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SOLUTION OF AN INTEGRO-DIFFERENTIAL EQUATION ARISING IN OSCILLATING MAGNETIC FIELDS USING HE'S HOMOTOPY PERTURBATION METHOD

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Abstract—In this research, an integro–differential equation which describes the charged particle motion for certain configurations of oscillating magnetic fields is considered. The homotopy perturbation method (HPM) is used for solving this equation. HPM is an analytical procedure for finding the solutions of problems which is based on the constructing a homotopy with an imbedding parameter p that is considered as a small parameter. The results of applying this procedure to the integro-differential equation with time-periodic coefficients show the high accuracy, simplicity and efficiency of this method.

1. INTRODUCTION

The integro-differential equation [23]

$$\frac{d^2y}{dt^2} = -a(t)y(t) + b(t)\int_0^t \cos(w_p s)y(s)ds + g(t), \tag{1}$$

where $a(t)$, $b(t)$ and $g(t)$ are given periodic functions of time may be easily found in the charged particle dynamics for some field configurations. Taking for instance the three mutually orthogonal magnetic field components $B_x = B_1 \sin(w_p t)$, $B_y = 0$, $B_z = B_0$, the nonrelativistic equations of motion for a particle of mass m and charge q in this field configuration are

$$m\frac{d^2x}{dt^2} = q\left(B_0\frac{dy}{dt}\right), \tag{2}$$

$$m\frac{d^2y}{dt^2} = q(B_1\sin(w_pt)\frac{dz}{dt} - B_0\frac{dx}{dt}, \tag{3}$$

$$m\frac{d^2z}{dt^2} = q\left(-B_1\sin(w_pt)\frac{dy}{dt}\right). \tag{4}$$

By integration of (2) and (4) and replacement of the time first derivatives of z and x in (3) one has (1) with

$$a(t) = w_c^2 + w_f^2\sin^2(w_pt), \qquad b(t) = w_f^2 w_p\sin(w_pt), \tag{5}$$

$$g(t) = w_f(\sin(w_pt))z'(0) + w_c^2 y(0) + w_c x'(0), \tag{6}$$

where $w_c = qB_0/m$ and $w_f = qB_1/m$. Making the additional simplification that $x'(0) = 0$ and $y(0) = 0$, equation (1) is finally written as

$$\frac{d^2y}{dt^2} = -\left(w_c^2 + w_f^2\sin^2(w_pt)y + w_f(\sin(w_pt)z'(0) + w_f^2 w_p(\sin(w_pt)\right) \int_0^t \cos(w_ps)y(s)ds. \tag{7}$$

In this study, we consider the equation (1) with the following initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta. \tag{8}$$

The solution of this equation is presented by means of homptopy perturbation method. The essential idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of "deformation", the solution of each of which is "close" to that at the previous stage of "deformation". Eventually at $p = 1$, the system takes the original form of equation and the final stage of "deformation" gives the desired solution. One of the most remarkable feature of the HPM is that usually only few perturbation terms are sufficient to obtain a reasonably accurate solution.

This paper is organized as follows: In Section 2, we describe the homotopy perturbation method briefly and apply this technique to equation (1). Section 3 contains the mathematical formulation of the approach. A numerical evaluation is included in Section 4 to demonstrate the validity and applicability of the method. Also a conclusion is given in Section 5. Finally some references are given at the end of this report.

2. THE HOMOTOPY PERTURBATION METHOD

Homptopy perturbation method was first proposed by the Chinese mathematician He [16–20]. This method has been employed to solve a large variety of linear and nonlinear problems such as fractional partial differential equations [32], the nonlinear Hirota-Satsuma coupled KdV partial differential equation [12], nonlinear boundary value problems [22], traveling wave solutions of nonlinear wave equations [21], Nonlinear convective-radiative cooling equation, nonlinear heat equation (porous media equation) and nonlinear heat equation with cubic nonlinearity [13], the Newton-like iteration methods for solving non-linear equations or improving the existing iteration methods [9], evaluating the efficiency of straight fins with temperature-dependent thermal conductivity and determining the temperature distribution within the fin [26], the inverse parabolic equations and computing an unknown time-dependent parameter [28], finding improved approximate solutions to conservative truly nonlinear oscillators [4], complicated integrals which cannot be expressed in terms of elementary functions or analytical formulae [10] and etc. The homotopy perturbation method is used in [14] to solve the nonlinear Fredholm integral equations of the first kind. Author of [34] applied this method to solve the system of Fredholm and Volterrs type integral equations. The homotopy perturbation method is employed to search for periodic solutions of Jacobi elliptic equations, which are widely studied in connection with nonlinear waves [5]. Author of [3] obtained an approximate analytic solution of the steady, laminar three-dimensional flow for an incompressible fluid past a stretching sheet using the homotopy perturbation method. It is worth pointing out that the flow studied in his research is governed by a boundary value problem consisting of a pair of non-linear differential equations. The homotopy perturbation method is extended in [30] and is used to solve a kind of nonlinear evolution with the help of symbolic computation system Maple. This technique is applied in [6] to obtain approximate solutions of Klein-Gordon and Sine-Gordon equations. Also an efficient way of choosing the initial approximation is given by these authors. The Blasius equation is solved in [1] using the homotopy perturbation method. This method is used in [2] to solve functional integral equations. Also comparison is made with an expansion method based on the Lagrange interpolation formula. The homotopy perturbation method is applied in [11] to solve pure strong nonlinear second-order differential equation. Using this approach the approximate analytic solution is obtained. Two types of differential equations are considered: with strong cubic and quadratic nonlinearity. The solution of thin flow problem with a third grade fluid is found in [29] using the traditional perturbation and the homotopy perturbation methods. Two solutions are in complete agreement. But the traditional perturbation method requires the presence of a small parameter which is not so with the solution obtained using the homotopy perturbation method . Authors of [24] solved the nonlinear matrix differential equations by homotopy perturbation method . A dynamic system and Burgers equation are taken as examples to illustrate its effectiveness and convenience. A reliable algorithm based on an adoption of the standard homotopy method is given in [15]. The homotopy perturbation method is treated as an algorithm in a sequence of intervals for finding accurate approximate solutions of linear and nonlinear system of ordinary differential equations. Also the results compared with the classical fourth order Runge-Kutta formula reveal that the new method is a promising tool for nonlinear systems of ordinary differential equations. He's homotopy perturbation method employed in [33] to solve the singular initial boundary value problems of Lane-Emden type equations. As is said in [33] homotopy perturbation method provides us with a freedom choice for construction of the homotopy. The results show that construction of the homotopy for the perturbation problem plays a significant role for the accuracy of the solution. Theoretically, any exactness can be achieved by an appropriate choice of the homotopy path and this is the verification of the flexibility of the method. The homotopy perturbation method is proposed in [8] to obtain approximate solutions of the time-dependent Emden-Fowler equations. Also an algorithm based on homotopy perturbation method is developed to overcome the difficulty of the singular point at $t = 0$ [8]. The analysis is accompanied by some linear and nonlinear singular initial-value problems. Homotopy perturbation method combined with averaging in [31] to solve Van der Pole oscillator with very strong nonlinearity. The result shows that the approximation obtained by this technique is valid uniformly even for very low parameters and is more accurate than the straightforward expansion solution. The homotopy perturbation method is applied in [7] to derive approximate solutions of nonlinear population dynamics models. The nonlinear models considered are the multispecies Lokta-Volterra equations. The results are compared with fourth-order Runge-Kutta method. The homotopy perturbation method is compared in [27] to some series solution of the Lane-Emden equation. Also He's homotopy perturbation technique and Wazwaz's two implementations of the Adomian method based on either the introduction of a new differential operator that overcomes the singularity of the Lane-Emden at the origin or the elimination of the first-order derivative term of the original equations. It is also shown that Adomian's decomposition technique can be introduced as a perturbation approach which coincides with He's homotopy perturbation method.

3. SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION

To illustrate the basic ideas of the homotopy perturbation method, we consider the following nonlinear differential equation :

$$A(y) - f(r) = 0, \quad r \in \Omega, \tag{9}$$

with the boundary conditions

$$B(y, \frac{\partial y}{\partial n}) = 0, \quad r \in \Gamma, \tag{10}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Therefore equation (9) can therefore be rewritten as follows:

$$L(y) + N(y) - f(r) = 0. \tag{11}$$

By the homotopy perturbation technique, we construct a homotopy $v(r, p) : \Omega \times [0, 1] \longrightarrow R$ which satisfies:

$$H(v, p) = (1-p)[L(v)-L(y_0)]+p[A(v)-f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega, \tag{12}$$

where $p \in [0, 1]$ is an embedding parameter and y_0 is an initial approximation of equation (9). Obviously, from these definition we will have:

$$H(v, 0) = L(v) - L(y_0) = 0, \qquad H(v, 1) = A(v) - f(r) = 0.$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $y_0(r)$ to $y(r)$. In topology, this is called deformation, and $L(v) - L(y_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of (12) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{13}$$

Setting $p = 1$, results in the approximate solution of (9):

$$y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots.$$

In order to solve the equation (1) using HPM, we construct the following homotopy:

$$\begin{aligned} H(v,p) &= \frac{d^2v}{dt^2} - \frac{d^2y_0}{dt^2} + p\frac{d^2y_0}{dt^2} \\ &\quad -p\left[-a(t)v(t) + b(t)\int_0^t \cos(w_p s)v(s)ds + g(t)\right] = 0. \end{aligned} \tag{14}$$

Substituting (13) in (14) and equating the coefficients of like powers of p, yield

$$p^0: \frac{d^2v_0}{dt^2} - \frac{d^2y_0}{dt^2} = 0, \tag{15}$$

$$p^1: \frac{d^2v_1}{dt^2} + \frac{d^2y_0}{dt^2} + a(t)v_0(t) - b(t)\int_0^t \cos(w_p s)v_0(s)ds - g(t) = 0, \tag{16}$$

$$p^n: \frac{d^2v_n}{dt^2} + a(t)v_{n-1}(t) - b(t)\int_0^t \cos(w_p s)v_{n-1}(s)ds = 0, \quad n \geq 2. \tag{17}$$

Then starting with an initial approximation y_0 and solving the above equations, we can identify v_n for $n = 1, 2, \cdots$ and therefore we obtain the n-th approximation of the exact solution as $y_n = v_0 + v_1 + \cdots + v_n$.

4. ILLUSTRATIVE TESTS

In this section, to illustrate the description above and to show the efficiency of the mentioned method for solving equation (1), we include some examples with known analytical solutions.

4.1 Test 1: Consider equation (1) with

$$\begin{aligned} w_p &= 2, \\ a(t) &= \cos(t), \qquad b(t) = \sin\left(\frac{t}{2}\right), \\ g(t) &= \cos(t) - t\sin(t) + \cos(t)(t\sin(t) + \cos(t)) \\ &\quad - \sin\left(\frac{t}{2}\right)\left(\frac{2}{9}\sin(3t) - \frac{t}{6}\cos(3t) + \frac{t}{2}\cos(t)\right), \end{aligned}$$

and

$$\alpha = 1, \qquad \beta = 0,$$

$y(t) = t\sin(t) + \cos(t)$ is the exact solution of this equation. To apply the homotopy perturbation method to this equation, based on the (15), (16) and (17), we obtain

$$p^0: \frac{d^2v_0}{dt^2} - \frac{d^2y_0}{dt^2} = 0, \tag{18}$$

$$
\begin{aligned}
p^1: \ & \frac{d^2 v_1}{dt^2} + \frac{d^2 y_0}{dt^2} + \cos(t) v_0 - \sin\left(\frac{t}{2}\right) \int_0^t \cos(y_p s) v_0(s) ds \\
& - \cos(t) + t\sin(t) \\
& - \cos(t)(t\sin(t) - \cos(t)) \\
& + \sin\left(\frac{t}{2}\right) \left(\frac{2}{9}\sin(3t) - \frac{t}{6}\cos(3t) + \frac{t}{2}\cos(t)\right) = 0, \qquad (19)
\end{aligned}
$$

$$
p^n: \ \frac{d^2 v_{n-1}}{dt^2} + \cos(t) v_{n-1} - \sin\left(\frac{t}{2}\right) \int_0^t \cos(w_p s) v_{n-1}(s) ds = 0. \qquad (20)
$$

We consider $y_0(t) = 1$ as initial approximation of the exact solution and regarding (18) we start with $v_0(t) = y_0(t)$. Since $v_0(0) = \alpha$ and $v_0'(0) = \beta$, and $y = v_0 + v_1 + v_2 + \cdots$, we can set $v_n(0) = 0$ and $v_n'(0) = 0$, $(n \geq 1)$ as initial conditions for equations (19) and (20).

By solving the above equations and getting $v_1, \ldots, v_6$, we calculate y_n for $n = 1, \ldots 6$. Numerical results obtained by these approximations are summarized in Table 1 and Figure 1.

Table 1. The norms $\|y - y_n\|_1$, $\|y - y_n\|_2$ and $\|y - y_n\|_\infty$ for $n = 1, 2, 3, 4, 5, 6$ in example 1.

n	$\|y - y_n\|_1$	$\|y - y_n\|_2$	$\|y - y_n\|_\infty$
1	0.00650855446568	0.01047363533934	0.02951608243029
2	1.355066399737036e-4	2.520126891825864e-4	8.507913309692073e-4
3	1.619208174116484e-6	3.362259352853569e-6	1.291966580742192e-5
4	1.255797313339171e-8	2.851022461480738e-8	1.212590543894969e-7
5	6.832408580405466e-11	1.672143477073295e-10	7.733280680049799e-10
6	1.753953346649812e-13	6.826965141905116e-13	3.382183422218077e-12

4.2 Test 2:

Test 2: As the second example, consider equation (1) with

$$
\begin{aligned}
w_p &= 1, \\
a(t) &= -\sin(t), \quad b(t) = \sin(t), \\
g(t) &= \frac{1}{9} e^{-\frac{t}{3}} - \sin(t)\left(e^{-\frac{t}{3}} + t\right) \\
& \quad - \sin(t)\left(-\frac{3}{10}\cos(t) e^{-\frac{t}{3}} + \frac{9}{10} e^{-\frac{t}{3}} \sin(t) + \cos(t) + t\sin(t) - \frac{7}{10}\right),
\end{aligned}
$$

and

$$
\alpha = 1, \quad \beta = \frac{2}{3}.
$$

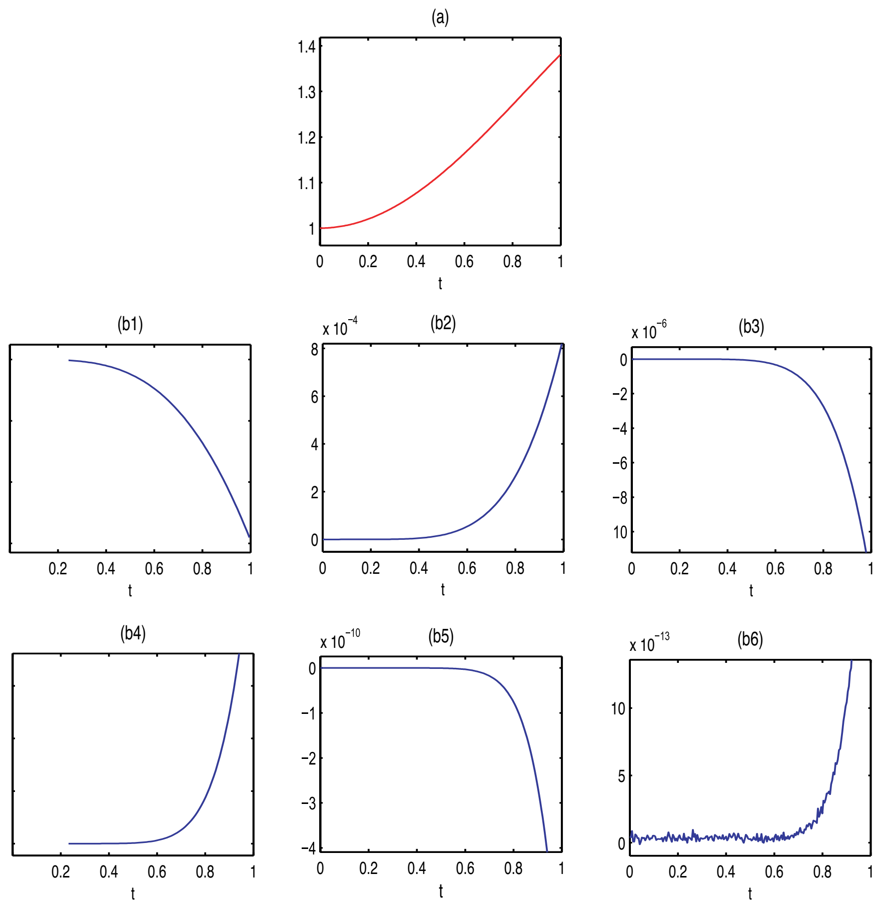

Figure 1. (a) the exact solution and the error function $y - y_n$ for (b1) $n = 1$, (b2) $n = 2$, (b3) $n = 3$, (b4) $n = 4$, (b5) $n = 5$ and (b6) $n = 6$ in example 1.

$y(t) = e^{-\frac{t}{3}} + t$ is the exact solution of this problem.

If we want to solve this equation by means of homotopy perturbation method, using (15), (16) and (17), we obtain

$$p^0 : \frac{d^2 v_0}{dt^2} - \frac{d^2 y_0}{dt^2} = 0, \tag{21}$$

$$p^1 : \frac{d^2 v_1}{dt^2} + \frac{d^2 y_0}{dt^2} - \sin(t) v_0 - \sin(t) \int_0^t \cos(y_p s) v_0(s) ds$$

$$
\begin{aligned}
&-\frac{1}{9}e^{-\frac{t}{3}}+\sin(t)(e^{-\frac{t}{3}}+t)\\
&+\sin(t)\left(-\frac{3}{10}\cos(t)e^{-\frac{t}{3}}+\frac{9}{10}e^{-\frac{t}{3}}\sin(t)+\cos(t)+t\sin(t)-\frac{7}{10}\right)\\
&= 0, \qquad (22)
\end{aligned}
$$

$$
p^n:\ \frac{d^2v_{n-1}}{dt^2}-\sin(t)v_{n-1}-\sin(t)\int_0^t\cos(w_ps)v_{n-1}(s)ds=0. \qquad (23)
$$

Considering $y_0(t) = 1 + \frac{2}{3}t$ and regarding (21), we start with $v_0(t) = y_0(t)$. Since $v_0(0) = \alpha$ and $v_0'(0) = \beta$, and $y = v_0 + v_1 + v_2 + \cdots$, we can set $v_n(0) = 0$ and $v_n'(0) = 0$, $(n \geq 1)$ as initial conditions for equations (22) and (23).

By getting $v_1, \ldots, v_6$, we obtain $y_1, \ldots, y_6$ as approximations of the exact solution. The error norms $\|y - y_n\|_1$, $\|y - y_n\|_2$ and $\|y - y_n\|_\infty$ for $n = 1, \ldots, 6$ are shown in Table 2. Also the exact solution and the error function $y(t) - y_n(t)$ for $n = 1, \ldots, 6$ are plotted in Figure 2.

Table 2. The norms $\|y - y_n\|_1$, $\|y - y_n\|_2$ and $\|y - y_n\|_\infty$ for $n = 1, 2, 3, 4, 5, 6$ in example 2.

n	$\|y-y_n\|_1$	$\|y-y_n\|_2$	$\|y-y_n\|_\infty$
1	4.789222307474183e-4	8.630464536533206e-4	0.00283817766336
2	5.768824461825676e-6	1.251027817714512e-5	5.102588734891278e-5
3	3.842183276690636e-8	9.525955825552416e-8	4.509285905229810e-7
4	1.622949042535507e-10	4.469333429047334e-10	2.371561169667302e-9
5	4.901130716282932e-13	1.435270009440732e-12	8.267164730568766e-12
6	4.329208149576017e-13	2.440936203513114e-13	7.326272469112508e-13

4.3 Test 3: In this example, consider equation (1) with

$$
\begin{aligned}
w_p &= 3,\\
a(t) &= 1, \quad b(t)=\sin(t)+\cos(t),\\
g(t) &= -t^3+t^2-11t+4-(\sin(t)+\cos(t))\\
&\quad\left(-\frac{t^3}{3}\sin(3t)-\frac{t^2}{3}\cos(3t)-\frac{13}{27}\cos(3t)-\frac{13}{9}t\sin(3t)\right.\\
&\quad\left.+\frac{t^2}{3}\sin(3t)+\frac{16}{27}\sin(3t)+\frac{2}{9}t\cos(3t)+\frac{13}{27}\right),
\end{aligned}
$$

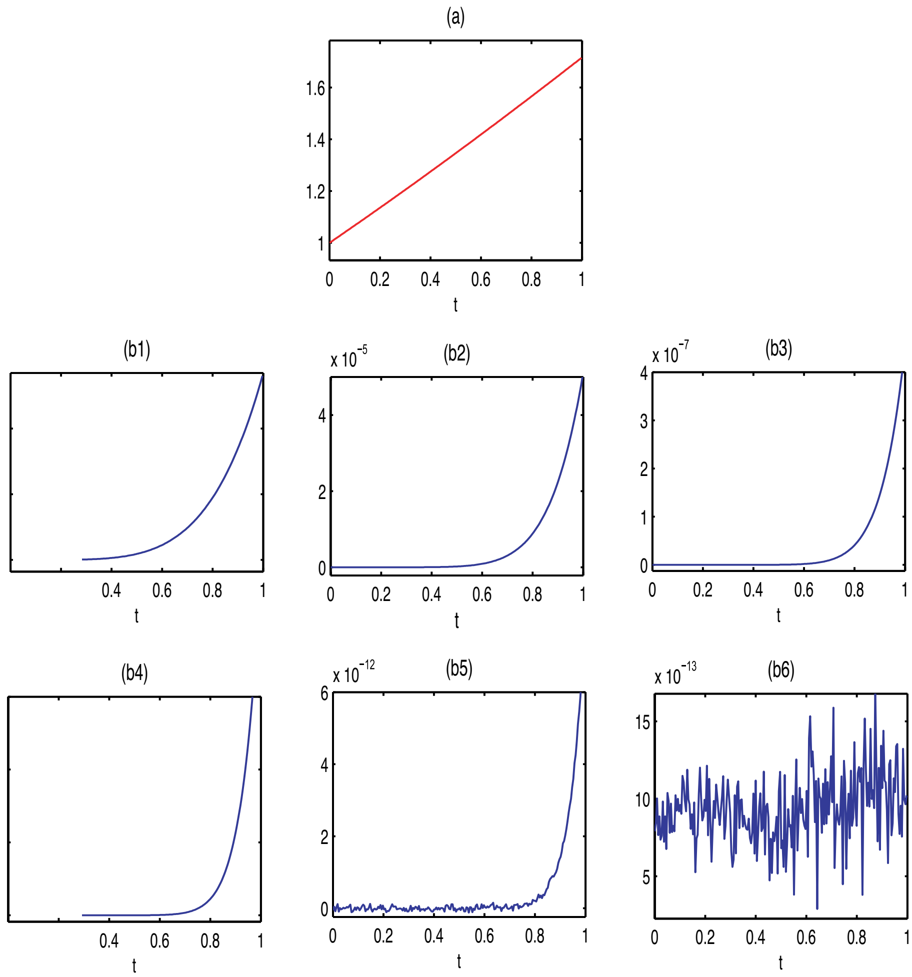

Figure 2. (a) the exact solution and the error function $y - y_n$ for (b1) $n = 1$, (b2) $n = 2$, (b3) $n = 3$, (b4) $n = 4$, (b5) $n = 5$ and (b6) $n = 6$ in example 2.

and

$$\alpha = 2, \qquad \beta = -5.$$

The exact solution of this problem is as follows

$$y(t) = -t^3 + t^2 - 5t + 2.$$

In order to solve this equation by means of homotopy perturbation method, according to (15), (16) and (17), we have:

$$p^0 : \frac{d^2v_0}{dt^2} - \frac{d^2y_0}{dt^2} = 0, \tag{24}$$

$$\begin{aligned} p^1 : \frac{d^2v_1}{dt^2} + \frac{d^2y_0}{dt^2} + v_0 \\ -(\sin(t) + \cos(t)) \left[\int_0^t \cos(y_p s) v_0(s) ds + \frac{t^3}{3} \sin(3t) \right. \\ + \frac{t^2}{3} \cos(3t) + \frac{13}{27} \cos(3t) + \frac{13}{9} t \sin(3t) - \frac{t^2}{3} \sin(3t) \\ \left. - \frac{16}{27} \sin(3t) - \frac{2}{9} t \cos(3t) - \frac{13}{27} \right] + t^3 - t^2 + 11t - 4 = 0, \end{aligned} \tag{25}$$

$$p^n : \frac{d^2v_{n-1}}{dt^2} + v_{n-1} - (\sin(t) + \cos(t)) \int_0^t \cos(w_p s) v_{n-1}(s) ds = 0. \tag{26}$$

We assume $y_0(t) = -5t + 2$ and set $v_0(t) = y_0(t)$. As previous examples, we solve the above equations with $v_n(0) = 0$ and $v'_n(0) = 0$, $(n \geq 1)$ as initial conditions and obtain $y_n = v_1 + \ldots + v_n$ for $n = 1, \ldots, 6$. Some numerical results are reported in Table 3 and Figure 3.

Table 3. The norms $\|y - y_n\|_1$, $\|y - y_n\|_2$ and $\|y - y_n\|_\infty$ for $n = 1, 2, 3, 4, 5, 6$ in example 3.

n	$\|y - y_n\|_1$	$\|y - y_n\|_2$	$\|y - y_n\|_\infty$
1	0.00753096458776	0.01165936443656	0.03061486265224
2	2.250303410711253e-4	4.265740431721630e-4	0.00149387389473
3	3.538054580471699e-6	7.678046394512791e-6	3.153239256392659e-5
4	3.480633346875790e-8	8.370424822549234e-8	3.854118226926664e-7
5	2.357215862005179e-10	16.162208867876271e-10	3.106503419956410e-9
6	1.168981793242157e-12	3.279510650660257e-12	1.783048708681179e-11

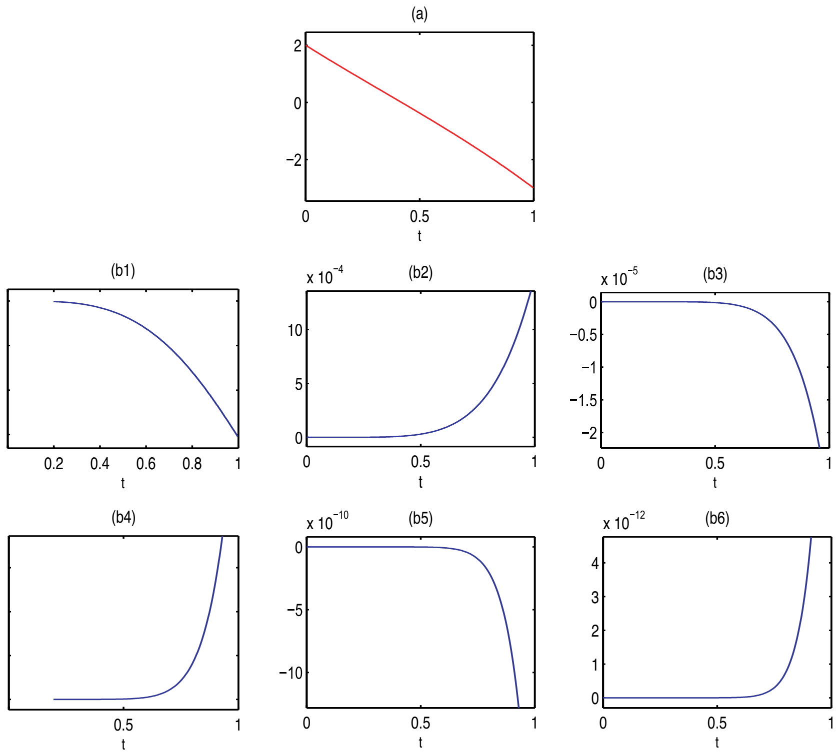

Figure 3. (a) the exact solution and the error function $y - y_n$ for (b1) $n = 1$, (b2) $n = 2$, (b3) $n = 3$, (b4) $n = 4$, (b5) $n = 5$ and (b6) $n = 6$ in example 3.

5. CONCLUSION

This article deals with the numerical solution of an integro–differential equation with time-periodic coefficients using He's homotopy perturbation method. This technique was tested on some examples and were seen to produce satisfactory results. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Furthermore this technique, in contrast to the traditional perturbation methods, does not require a small parameter and the approximations obtained by the proposed method are uniformly valid not only for small parameters, but also for very large parameters. The numerical results obtained in this research are indistinguishable due to the fact that this approach justifies its efficiency and presents quite promising results and provides a high degree of accuracy only with few iterations, without any need to restrictive assumptions. The use of the technique presented in this paper to solve some other models including the problems described in [35–45] can be an interesting investigation.

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