dv + \sum_v \boldsymbol{e}_v(\boldsymbol{r}) \int_V \boldsymbol{E}(\boldsymbol{r}, t) \cdot \boldsymbol{e}_v(\boldsymbol{r}) dv \\ &= \sum_n V_n(t) \boldsymbol{e}_n(\boldsymbol{r}) + \sum_v V_v(t) \boldsymbol{e}_v(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}, t) &= \sum_n \boldsymbol{h}_n(\boldsymbol{r}) \int_V \boldsymbol{H}(\boldsymbol{r}, t) \cdot \boldsymbol{h}_n(\boldsymbol{r}) dv + \sum_\tau \boldsymbol{h}_\tau(\boldsymbol{r}) \int_V \boldsymbol{H}(\boldsymbol{r}, t) \cdot \boldsymbol{h}_\tau(\boldsymbol{r}) dv \\ &= \sum_n I_n(t) \boldsymbol{h}_n(\boldsymbol{r}) + \sum_\tau I_\tau(t) \boldsymbol{h}_\tau(\boldsymbol{r}) \end{aligned} \tag{16}$$

$$\begin{aligned} \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) &= \sum_n \boldsymbol{h}_n(\boldsymbol{r}) \int_V \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) \cdot \boldsymbol{h}_n(\boldsymbol{r}) dv \\ &\quad + \sum_\tau \boldsymbol{h}_\tau(\boldsymbol{r}) \int_V \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) \cdot \boldsymbol{h}_\tau(\boldsymbol{r}) dv \\ \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) &= \sum_n \boldsymbol{e}_n(\boldsymbol{r}) \int_V \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) \cdot \boldsymbol{e}_n(\boldsymbol{r}) dv \\ &\quad + \sum_v \boldsymbol{e}_v(\boldsymbol{r}) \int_V \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) \cdot \boldsymbol{e}_v(\boldsymbol{r}) dv \end{aligned} \tag{17}$$

where the subscript n denotes the modes belonging to category II, and the Greek subscript v and τ for the modes belonging to category I or

III, and

$$\begin{aligned} V_{n(v)}(t) &= \int\limits_V \boldsymbol{E}(\boldsymbol{r},t)\cdot \boldsymbol{e}_{n(v)}(\boldsymbol{r})dv \\ I_{n(\tau)}(t) &= \int\limits_V \boldsymbol{H}(\boldsymbol{r},t)\cdot \boldsymbol{h}_{n(\tau)}(\boldsymbol{r})dv \end{aligned} \tag{18}$$

Making use of the following calculations

$$\begin{aligned} \int\limits_V \nabla\times\boldsymbol{E}\cdot\boldsymbol{h}_n dv &= \int\limits_V \boldsymbol{E}\cdot\nabla\times\boldsymbol{h}_n dv + \int\limits_S (\boldsymbol{E}\times\boldsymbol{h}_n)\cdot\boldsymbol{u}_n ds = k_n V_n \\ \int\limits_V \nabla\times\boldsymbol{E}\cdot\boldsymbol{h}_\tau dv &= \int\limits_V \boldsymbol{E}\cdot\nabla\times\boldsymbol{h}_\tau dv + \int\limits_S (\boldsymbol{E}\times\boldsymbol{h}_\tau)\cdot\boldsymbol{u}_n ds = 0 \\ \int\limits_V \nabla\times\boldsymbol{H}\cdot\boldsymbol{e}_n ds &= \int\limits_V \boldsymbol{H}\cdot\nabla\times\boldsymbol{e}_n dv + \int\limits_S (\boldsymbol{H}\times\boldsymbol{e}_n)\cdot\boldsymbol{u}_n ds = k_n I_n \\ \int\limits_V \nabla\times\boldsymbol{H}\cdot\boldsymbol{e}_v ds &= \int\limits_V \boldsymbol{H}\cdot\nabla\times\boldsymbol{e}_v dv + \int\limits_S (\boldsymbol{H}\times\boldsymbol{e}_v)\cdot\boldsymbol{u}_n ds = 0 \end{aligned}$$

(17) can be written as

$$\nabla\times\boldsymbol{E} = \sum_n k_n V_n \boldsymbol{h}_n, \qquad \nabla\times\boldsymbol{H} = \sum_n k_n I_n \boldsymbol{e}_n$$

Substituting the above expansions into (15) leads to

$$\begin{aligned} \sum_n k_n I_n \boldsymbol{e}_n &= \varepsilon\sum_n \boldsymbol{e}_n \frac{\partial V_n}{\partial t} + \varepsilon\sum_v \boldsymbol{e}_v \frac{\partial V_v}{\partial t} + \sigma\sum_n \boldsymbol{e}_n V_n + \sigma\sum_v \boldsymbol{e}_v V_v + \boldsymbol{J} \\ \sum_n k_n V_n \boldsymbol{h}_n &= -\mu\sum_n \boldsymbol{h}_n \frac{\partial I_n}{\partial t} - \mu\sum_\tau \boldsymbol{h}_\tau \frac{\partial I_\tau}{\partial t} - \boldsymbol{J}_m \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial V_n}{\partial t} + \frac{\sigma}{\varepsilon}V_n - \frac{k_n}{\varepsilon}I_n &= -\frac{1}{\varepsilon}\int\limits_V \boldsymbol{J}\cdot\boldsymbol{e}_n dv \\ \frac{\partial V_v}{\partial t} + \frac{\sigma}{\varepsilon}V_v &= -\frac{1}{\varepsilon}\int\limits_V \boldsymbol{J}\cdot\boldsymbol{e}_v dv \\ \frac{\partial I_n}{\partial t} + \frac{k_n}{\mu}V_n &= -\frac{1}{\mu}\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_n dv \\ \frac{\partial I_\tau}{\partial t} &= -\frac{1}{\mu}\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_\tau dv \end{aligned} \tag{19}$$

From the above equations one may obtain

$$\begin{aligned}\frac{\partial^2 I_n}{\partial t^2}+2\gamma\frac{\partial I_n}{\partial t}+\omega_n^2 I_n &= \omega_n S_n^I\\ \frac{\partial^2 V_n}{\partial t^2}+2\gamma\frac{\partial V_n}{\partial t}+\omega_n^2 V_n &= \omega_n S_n^V\end{aligned}\tag{20}$$

where $\omega_n = k_n c,\ \gamma = \sigma/2\varepsilon$ and

$$\begin{aligned}S_n^I &= c\int\limits_V \boldsymbol{J}\cdot\boldsymbol{e}_n dv - \frac{1}{k_n\eta}\frac{\partial}{\partial t}\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_n dv - \frac{\sigma c}{k_n}\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_n dv\\ S_n^V &= -\frac{\eta}{k_n}\frac{\partial}{\partial t}\int\limits_V \boldsymbol{J}\cdot\boldsymbol{e}_n dv - c\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_n dv\end{aligned}$$

To find I_n and V_n, one may use the retarded Green's function defined by

$$\begin{cases}\dfrac{\partial^2 G_n(t,t')}{\partial t^2}+2\gamma\dfrac{\partial G_n(t,t')}{\partial t}+\omega_n^2 G_n(t,t') = -\delta(t-t')\\ G_n(t,t')\Big|_{t<t'} = 0\end{cases}\tag{21}$$

It is easy to show that the solution of (21) is [8]

$$G_n(t,t') = -\frac{e^{-\gamma(t-t')}}{\sqrt{\omega_n^2-\gamma^2}}\sin\sqrt{\omega_n^2-\gamma^2}(t-t')H(t-t')\tag{22}$$

Therefore the general solution of I_n is given by

$$\begin{aligned}I_n(t) &= -\int\limits_{-\infty}^{\infty} G_n(t,t')\omega_n S_n^I(t')dt'\\ &\quad+\left[G_n(t,t')\frac{\partial I_n(t')}{\partial t'} - I_n(t')\frac{\partial G_n(t,t')}{\partial t'}\right]_{t'=-\infty}^{t'=\infty}\end{aligned}\tag{23}$$

If the source is turned on at $t = 0$, one may let $V_n(0^-) = I_n(0^-) = 0$ due to causality. Considering the third equation of (19), the term in the square bracket vanishes and the above equation reduces to

$$I_n(t) = -\int\limits_{-\infty}^{\infty} G_n(t,t')\omega_n S_n^I(t')dt'$$

$$= \frac{\omega_n}{\sqrt{\omega_n^2-\gamma^2}}\int\limits_{0^-}^{t} e^{-\gamma(t-t')}\sin\sqrt{\omega_n^2-\gamma^2}(t-t')$$
$$\times\left[c\int\limits_V \boldsymbol{J}\cdot\boldsymbol{e}_n dv-\frac{1}{k_n\eta}\frac{\partial}{\partial t'}\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_n dv\right]dt' \tag{24}$$

Similarly

$$\begin{aligned} V_n(t) &= -\int\limits_{-\infty}^{\infty} G_n(t,t')\omega_n S_n^V(t')dt' \\ &= \frac{\omega_n}{\sqrt{\omega_n^2-\gamma^2}}\int\limits_{0^-}^{t} e^{-\gamma(t-t')}\sin\omega_n(t-t') \\ &\quad\times\left[-\frac{\eta}{k_n}\frac{\partial}{\partial t'}\int\limits_V \boldsymbol{J}\cdot\boldsymbol{e}_n dv-c\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_n dv\right]dt' \end{aligned} \tag{25}$$

Note that

$$\begin{aligned} V_v(t) &= -\frac{1}{\varepsilon}e^{-\gamma t}\int\limits_{0^-}^{t} dt' e^{\gamma t'}\left[\int\limits_V \boldsymbol{J}\cdot\boldsymbol{e}_v dv\right] \\ I_\tau(t) &= -\frac{1}{\mu}\int\limits_{0^-}^{t} dt'\int\limits_V \boldsymbol{J}_m\cdot\boldsymbol{h}_\tau dv \end{aligned} \tag{26}$$

In order to validate the theory, let us consider a cavity excited by an infinitesimal electric and magnetic dipole at $\boldsymbol{r}_0$ [3, pp. 538]

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{P}_0 f(t)\delta(\boldsymbol{r}-\boldsymbol{r}_0) \\ \boldsymbol{M} &= \boldsymbol{M}_0 f(t)\delta(\boldsymbol{r}-\boldsymbol{r}_0) \end{aligned}$$

The fields in the cavity satisfy

$$\begin{aligned} \nabla\times\boldsymbol{H}(\boldsymbol{r},t) &= \varepsilon_0\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}+\boldsymbol{P}_0 f'(t)\delta(\boldsymbol{r}-\boldsymbol{r}_0), \qquad \boldsymbol{r}\in V \\ \nabla\times\boldsymbol{E}(\boldsymbol{r},t) &= -\mu_0\frac{\partial\boldsymbol{H}(\boldsymbol{r},t)}{\partial t}-\boldsymbol{M}_0\mu_0 f'(t)\delta(\boldsymbol{r}-\boldsymbol{r}_0), \quad \boldsymbol{r}\in V \end{aligned} \tag{27}$$

Comparing (27) with (15), the following identifications may be made

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{r},t) &= \boldsymbol{P}_0 f'(t)\delta(\boldsymbol{r}-\boldsymbol{r}_0), \qquad \boldsymbol{r}\in V \\ \boldsymbol{J}_m(\boldsymbol{r},t) &= \boldsymbol{M}_0\mu_0 f'(t)\delta(\boldsymbol{r}-\boldsymbol{r}_0), \quad \boldsymbol{r}\in V \end{aligned} \tag{28}$$

Introducing these into (24) gives

$$\begin{aligned} I_n(t) &= c\boldsymbol{P}_0 \cdot \boldsymbol{e}_n(\boldsymbol{r}_0) \int_{0^-}^{t} \sin \omega_n (t-t') f'(t') dt' \\ &\quad - \frac{c\mu_0}{\eta} \boldsymbol{M}_0 \cdot \boldsymbol{h}_n(\boldsymbol{r}_0) \int_{0^-}^{t} \cos \omega_n (t-t') f'(t') dt' \\ &= c\omega_n \boldsymbol{P}_0 \cdot \boldsymbol{e}_n(\boldsymbol{r}_0) \int_{0^-}^{t} \cos \omega_n (t-t') f(t) dt' \\ &\quad - \boldsymbol{M}_0 \cdot \boldsymbol{h}_n(\boldsymbol{r}_0) \left[f(t) - \omega_n \int_{0^-}^{t} \sin \omega_n (t-t') f(t') dt' \right] \end{aligned} \tag{29}$$

If the excitation waveform is sinusoidal which is turned on at $t = 0$, i.e., $f(t) = H(t) \sin \omega t$, then (29) may be written as

$$\begin{aligned} I_n(t) = & \frac{-\omega}{\omega^2 - \omega_n^2} [c\omega_n \boldsymbol{P}_0 \cdot \boldsymbol{e}_n(\boldsymbol{r}_0) \cos \omega t + \omega \boldsymbol{M}_0 \cdot \boldsymbol{h}_n(\boldsymbol{r}_0) \sin \omega t] \\ & + \frac{\omega}{\omega^2 - \omega_n^2} [c\omega_n \boldsymbol{P}_0 \cdot \boldsymbol{e}_n(\boldsymbol{r}_0) \cos \omega t + \omega \boldsymbol{M}_0 \cdot \boldsymbol{h}_n(\boldsymbol{r}_0) \sin \omega_n t] \end{aligned} \tag{30}$$

Similarly one can obtain the expressions of $V_n(t)$.

From the time-harmonic theory in which $f(t) = \sin \omega t$, one would obtain [3, 7.130b]

$$I_n(t) = \frac{-\omega}{\omega^2 - \omega_n^2} [c\omega_n \boldsymbol{P}_0 \cdot \boldsymbol{e}_n(\boldsymbol{r}_0) \cos \omega t + \omega \boldsymbol{M}_0 \cdot \boldsymbol{h}_n(\boldsymbol{r}_0) \sin \omega t] \tag{31}$$

for a lossless cavity. Thus the second term of the right-hand side of (30) does not occur in (31). Also note that (31) is sinusoidal but (30) is not. Furthermore (30) does not approach to (31) as $t \to \infty$, which contradicts our usual understanding. In fact, the response (30) based on the time-domain analysis is not sinusoidal if the frequency of the excitation sinusoidal waveform does not coincide with any resonant frequencies, whereas the response (31) based on the time-harmonic theory is always sinusoidal. Therefore the time-harmonic analysis for a metal cavity suffers a theoretical drawback that its solution does not correspond to any practical situation in which a source is always turned on at a finite instant of time. Apparently this theoretical drawback is due to the lack of causality in the time-harmonic theory.

It can also be seen from (31) that $I_n(t)$ becomes singular when ω approaches ω_n, which implies that the fields are infinite everywhere inside the cavity. This phenomenon is discussed in Bladel's book [5] and is compared to a lossless resonant LC network. In Collin's book [3], these singularities do not occur because of the introduction of losses in the metal cavity. However, the time-domain solution (30) has no singularities even for a lossless cavity. In fact (30) may be rewritten as

$$\begin{aligned} I_n(t) \;=\; & \left[\frac{\omega\omega_n t}{\omega+\omega_n} c\boldsymbol{P}_0\cdot\boldsymbol{e}_n(\boldsymbol{r}_0)\sin\frac{\omega+\omega_n}{2}t \right. \\ & \left. -\frac{\omega^2 t}{\omega+\omega_n}\boldsymbol{M}_0\cdot\boldsymbol{h}_n(\boldsymbol{r}_0)\cos\frac{\omega+\omega_n}{2}t\right]\frac{\sin\dfrac{\omega-\omega_n}{2}t}{\dfrac{\omega-\omega_n}{2}t} \end{aligned}$$

As ω approaches ω_n, the above becomes

$$I_n(t)=\frac{\omega_n t}{2}[c\boldsymbol{P}_0\cdot\boldsymbol{e}_n(\boldsymbol{r}_0)\sin\omega_n t-\boldsymbol{M}_0\cdot\boldsymbol{h}_n(\boldsymbol{r}_0)\cos\omega_n t] \qquad (32)$$

for a finite time t, and no singularities appear in (32). The above phenomenon can be explained by the uniqueness theorem of electromagnetic field. For a bounded region (such as a cavity), the time-harmonic Maxwell equations have a unique solution if and only if the region is filled with lossy medium, while the time-domain Maxwell equations always have a unique solution even if the medium is lossless [9, 10]. Therefore introducing losses in the metal cavity is required by the time-harmonic electromagnetic theory, which guarantees that the solution is unique and has no singularities.

Therefore the time-domain solution gives a more reasonable picture for the physical process inside a metal cavity. More examples in Section 5 will demonstrate this point.

4. TRANSIENT FIELDS IN A WAVEGUIDE CAVITY FILLED WITH LOSSY MEDIUM

The evaluation of modal functions in an arbitrary metal cavity is not an easy task. When the metal cavity consists of a section of a uniform metal waveguide, the analysis of the transient process in the metal cavity can be carried out by means of the time-domain theory of waveguide [6].

4.1. Field Expansions in a Waveguide Filled with Lossy Medium

Consider a waveguide cavity with a perfect electric wall of length L, as shown in Figure 2. The transient electromagnetic fields inside the

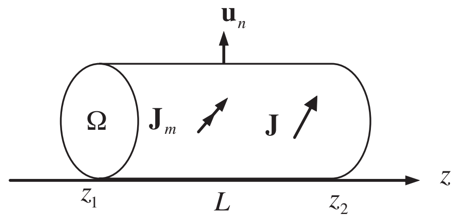

Figure 2. A metal cavity formed by a waveguide of cross-section Ω.

waveguide cavity with current source $\boldsymbol{J}$ and $\boldsymbol{J}_m$ can be expressed as [6]

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \sum_{n=1}^{\infty} v_n(z,t)\boldsymbol{e}_{tn}(\boldsymbol{\rho}) + \boldsymbol{u}_z \sum_{n=1}^{\infty} \frac{\nabla\cdot\boldsymbol{e}_{tn}(\boldsymbol{\rho})}{k_{cn}}\overline{e}_{zn} \\ \boldsymbol{H}(\boldsymbol{r},t) &= \sum_{n=1}^{\infty} i_n(z,t)\boldsymbol{u}_z\times\boldsymbol{e}_{tn}(\boldsymbol{\rho}) + \boldsymbol{u}_z\frac{1}{\sqrt{\Omega}}\int_\Omega \frac{\boldsymbol{u}_z\cdot\boldsymbol{H}}{\sqrt{\Omega}} \\ &\quad + \sum_{n=1}^{\infty}\frac{\nabla\times\boldsymbol{e}_{tn}(\boldsymbol{\rho})}{k_{cn}}\overline{h}_{zn}\end{aligned} \tag{33}$$

where $\boldsymbol{\rho} = (x,y)$ is the position vector in the waveguide cross-section Ω; $\boldsymbol{e}_{tn}$ are the transverse vector modal functions, and

$$\begin{aligned} v_n(z,t) &= \int_\Omega \boldsymbol{E}\cdot\boldsymbol{e}_{tn}d\Omega, \quad i_n(z,t) = \int_\Omega \boldsymbol{H}\cdot\boldsymbol{u}_z\times\boldsymbol{e}_{tn}d\Omega \\ \overline{h}_{zn}(z,t) &= \int_\Omega \boldsymbol{H}\cdot\left(\frac{\nabla\times\boldsymbol{e}_{tn}}{k_{cn}}\right)d\Omega, \quad \overline{e}_{zn}(z,t) = \int_\Omega \boldsymbol{u}_z\cdot\boldsymbol{E}\left(\frac{\nabla\cdot\boldsymbol{e}_{tn}}{k_{cn}}\right)d\Omega\end{aligned}$$

Similar to the time-domain theory of waveguide filled with lossless medium [6], the modal voltage and current for TEM mode satisfy the one-dimensional wave equation

$$\begin{aligned}&\frac{\partial^2 v_n^{TEM}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 v_n^{TEM}}{\partial t^2} - \sigma\frac{\eta}{c}\frac{\partial v_n^{TEM}}{\partial t} \\ &= \frac{\eta}{c}\frac{\partial}{\partial t}\int_\Omega \boldsymbol{J}\cdot\boldsymbol{e}_{tn}d\Omega - \frac{\partial}{\partial z}\int_\Omega \boldsymbol{J}_m\cdot\boldsymbol{u}_z\times\boldsymbol{e}_{tn}d\Omega \\ &\frac{\partial^2 i_n^{TEM}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 i_n^{TEM}}{\partial t^2} - \sigma\frac{\eta}{c}\frac{\partial i_n^{TEM}}{\partial t} \\ &= \sigma\int_\Omega \boldsymbol{J}_m\cdot\boldsymbol{u}_z\times\boldsymbol{e}_{tn}d\Omega - \frac{\partial}{\partial z}\int_\Omega \boldsymbol{J}\cdot\boldsymbol{e}_{tn}d\Omega + \frac{1}{c\eta}\frac{\partial}{\partial t}\int_\Omega \boldsymbol{J}_m\cdot\boldsymbol{u}_z\times\boldsymbol{e}_{tn}d\Omega\end{aligned} \tag{34}$$

Once v_v^{TEM} (or i_n^{TEM}) is determined, i_n^{TEM} (or v_n^{TEM}) can be determined by time integration of v_n^{TEM} (or i_n^{TEM}). The modal voltage v_n^{TE} satisfies the following hyperbolic equation

$$\begin{aligned}
&\frac{\partial^2 v_n^{TE}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 v_n^{TE}}{\partial t^2} - \sigma\frac{\eta}{c}\frac{\partial v_n^{TE}}{\partial t} - k_{cn}^2 v_n^{TE} \\
=\ &\frac{\eta}{c}\frac{\partial}{\partial t}\int_\Omega \boldsymbol{J}\cdot\boldsymbol{e}_{tn} d\Omega - \frac{\partial}{\partial z}\int_\Omega \boldsymbol{J}_m\cdot\boldsymbol{u}_z\times\boldsymbol{e}_{tn} d\Omega \\
&+k_{cn}\int_\Omega (\boldsymbol{u}_z\cdot\boldsymbol{J}_m)\left(\frac{\boldsymbol{u}_z\cdot\nabla\times\boldsymbol{e}_{tn}}{k_{cn}}\right) d\Omega \qquad (35)
\end{aligned}$$

When $\sigma = 0$, the above equation reduces to Klein-Gordon equation. (35) will be called modified Klein-Gordon equation. The modal current i_n^{TE} can be determined by a time integration of $\partial v_n^{TE}/\partial z$

$$\begin{aligned}
i_n^{TE}(z,t) \ =\ & -\frac{\eta}{c}\int_{-\infty}^{t} \frac{\partial v_n^{TE}(z,t')}{\partial z} dt' \\
& -\frac{\eta}{c}\int_{-\infty}^{t} \left\{\int_\Omega \boldsymbol{J}_m(\boldsymbol{r},t')\cdot[\boldsymbol{u}_z\times\boldsymbol{e}_{tn}(\boldsymbol{\rho})] d\Omega(\boldsymbol{\rho})\right\} dt' \quad (36)
\end{aligned}$$

The modal current i_n^{TM} also satisfies the modified Klein-Gordon equation.

$$\begin{aligned}
&\frac{\partial^2 i_n^{TM}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 i_n^{TM}}{\partial t^2} - \sigma\frac{\eta}{c}\frac{\partial i_n^{TM}}{\partial t} - k_{cn}^2 i_n^{TM} \\
=\ &\sigma\int_\Omega \boldsymbol{J}_m\cdot\boldsymbol{u}_z\times e_{tn} d\Omega - \frac{\partial}{\partial z}\int_\Omega \boldsymbol{J}\cdot\boldsymbol{e}_{tn} d\Omega \\
&+\frac{1}{c\eta}\frac{\partial}{\partial t}\int_\Omega \boldsymbol{J}_m\cdot\boldsymbol{u}_z\times\boldsymbol{e}_{tn} d\Omega - k_{cn}\int_\Omega \boldsymbol{u}_z\cdot\boldsymbol{J}\left(\frac{\nabla\cdot\boldsymbol{e}_{tn}}{k_{cn}}\right) d\Omega \quad (37)
\end{aligned}$$

The modal voltage v_n^{TM} can then be determined by a time integration of $\partial i_n^{TM}/\partial z$

$$v_n^{TM}(z,t) = -c\eta\int_{-\infty}^{t} \frac{\partial i_n^{TM}(z,t')}{\partial z} dt' - c\eta\int_{-\infty}^{t} \left[\int_\Omega \boldsymbol{J}(\boldsymbol{r},t')\cdot\boldsymbol{e}_{tn}(\boldsymbol{\rho}) d\Omega(\boldsymbol{\rho})\right] dt' \qquad (38)$$

4.2. Retarded Green's Function of Modified Klein-Gordon Equation

Since the tangential electric field on the electric conductor must be zero, the time-domain voltage satisfies the homogeneous Dirichlet boundary conditions

$$v_n(z,t)|_{z=z_1} = v_n(z,t)|_{z=z_2} = 0 \tag{39}$$

Making use of the following relation [6]

$$\begin{aligned} -\frac{\partial i_n(z,t)}{\partial z} + k_{cn}\overline{h}_{zn}(z,t) = \frac{1}{c\eta}\frac{\partial v_n(z,t)}{\partial t} + \sigma v_n(z,t) \\ + \int\limits_{\Omega} \boldsymbol{J}(\boldsymbol{\rho},z,t)\cdot \boldsymbol{e}_{tn}(\boldsymbol{\rho})d\Omega(\boldsymbol{\rho}) \end{aligned}$$

and considering the boundary condition that the normal component of the magnetic field on an electric conductor must be zero, the time-domain current must satisfy the homogeneous Neumann boundary conditions

$$\left.\frac{\partial i_n(z,t)}{\partial z}\right|_{z=z_1} = \left.\frac{\partial i_n(z,t)}{\partial z}\right|_{z=z_2} = 0 \tag{40}$$

In order to solve (34), (35) and (37) subject to the boundary conditions (39) and (40), one may introduce the following retarded Green's functions for the modified Klein-Gordon equation

$$\begin{cases} \left(\dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \sigma\dfrac{\eta}{c}\dfrac{\partial}{\partial t} - k_{cn}^2\right) G_n^v(z,t;z',t') = -\delta(z-z')\delta(t-t') \\ G_n^v(z,t;z',t')|_{t<t'} = 0 \\ G_n^v(z,t;z',t')|_{z=z_1} = G_n^v(z,t;z',t')|_{z=z_2} = 0 \end{cases} \tag{41}$$

and

$$\begin{cases} \left(\dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \sigma\dfrac{\eta}{c}\dfrac{\partial}{\partial t} - k_{cn}^2\right) G_n^i(z,t;z',t') = -\delta(z-z')\delta(t-t') \\ G_n^i(z,t;z',t')|_{t<t'} = 0 \\ \left.\dfrac{\partial G_n^i(z,t;z',t')}{\partial z}\right|_{z=z_1} = \left.\dfrac{\partial G_n^i(z,t;z',t')}{\partial z}\right|_{z=z_2} = 0 \end{cases} \tag{42}$$

for the modal voltage and modal current respectively. Note that the retarded Green's functions satisfy the causality condition. Taking the

Fourier transform with respect to time

$$\tilde{G}_n^{v,i}(z,\omega;z',t') = \int_{-\infty}^{\infty} G_n^{v,i}(z,t;z',t')e^{-j\omega t}dt$$

gives

$$\left(\frac{\partial^2}{\partial z^2} + \beta_n^2\right)\tilde{G}_n^{v,i}(z,\omega;z',t') = -e^{-j\omega t'}\delta(z-z') \tag{43}$$

where $\beta_n^2 = (\omega/c)^2 - k_{cn}^2 - j\sigma\eta\omega/c$. The above equations can be solved by the method of eigenfunctions, i.e.,

$$\begin{aligned}\tilde{G}_n^{v}(z,\omega;z',t') &= \sum_{m=1}^{\infty} g_m^v \sqrt{\frac{2}{L}}\sin\frac{m\pi}{L}(z-z_1)\\ \tilde{G}_n^{i}(z,\omega;z',t') &= \sum_{m=1}^{\infty} g_m^i \sqrt{\frac{\varepsilon_m}{L}}\cos\frac{m\pi}{L}(z-z_1)\end{aligned}$$

where $L = z_2 - z_1$, and $\varepsilon_m = 1(m=0),\ \varepsilon_m = 2(m \neq 0)$. Substituting these into (43) leads to

$$\begin{aligned}g_m^v &= \frac{-1}{\beta_n^2 - (m\pi/L)^2}\sqrt{\frac{2}{L}}\sin\frac{m\pi}{L}(z'-z_1)e^{-j\omega t'}\\ g_m^i &= \frac{-1}{\beta_n^2 - (m\pi/L)^2}\sqrt{\frac{\varepsilon_m}{L}}\cos\frac{m\pi}{L}(z'-z_1)e^{-j\omega t'}\end{aligned}$$

Thus

$$\tilde{G}_n^{v}(z,\omega;z',t') = \sum_{m=1}^{\infty}\frac{-1}{\beta_n^2-(m\pi/L)^2}\frac{2}{L}\sin\frac{m\pi}{L}(z-z_1)\sin\frac{m\pi}{L}(z'-z_1)e^{-j\omega t'}$$

$$\tilde{G}_n^{i}(z,\omega;z',t') = \sum_{m=0}^{\infty}\frac{-1}{\beta_n^2-(m\pi/L)^2}\frac{\varepsilon_m}{L}\cos\frac{m\pi}{L}(z-z_1)\cos\frac{m\pi}{L}(z'-z_1)e^{-j\omega t'}$$

Taking the inverse Fourier transform

$$G_n^{v,i}(z,t;z',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}_n^{v,i}(z,\omega;z',t')e^{j\omega t}d\omega$$

one may obtain

$$G_n^{v}(z,t;z',t') = -\sum_{m=1}^{\infty}\frac{c^2}{\pi L}\sin\frac{m\pi}{L}(z-z_1)\sin\frac{m\pi}{L}(z'-z_1)$$

$$\times \int_{-\infty}^{\infty} \frac{e^{j\omega(t-t')}d\omega}{\omega^2 - (ck_{cn})^2 - (m\pi c/L)^2 - j\omega\sigma/\varepsilon}$$

$$G_n^i(z,t;z',t') = -\sum_{m=0}^{\infty} \frac{\varepsilon_m c^2}{2\pi L} \cos\frac{m\pi}{L}(z-z_1)\cos\frac{m\pi}{L}(z'-z_1) \times \int_{-\infty}^{\infty} \frac{e^{j\omega(t-t')}d\omega}{\omega^2 - (ck_{cn})^2 - (m\pi c/L)^2 - j\omega\sigma/\varepsilon}$$

The integral in the summation can be evaluated by the residue theorem [6]. The results are

$$G_n^v(z,t;z',t') = \sum_{m=1}^{\infty} \frac{2c}{L} \sin\frac{m\pi}{L}(z-z_1)\sin\frac{m\pi}{L}(z'-z_1) \times e^{-\gamma(t-t')} \frac{\sin\left[c(t-t')\sqrt{k_{cn}^2 + (m\pi/L)^2 - \gamma^2}\right]}{\sqrt{k_{cn}^2 + (m\pi/L)^2 - \gamma^2}} H(t-t') \tag{44}$$

$$G_n^i(z,t;z',t') = \sum_{m=0}^{\infty} \frac{\varepsilon_m c}{L} \cos\frac{m\pi}{L}(z-z_1)\cos\frac{m\pi}{L}(z'-z_1) \times e^{-\gamma(t-t')} \frac{\sin\left[c(t-t')\sqrt{k_{cn}^2 + (m\pi/L)^2 - \gamma^2}\right]}{\sqrt{k_{cn}^2 + (m\pi/L)^2 - \gamma^2}} H(t-t') \tag{45}$$

where $\gamma = \sigma/2\varepsilon$. If one of the ends of the waveguide cavity extends to infinity, say, $z_2 \to \infty$, the discrete values $m\pi/L$ become a continuum. In this case, (44) and (45) can be rewritten as

$$G_n^v(z,t;z',t')\Big|_{z_2\to\infty} = -\frac{c}{\pi}e^{-\gamma(t-t')} \times \int_0^{\infty} [\cos k(z+z'-2z_1) - \cos k(z-z')] \frac{\sin\left[c(t-t')\sqrt{k_{cn}^2 + k^2 - \gamma^2}\right]}{\sqrt{k_{cn}^2 + k^2 - \gamma^2}} dk$$

$$G_n^i(z,t;z',t')\Big|_{z_2\to\infty} = \frac{c}{\pi}e^{-\gamma(t-t')} \times \int_0^{\infty} [\cos k(z+z'-2z_1) + \cos k(z-z')] \frac{\sin\left[c(t-t')\sqrt{k_{cn}^2 + k^2 - \gamma^2}\right]}{\sqrt{k_{cn}^2 + k^2 - \gamma^2}} dk$$

where $k = \omega/c$. These integrations may be carried out by using

$$\int_0^\infty \left(\frac{\sin q\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)\cos bx dx = \frac{\pi}{2}J_0\left(a\sqrt{q^2-b^2}\right)H(q-b)$$
$$a>0,\ q>0,\ b>0$$

and the results are

$$\begin{aligned}
&e^{\gamma(t-t')}G_n^v(z,t;z',t')|_{z_2\to\infty} = \\
&-\frac{c}{2}J_0\left[\left(k_{cn}^2-\gamma^2\right)^{1/2}\sqrt{c^2(t-t')^2-|z+z'-2z_1|^2}\right] \\
&\times H[c(t-t')-|z+z'-2z_1|] \\
&+\frac{c}{2}J_0\left[\left(k_{cn}^2-\gamma^2\right)^{1/2}\sqrt{c^2(t-t')^2-|z-z'|^2}\right]H[c(t-t')-|z-z'|]
\end{aligned} \tag{46}$$

$$\begin{aligned}
&e^{\gamma(t-t')}G_n^i(z,t;z',t')|_{z_2\to\infty} = \\
&\frac{c}{2}J_0\left[\left(k_{cn}^2-\gamma^2\right)^{1/2}\sqrt{c^2(t-t')^2-|z+z'-2z_1|^2}\right] \\
&\times H[c(t-t')-|z+z'-2z_1|] \\
&+\frac{c}{2}J_0\left[\left(k_{cn}^2-\gamma^2\right)^{1/2}\sqrt{c^2(t-t')^2-|z-z'|^2}\right]H[c(t-t')-|z-z'|]
\end{aligned} \tag{47}$$

4.3. Solution of Inhomogeneous Klein-Gordon Equation

The retarded Green's functions can be used to solve the modified Klein-Gordon equation. Consider the inhomogeneous Klein-Gordon equations

$$\begin{aligned}
\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\sigma\frac{\eta}{c}\frac{\partial}{\partial t}-k_{cn}^2\right)v_n(z,t) &= f(z,t),\ z_1<z<z_2 \\
\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\sigma\frac{\eta}{c}\frac{\partial}{\partial t}-k_{cn}^2\right)i_n(z,t) &= g(z,t),\ z_1<z<z_2
\end{aligned}$$

with the boundary conditions (39) and (40). It is easy to show that the solutions of the above equations are

$$\begin{aligned} v_n(z,t) &= -\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} f(z',t') G_n^v(z,t;z',t')dt', \;\; z \in (z_1, z_2) \\ i_n(z,t) &= -\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} g(z',t') G_n^i(z,t;z',t')dt', \;\; z \in (z_1, z_2) \end{aligned} \tag{48}$$

Thus the solutions of (34), (35) and (37) can be obtained from (48) as

$$\begin{aligned} v_n^{TE}(z,t) = &-\frac{\eta}{c}\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t'} \int_{\Omega} \boldsymbol{J}(\boldsymbol{\rho}',z',t') \cdot \boldsymbol{e}_{tn}(\boldsymbol{\rho}') d\Omega(\boldsymbol{\rho}')\right] \\ &\times G_n^v(z,t;z',t')dt' \\ &-\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} \left[\int_{\Omega} \boldsymbol{J}_m(\boldsymbol{\rho}',z',t') \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn}(\boldsymbol{\rho}') d\Omega(\boldsymbol{\rho}')\right] \frac{\partial G_n^v(z,t;z',t')}{\partial z'} dt' \\ &-k_{cn}\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} \left\{\int_{\Omega} \boldsymbol{u}_z \cdot \boldsymbol{J}_m(\boldsymbol{\rho}',z',t') \left[\frac{\boldsymbol{u}_z \cdot \nabla \times \boldsymbol{e}_{tn}(\boldsymbol{\rho}')}{k_{cn}}\right] d\Omega(\boldsymbol{\rho}')\right\} \\ &\times G_n^v(z,t;z',t')dt' \end{aligned} \tag{49}$$

$$\begin{aligned} i_n^{TM}(z,t) = &-\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} \left[\int_{\Omega} \boldsymbol{J}(\boldsymbol{\rho}',z',t') \cdot \boldsymbol{e}_{tn}(\boldsymbol{\rho}') d\Omega(\boldsymbol{\rho}')\right] \frac{\partial G_n^i(z,t;z',t')}{\partial z'} dt' \\ &-\frac{1}{c\eta}\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t'} \int_{\Omega} \boldsymbol{J}_m(\boldsymbol{\rho}',z',t') \cdot \boldsymbol{u}_z \times \boldsymbol{e}_{tn}(\boldsymbol{\rho}') d\Omega(\boldsymbol{\rho}')\right] G_n^i(z,t;z',t')dt' \\ &+k_{cn}\int_{z_1}^{z_2} dz' \int_{-\infty}^{\infty} \left\{\int_{\Omega} \boldsymbol{u}_z \cdot \boldsymbol{J}(\boldsymbol{\rho}',z',t') \left[\frac{\nabla \cdot \boldsymbol{e}_{tn}(\boldsymbol{\rho}')}{k_{cn}}\right] d\Omega(\boldsymbol{\rho}')\right\} G_n^i(z,t;z',t')dt' \end{aligned} \tag{50}$$

In deriving the above expressions it has been assumed that all sources are confined in (z_1, z_2). It should be notified that if the magnetic current $\boldsymbol{J}_m$ approaches to z_1 or z_2 so that it is tightly pressed on the electric wall $z = z_1$ or $z = z_2$, the time-domain voltage and current will not satisfy the homogeneous boundary conditions (39) and (40) at $z = z_1$ or $z = z_2$.

5. APPLICATIONS

The time-domain theory developed above may be used to study the transient process inside a cavity resonator. It will be shown that the time-domain response of the cavity to an arbitrary excitation waveform turned on at a finite instant of time will be severely distorted. For example, the response of a cavity to a sinusoidal excitation turned on at a finite instant of time is not sinusoidal in general, which is totally different from the prediction of time-harmonic theory. To obtain a sinusoidal oscillation in the cavity, the frequency of the excitation sinusoidal wave must coincide with one of the resonant frequencies.

5.1. A Shorted Rectangular Waveguide

Let us investigate the transient process in a shorted rectangular waveguide excited by a line current extending across the waveguide centered at $x = x_0 = a/2,\ z = z_0$

$$\boldsymbol{J}(\boldsymbol{r}, t) = \boldsymbol{u}_y \delta(x - x_0)\delta(z - z_0) f(t) \tag{51}$$

as shown in Figure 3. By the symmetry of the structure and excitation, only TE_{n0} mode will be excited, which are

$$\boldsymbol{e}_{tn}(x, y) = \boldsymbol{e}_{n0}^{TE}(x, y) = -\boldsymbol{u}_y \left(\frac{2}{ab}\right)^{1/2} \sin\frac{n\pi x}{a},\ n = 1, 2, 3\cdots \tag{52}$$

with $k_{cn} = n\pi/a$. It follows from (46) and (49) that

$$\begin{aligned} v_n^{TE}(z, t) &= \frac{b\eta}{2}\left(\frac{2}{ab}\right)^{1/2} \sin\frac{n\pi}{a} x_0 \\ &\quad \times \left\{ \int_{-\infty}^{t-|z-z_0|/c} \frac{df(t')}{dt'} J_0\left[k_{cn} c \sqrt{(t-t')^2 - |z-z_0|^2/c^2}\right] dt' \right. \end{aligned}$$

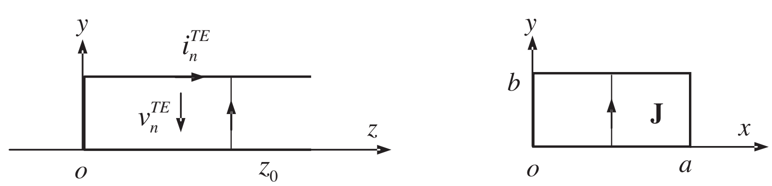

Figure 3. A shorted rectangular waveguide excited by a centered current source.

$$- \int_{-\infty}^{t-|z+z_0|/c} \frac{df(t')}{dt'} J_0 \left[k_{cn} c \sqrt{(t-t')^2 - |z+z_0|^2/c^2} \right] dt' \Bigg\}$$

Assuming that $f(t) = H(t) \sin \omega t$, the time-domain voltage may be written as

$$\begin{aligned} v_n^{TE}(z,t) &= \frac{b\eta}{2} \left(\frac{2}{ab} \right)^{1/2} ka \sin \frac{n\pi}{a} x_0 \\ &\times \Bigg\{ \int_{|z-z_0|/a}^{ct/a} \cos ka(ct/a - u) J_0 \left[k_{cn} a \sqrt{u^2 - |z-z_0|^2/a^2} \right] du \\ &- \int_{|z+z_0|/a}^{ct/a} \cos ka(ct/a - u) J_0 \left[k_{cn} a \sqrt{u^2 - |z+z_0|^2/a^2} \right] du \Bigg\} \end{aligned}$$

The time-domain response $v_n^{TE}(z,t)$ may be divided into the sum of a steady-state part and a transient part

$$v_n^{TE}(z,t) = v_n^{TE}(z,t)\Big|_{steady} + v_n^{TE}(z,t)\Big|_{transient}$$

where

$$\begin{aligned} v_n^{TE}(z,t)\Big|_{steady} &= \frac{b\eta}{2} \left(\frac{2}{ab} \right)^{1/2} ka \sin \frac{n\pi}{a} x_0 \\ &\times \Bigg\{ \int_{|z-z_0|/a}^{\infty} \cos ka(ct/a - u) J_0 \left[k_{cn} a \sqrt{u^2 - |z-z_0|^2/a^2} \right] du \\ &- \int_{|z+z_0|/a}^{\infty} \cos ka(ct/a - u) J_0 \left[k_{cn} a \sqrt{u^2 - |z+z_0|^2/a^2} \right] du \Bigg\} \\ v_n^{TE}(z,t)\Big|_{transient} &= -\frac{b\eta}{2} \left(\frac{2}{ab} \right)^{1/2} ka \sin \frac{n\pi}{a} x_0 \\ &\times \Bigg\{ \int_{ct/a}^{\infty} \cos ka(ct/a - u) J_0 \left[k_{cn} a \sqrt{u^2 - |z-z_0|^2/a^2} \right] du \\ &- \int_{ct/a}^{\infty} \cos ka(ct/a - u) J_0 \left[k_{cn} a \sqrt{u^2 - |z+z_0|^2/a^2} \right] du \Bigg\} \end{aligned}$$

The transient part approaches to zero as $t \to \infty$. The integrals in the steady part can be carried out by use of the following calculations [11]

$$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\sin cx dx = \begin{cases} 0, & 0<c<b \\ \cos\left(a\sqrt{c^2-b^2}\right)/\sqrt{c^2-b^2}, & 0<b<c \end{cases}$$

$$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos cx dx = \begin{cases} \exp\left(-a\sqrt{b^2-c^2}\right)/\sqrt{b^2-c^2}, & 0<c<b \\ -\sin\left(a\sqrt{c^2-b^2}\right)/\sqrt{c^2-b^2}, & 0<b<c \end{cases}$$

Thus

$$\begin{aligned} v_n^{TE}(z,t)\Big|_{steady} &= \frac{b\eta}{2}\left(\frac{2}{ab}\right)^{1/2} ka \sin\frac{n\pi}{a}\frac{1}{\sqrt{|(ka)^2-(k_{cn}a)^2|}} \\ &\cdot\begin{cases} \sin\left(ka\dfrac{ct}{a}-\dfrac{|z-z_0|}{a}\sqrt{(ka)^2-(k_{cn}a)^2}\right) \\ -\sin\left(ka\dfrac{ct}{a}-\dfrac{|z+z_0|}{a}\sqrt{(ka)^2-(k_{cn}a)^2}\right), & k>k_{cn} \\ \cos\left(ka\dfrac{ct}{a}\right)\exp\left[-\dfrac{|z-z_0|}{a}\sqrt{(k_{cn}a)^2-(ka)^2}\right] \\ -\cos\left(ka\dfrac{ct}{a}\right)\exp\left[-\dfrac{|z+z_0|}{a}\sqrt{(k_{cn}a)^2-(ka)^2}\right] & k<k_{cn} \end{cases} \end{aligned}$$

In the region $0 < z < z_0$, the above equation may be rewritten as

$$\begin{aligned} v_n^{TE}(z,t)\Big|_{steady} &= \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{1/2}\sin\frac{n\pi}{2}\frac{\eta k}{\beta_n} \\ &\cdot\begin{cases} 2\sin(\beta_n z)\cos(\omega t-\beta_n z_0), & k>k_{cn} \\ \cos(\omega t)\exp[\beta_n(z-z_0)]- \\ \cos(\omega t)\exp[-\beta_n(z+z_0)], & k<k_{cn} \end{cases} \end{aligned}$$

where $\beta_n = (|k^2-k_{cn}^2|)^{1/2}$. Therefore the time-domain voltage for the TE_{n0} mode in the shorted waveguide is a standing wave if the operating frequency is higher than cut-off frequency of the TE_{n0} mode, which is a well-known result in time-harmonic theory.

The time-domain current can be determined by (36)

$$\begin{aligned} i_n^{TE}(z,t) &= \left(\frac{2b}{a}\right)^{1/2}\sin\frac{n\pi}{2} \\ &\times\left[-\frac{1}{2}\sin\omega(t-|z+z_0|/c)-\frac{1}{2}\sin\omega(t-|z-z_0|/c)\right] \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{2b}{a}\right)^{1/2}\sin\frac{n\pi}{2}\\
&\times\left\{\frac{k_{cn}(z+z_0)}{2}\int\limits_0^{t-|z+z_0|/c}\frac{J_1\left[k_{cn}c\sqrt{(t-t')^2-|z+z_0|^2/c^2}\right]}{\sqrt{(t-t')^2-|z+z_0|^2/c^2}}\sin\omega t'dt'\right.\\
&\left.-\frac{k_{cn}(z-z_0)}{2}\int\limits_0^{t-|z-z_0|/c}\frac{J_1\left[k_{cn}c\sqrt{(t-t')^2-|z-z_0|^2/c^2}\right]}{\sqrt{(t-t')^2-|z-z_0|^2/c^2}}\sin\omega t'dt'\right\}
\end{aligned}
$$

The steady state part of $i_n^{TE}(z,t)$ is given by

$$
\begin{aligned}
i_n^{TE}(z,t)\Big|_{steady}&=\left(\frac{2b}{a}\right)^{1/2}\sin\frac{n\pi}{2}\\
&\times\left[-\frac{1}{2}\sin\omega(t-|z+z_0|/c)-\frac{1}{2}\sin\omega(t-|z-z_0|/c)\right]\\
&+\left(\frac{2b}{a}\right)^{1/2}\sin\frac{n\pi}{2}\\
&\times\left\{\frac{k_{cn}(z+z_0)}{2}\int\limits_{|z+z_0|/c}^{\infty}\frac{J_1\left[k_{cn}c\sqrt{u^2-|z+z_0|^2/c^2}\right]}{\sqrt{u^2-|z+z_0|^2/c^2}}\sin\omega(t-u)du\right.\\
&\left.-\frac{k_{cn}(z-z_0)}{2}\int\limits_{|z-z_0|/c}^{\infty}\frac{J_1\left[k_{cn}c\sqrt{u^2-|z-z_0|^2/c^2}\right]}{\sqrt{u^2-|z-z_0|^2/c^2}}\sin\omega(t-u)du\right\}
\end{aligned}
$$

Assuming that $k > k_{cn}$ and making use of the following calculations

$$
\begin{aligned}
&\int\limits_a^{\infty}\frac{\sin cx}{\sqrt{x^2-a^2}}J_v\left(b\sqrt{x^2-a^2}\right)dx=\\
&\qquad\frac{\pi}{2}J_{v/2}\left[\frac{a}{2}\left(c-\sqrt{c^2-b^2}\right)\right]J_{-v/2}\left[\frac{a}{2}\left(c+\sqrt{c^2-b^2}\right)\right]\\
&\int\limits_a^{\infty}\frac{\cos cx}{\sqrt{x^2-a^2}}J_v\left(b\sqrt{x^2-a^2}\right)dx=\\
&\qquad-\frac{\pi}{2}J_{v/2}\left[\frac{a}{2}\left(c-\sqrt{c^2-b^2}\right)\right]N_{-v/2}\left[\frac{a}{2}\left(c+\sqrt{c^2-b^2}\right)\right]\\
&\qquad(a>0,\ 0<b<c)
\end{aligned}
$$

one may obtain

$$i_n^{TE}(z,t)\Big|_{steady} = -\frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{1/2} 2\sin\frac{n\pi}{2}\cos(\beta_n z)\sin(\omega t - \beta_n z_0)$$

Let $V_n^{TE}(z)$ and $I_n^{TE}(z)$ be the phasors of $v_n^{TE}(z,t)|_{steady}$ and $i_n^{TE}(z,t)|_{steady}$ respectively, then

$$\begin{aligned} V_n^{TE}(z) &= \frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{1/2}\frac{\eta k}{\beta_n}2\sin\frac{n\pi}{2}\sin(\beta_n z)e^{-j\beta_n z_0}, \quad k > k_{cn} \\ I_n^{TE}(z) &= j\frac{1}{\sqrt{2}}\left(\frac{b}{a}\right)^{1/2} 2\sin\frac{n\pi}{2}\cos(\beta_n z)e^{-j\beta_n z_0}, \quad k > k_{cn} \end{aligned}$$

Since the current is assumed to be in positive z-direction, the impedance at $z \in (0, z_0)$ is thus given by

$$Z_n(z) = \frac{V_n^{TE}(z)}{-I_n^{TE}(z)} = j\frac{\eta k}{\beta_n}\tan(\beta_n z), \quad k > k_{cn}$$

which is a well-known result and validates the time-domain theory.

5.2. A Rectangular Waveguide Cavity

Let the shorted waveguide shown in Figure 3 be closed by a perfect conducting wall at $z = L$ with $L > z_0$ (Figure 4) and the excitation source be given by (51). By symmetry, only TE_{n0} mode will be excited. It follows from (44) and (49) that

$$\begin{aligned} v_n^{TE}(z,t) = \frac{2\eta}{L}\left(\frac{2b}{a}\right)^{1/2}\sin\frac{n\pi x_0}{a}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi}{L}z\sin\frac{m\pi}{L}z_0}{\sqrt{(n\pi/a)^2+(m\pi/L)^2}} \\ \times\int_{-\infty}^{\infty}\frac{df(t')}{dt'}\sin\left[c(t-t')\sqrt{(n\pi/a)^2+(m\pi/L)^2}\right]H(t-t')dt' \end{aligned} \tag{53}$$

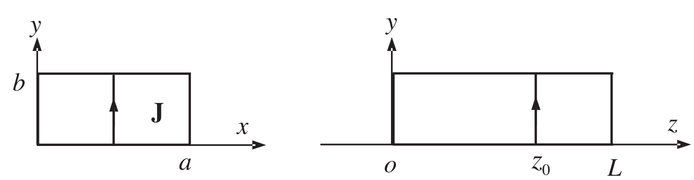

Figure 4. A rectangular waveguide cavity excited by a current source.

Again if it is assumed that $f(t) = H(t)\sin\omega t$, the above expression may be written as

$$\begin{aligned} v_n^{TE}(z,t) &= \frac{2\eta\omega}{L}\left(\frac{2b}{a}\right)^{1/2}\sin\frac{n\pi x_0}{a}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi}{L}z\sin\frac{m\pi}{L}z_0}{\sqrt{(n\pi/a)^2+(m\pi/L)^2}} \\ &\quad\times\int_{-\infty}^{t}\sin\left[c(t-t')\sqrt{(n\pi/a)^2+(m\pi/L)^2}\right]\cos\omega t' dt' \\ &= -\frac{2\eta k}{L}\left(\frac{2b}{a}\right)^{1/2}\sin\frac{n\pi x_0}{a}\sum_{m=1}^{\infty}\sin\frac{m\pi}{L}z\sin\frac{m\pi}{L}z_0 \\ &\quad\times\frac{\cos kct-\cos\left[ct\sqrt{(n\pi/a)^2+(m\pi/L)^2}\right]}{k^2-(n\pi/a)^2+(m\pi/L)^2} \end{aligned} \tag{54}$$

where $k = \omega/c$. The electromagnetic field is given by (33)

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},t) &= \boldsymbol{u}_y E_y = \sum_{n=1}^{\infty} v_n^{TE}(z,t)\boldsymbol{e}_{n0}^{TE}(x,y) \\ &= \boldsymbol{u}_y\frac{4\eta k}{La}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{n\pi x_0}{a}\sin\frac{m\pi}{L}z\sin\frac{m\pi}{L}z_0 \\ &\quad\times\frac{\cos kct-\cos\left[ct\sqrt{(n\pi/a)^2+(m\pi/L)^2}\right]}{k^2-(n\pi/a)^2+(m\pi/L)^2} \end{aligned} \tag{55}$$

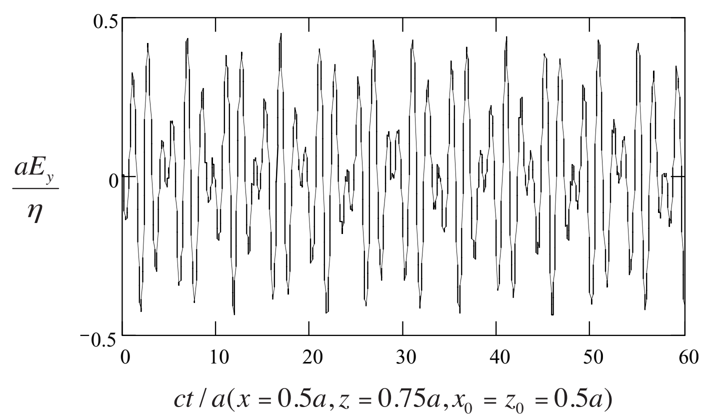

Figure 5. Electric field excited by a sinusoidal wave $f(t) = H(t)\sin\omega t$ when $k = \omega/c \neq \sqrt{(n\pi/a)^2 + (m\pi/L)^2}$ $(ka = \pi)$.

where $\boldsymbol{e}_{n0}^{TE}(x, y)$ are given by (52). As can be seen from (54) and (55), one cannot separate the response of a closed cavity into a transient part and a steady-state part. Figure 5 shows the normalized electric field $a\boldsymbol{E}(\boldsymbol{r}, t)/\eta$ at $x = 0.5a,\ z = 0.75a$, excited by a source defined by (51) with sinusoidal waveform $f(t) = H(t)\sin\omega t$ with $a = b = L$. It is assumed that k is below any resonant wavenumbers $\sqrt{(n\pi/a)^2 + (m\pi/L)^2}$ $(m, n \geq 1)$ with $k = \pi$. It can be seen from the plot that the electric field does not approach to a pure sinusoidal wave as $t \to \infty$. It should be noted that (54), (55) are finite as k approaches to any of the resonant wavenumbers $\sqrt{(n\pi/a)^2 + (m\pi/L)^2}$. For example, when k approaches to $\sqrt{(\pi/a)^2 + (\pi/L)^2}$, the singular term for $(n, m) = (1, 1)$ in (55) becomes

$$\frac{\cos kct - \cos\left[ct\sqrt{(\pi/a)^2 + (\pi/L)^2}\right]}{k^2 - (\pi/a)^2 + (\pi/L)^2} = -\frac{ct\sin\left(ct\sqrt{(\pi/a)^2 + (\pi/L)^2}\right)}{2\sqrt{(\pi/a)^2 + (\pi/L)^2}} \tag{56}$$

which is a finite number for a finite t.

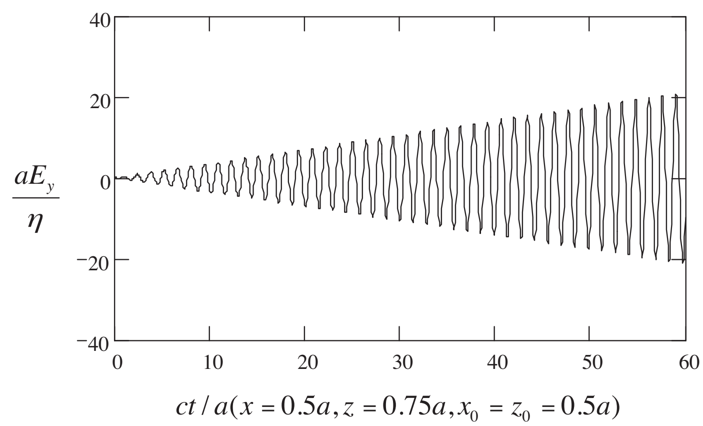

Figure 6. Electric field exited by a sinusoidal wave $f(t) = H(t)\sin\omega t$ when $k = \omega/c = \sqrt{(\pi/a)^2 + (\pi/L)^2}$.

Figure 6 shows the normalized electric field $a\boldsymbol{E}(\boldsymbol{r}, t)/\eta$ at $x = 0.5a,\ z = 0.75a$, excited by a sinusoidal wave $f(t) = H(t)\sin\omega t$ when $k = \sqrt{(\pi/a)^2 + (\pi/L)^2}$ with $a = b = L$. In this case a sinusoidal wave will be gradually built up as $t \to \infty$. Therefore the response of a metal cavity is a sinusoidal wave if and only if the frequency of the exciting sinusoidal wave coincides with one of the resonant frequencies of the metal cavity.

If the excitation waveform is a unit step function, i.e., $f(t) = H(t)$, (53) becomes

$$v_n^{TE}(z,t) = \frac{2\eta}{L}\left(\frac{2b}{a}\right)^{1/2} \sin\frac{n\pi x_0}{a} \sum_{m=1}^{\infty} \sin\frac{m\pi}{L}z \sin\frac{m\pi}{L}z_0 \times \frac{\sin\left[ct\sqrt{(n\pi/a)^2+(m\pi/L)^2}\right]}{\sqrt{(n\pi/a)^2+(m\pi/L)^2}} \tag{57}$$

and the electric field is

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{u}_y E_y = -\boldsymbol{u}_y \frac{4\eta}{La} \times \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\frac{n\pi x}{a} \sin\frac{n\pi x_0}{a} \sin\frac{m\pi}{L}z \sin\frac{m\pi}{L}z_0 \times \frac{\sin\left[ct\sqrt{(n\pi/a)^2+(m\pi/L)^2}\right]}{\sqrt{(n\pi/a)^2+(m\pi/L)^2}} \tag{58}$$

Figure 7 shows the normalized electric field $a\boldsymbol{E}(\boldsymbol{r},t)/\eta$ at $x = 0.5a$, $z = 0.75a$, excited by the unit step waveform. Note that the response of the metal cavity is not a unit step function.

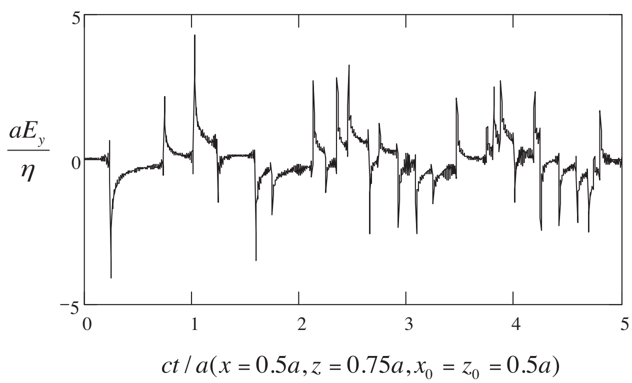

Figure 7. Electric field excited by a unit step waveform $f(t) = H(t)$.

5.3. A Coaxial Waveguide Cavity

A coaxial waveguide cavity of length L consisting of an inner conductor of radius a and an outer conductor of radius b is shown in Figure 8. Let the coaxial waveguide be excited by a magnetic ring current located at

$z = z_0$

$$\boldsymbol{J}_m(\boldsymbol{r}, t) = \boldsymbol{u}_\theta f(t)\delta(z - z_0)\delta(\rho - \rho_0),\ a < \rho_0 < b,\ 0 < z_0 < L$$

where (ρ, φ, z) are the polar coordinates and $\boldsymbol{u}_\theta$ is the unit vector in θ direction. According to the symmetry, only TEM mode and those TM_{0q} modes that are independent of φ will be excited. The orthonormal vector mode functions for these modes are given by [12]

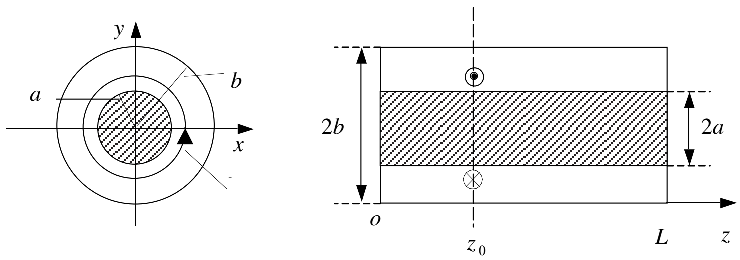

Figure 8. Cross-section of a coaxial waveguide.

$$\begin{aligned} k_{c1} &= 0,\ \boldsymbol{e}_{t1}(\rho, \varphi) = \boldsymbol{u}_\rho e_1(\rho),\ e_1(\rho) = \frac{1}{\rho\sqrt{2\pi \ln c_1}} \\ k_{cn} &= \frac{\chi_n}{a},\ \boldsymbol{e}_{tn}(\rho, \varphi) = \boldsymbol{u}_\rho e_n(\rho) \\ e_n(\rho) &= \frac{\sqrt{\pi}}{2}\frac{\chi_n}{a}\frac{J_1(\chi_n\rho/a)N_0(\chi_n) - N_1(\chi_n\rho/a)J_0(\chi_n)}{\sqrt{J_0^2(\chi_n)/J_0^2(c_1\chi_n) - 1}},\ n \geq 2 \end{aligned}$$

where $c_1 = b/a$, $\boldsymbol{u}_\rho$ is the unit vector in ρ direction, and χ_n is the nth nonvanishing root of the eqauation $J_0(\chi_n c_1)N_0(\chi_n) - N_0(\chi_n c_1)J_0(\chi_n) = 0$. It follows from (45) and (50) that

$$\begin{aligned} i_n^{TM}(z, t) &= -\frac{2\pi}{c\eta} e_n(\rho_0)\rho_0 \int_{-\infty}^{\infty} \frac{df(t')}{dt'} G_n^i(z, t; z_0, t')dt' \\ &= -\frac{2\pi}{\eta L}\rho_0 e_n(\rho_0) \sum_{m=0}^{\infty} \varepsilon_m \frac{\cos\dfrac{m\pi}{L}z \cos\dfrac{m\pi}{L}z_0}{\sqrt{(\chi_n/a)^2 + (m\pi/L)^2}} \\ &\quad \times \int_{-\infty}^{t} \frac{df(t')}{dt'} \sin\left[c(t - t')\sqrt{(\chi_n/a)^2 + (m\pi/L)^2}\right] dt' \quad (59) \end{aligned}$$

Let $f(t) = H(t)\sin\omega t$, the above expression may be written as

$$
\begin{aligned}
i_n^{TM}(z,t) &= -\frac{2kc\pi}{\eta L}\rho_0 e_n(\rho_0)\sum_{m=0}^{\infty}\varepsilon_m \frac{\cos\dfrac{m\pi}{L}z\cos\dfrac{m\pi}{L}z_0}{\sqrt{(\chi_n/a)^2+(m\pi/L)^2}} \\
&\quad \times \int_0^t \sin\left[c(t-t')\sqrt{(\chi_n/a)^2+(m\pi/L)^2}\right]\cos\omega t' dt' \\
&= \frac{2k\pi}{\eta L}\rho_0 e_n(\rho_0)\sum_{m=0}^{\infty}\varepsilon_m \cos\frac{m\pi}{L}z\cos\frac{m\pi}{L}z_0 \\
&\quad \times \frac{\cos kct - \cos\left[c(t-t')\sqrt{(\chi_n/a)^2+(m\pi/L)^2}\right]}{k^2-(\chi_n/a)^2+(m\pi/L)^2}
\end{aligned}
\tag{60}
$$

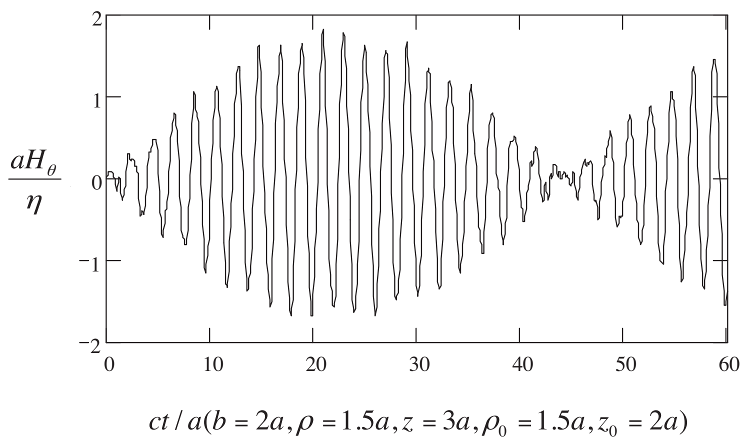

Figure 9. Magnetic field exited by a sinusoidal wave $f(t) = H(t)\sin\omega t$ when $ka = 3\,(k \neq \sqrt{(\chi_n/a)^2 + (m\pi/L)^2})$.

The magnetic field $\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{u}_\theta H_\theta$ in the coaxial waveguide can be determined from (33). Figure 9 shows the magnetic field at $\rho = (a + b)/2$ and $z = 3a$ when $ka = 3$ with the assumption that $b = 2a,\ L = 2b,\ \rho_0 = (a + b)/2,\ z_0 = L/2$. In this case k is not equal to any resonant wavenumbers $\sqrt{(\chi_{/}a)^2 + (m\pi/L)^2}\,(m, n \geq 1)$ and the field response is not a sinusoidal wave. When the frequency of the excitation waveform approaches to one of the resonant frequencies, say $ka = \chi_2$, a sinusoidal wave will gradually build up inside the coaxial waveguide cavity as shown in Figure 10.

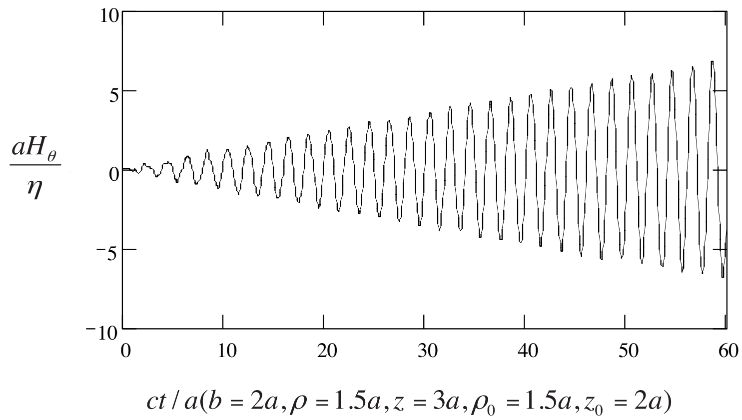

Figure 10. Magnetic field exited by a sinusoidal wave $f(t) = H(t)\sin\omega t$ when $ka = \chi_2 = 3.123$.

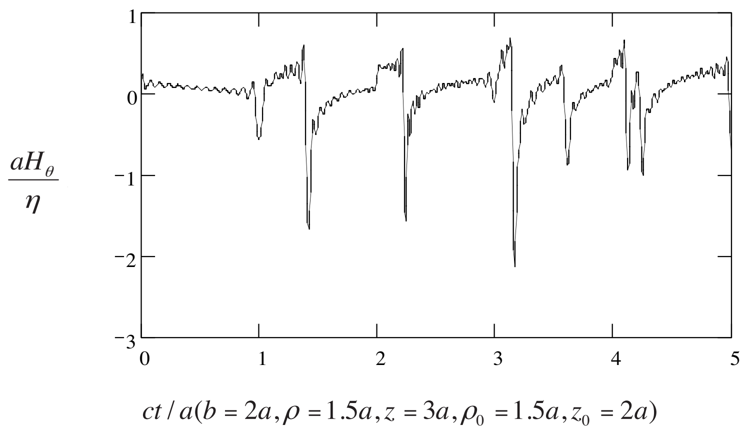

Figure 11. Magnetic field exited by a sinusoidal wave $f(t) = H(t)$.

If the coaxial waveguide cavity is excited by unit step waveform $f(t) = H(t)$, (59) may be written as

$$\begin{aligned} i_n^{TM}(z,t) &= -\frac{2\pi}{\eta L}\rho_0 e_n(\rho_0)\sum_{m=0}^{\infty}\varepsilon_m \frac{\cos\dfrac{m\pi}{L}z\cos\dfrac{m\pi}{L}z_0}{\sqrt{(\chi_n/a)^2+(m\pi/L)^2}} \\ &\quad\times\sin\left[ct\sqrt{(\chi_n/a)^2+(m\pi/L)^2}\right] \end{aligned}$$

The field response is shown in Figure 11, and is no longer a unit step waveform.

6. CONCLUSION

This paper provides a thorough discussion on the time-domain theory of metal cavities. A rigorous proof of the completeness of the vector modal functions inside an arbitrary metal cavity has been presented on the basis of the theory of symmetric operators. The major topic of this paper is the waveguide cavity resonator, which has been examined by using the time-domain theory of the waveguide developed previously. The fields inside the waveguide cavity have been expanded in terms of the vector modal functions of the corresponding waveguide, and the field expansion coefficients are shown to satisfy the modified Klein-Gordon equations subject to the homogeneous Dirichlet or Neumann boundary condition, which can then be solved by the method of retarded Green's function. Such an approach guarantees that the solutions satisfy the causality condition. Some numerical examples have been presented to demonstrate the time-domain theory. It has been shown that, for a closed cavity, the time-domain response to a sinusoidal waveform turned on at a finite instant of time is generally not sinusoidal even when the time tends to infinity. A sinusoidal response can develop inside the cavity only when the frequency of the excitation sinusoidal wave coincides with one of the resonant frequencies of the metal cavity.

The traditional time-harmonic theory, however, always yields a sinusoidal response if the excitation waveform is sinusoidal. Therefore the time-harmonic theory fails to give a correct theoretical prediction for a closed cavity whenever the source is turned on at a finite instant of time which corresponds to all practical situations. In addition the singularities occur in the time-harmonic theory for a lossless metal cavity if the frequency of the excitation source coincides with one of the resonant frequencies. In this case, the field distributions are infinite everywhere inside the cavity, leading to an awkward situation. On the other hand, the time-domain theory always gives a finite solution and is thus more appropriate to describe what happens in a closed metal cavity, which is the foundation for a number of important applications in microwave engineering and is also helpful in studying various cavity-related problems [13–20].

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