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FRACTIONAL BOUNDARY CONDITIONS IN PLANE WAVE DIFFRACTION ON A STRIP

E. I. Veliev and M. V. Ivakhnychenko

IRE NAS of Ukraine
12 Proskury, Kharkov 61085, Ukraine

T. M. Ahmedov

Institute of Mathematics NAS of Azerbaijan
9 F.Agaeva, AZ 1141, Baku, Azerbaijan

Abstract—New fractional boundary conditions (FBC) on plane boundaries are introduced. FBC act as intermediate case between perfect electric conductor and perfect magnetic conductor. In certain sense FBC are analogue of commonly used impedance boundary conditions with pure imaginary impedance. The relation between fractional order and impedance is shown. Plane wave diffraction problem by a strip described by FBC is formulated and solved using new method which extends known methods. Numerical results for physical characteristics are presented. Analyzing the scattering properties of the fractional strip new features are observed. FBC has one important special case where the fractional order equals to 1/2. For this special case the solution of diffraction problem can be found in analytical form for any value of wavenumber. Also for small values of wavenumber monostatic radar cross section has new specific resonances which are absent for other values of fractional order.

1. INTRODUCTION

During last fifteen years N. Engheta has been developing the method of fractional operators to solve a wide class of problems in electromagnetic theory [1–3]. Engheta explored such fractional operators as fractional derivative, fractional integral and fractional curl operator. The fractional derivative is denoted as $D_y^\nu f(y)$ and defined by the integral of Riemann-Liouville [4]

$$D_y^\nu f(y) \equiv {}_{-\infty}D_y^\nu f(y) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dy} \int_{-\infty}^{y} \frac{f(t)}{(y-t)^\nu} dt, \tag{1}$$

where the fractional order ν is changed between $0 < \nu < 1$, $\Gamma(\nu)$ is the Gamma function.

Engheta introduced the concept "Fractional paradigm in electrodynamics" [3] which means the following: if the function $f(y)$ and its first derivative $f'(y)$ describe two canonical states of the electromagnetic field, then the fractional derivative of this function $D_y^\nu f(y)$ describes an intermediate state of the field between the canonical states, i.e.,

$$D_y^\nu f(y) = \begin{cases} f(y), & \nu = 0, \\ D_y^\nu f(y), & 0 < \nu < 1 \\ f'(y), & \nu = 1 \end{cases} \tag{2}$$

Fractional operators in electromagnetics were studied by many authors. Fractional dual solutions obtained by fractional curl operator and corresponding sources were discussed in [5–8]. Then fractional field was analyzed in various electromagnetic problems: wave propagation in chiral media [9–11], waveguides [12], reflection and scattering problems [13–17].

Following the fractional paradigm in electromagnetics the new fractional boundary conditions (FBC) are introduced. The FBC can be considered as intermediate conditions between the well known Dirichlet and Newmann boundary conditions. In a two-dimensional case FBC for the function $U(x, y)$ on the boundary $y = 0$ can be defined as follows,

$$D_y^\nu U(x, y)|_{y=0} = 0 \tag{3}$$

In the case of diffraction problem of a plane wave the function $U(x, y)$ denotes the component $E_z(x, y)$ or $H_z(x, y)$ of the electric or magnetic field depending on the polarization.

Let us assume that $U(x, y)$ describes the z-component of the electric field, i.e., $U(x, y) \equiv E_z(x, y)$. FBC (3) describes an intermediate boundary between the Perfect Electric Conductor (PEC) and the Perfect Magnetic Conductor (PMC). Indeed, if $\nu = 0$ then $D_y^\nu E_z(x, y)|_{\nu=0,y=0} = E_z(x, y)|_{y=0} = 0$ and this corresponds to the PEC boundary, and if $\nu = 1$ we obtain $D_y^\nu E_z(x, y)|_{\nu=1,y=0} = \partial/\partial y E_z(x, y)|_{y=0} = 0$ that describes the PMC boundary [18].

In a particular case FBC (3) can describe the special kind of impedance boundary [18] if the following relation is supposed

$$D_y^\nu E_z(x,y)|_{y=\pm 0} = \left[E_z(x,y) \pm \frac{\eta_\nu}{\imath k}\frac{\partial}{\partial y}E_z(x,y)\right]\Big|_{y=\pm 0} = 0 \tag{4}$$

where η_ν is an impedance. The fractional derivative (1) can be simplified if applied to plane waves in the exponential form, i.e., $D_y^\nu e^{-\imath ky} = (-\imath k)^\nu e^{-\imath ky}$. If $E_z(x,y)$ describes a plane wave incident to the boundary $y = 0$ by the angle θ_0, i.e., $E_z(x,y) = e^{-\imath k(x\cos\theta_0 + y\sin\theta_0)}$, where $k = 2\pi/\lambda$ is the wave number, then from equation (4) the relation between fractional order ν and impedance η_ν can be derived as

$$\nu = \frac{1}{\imath\pi}\ln\left(\frac{1-\eta_\nu\sin\theta_0}{1+\eta_\nu\sin\theta_0}\right), \quad \eta_\nu = \frac{1}{\imath\sin\theta_0}\tan\frac{\pi\nu}{2} \tag{5}$$

The same relation was obtained by N. Engheta [5, 16]. As it follows from (5) for values of the fractional order $0 < \nu < 1$ the special impedance η_ν is always a pure imaginary value.

2. PROBLEM FORMULATION

A two-dimensional problem of electromagnetic wave diffraction by a plane strip with FBC (3) is in the focus of our study. Let us assume that an E-polarized plane wave, described by the function $U_0(x,y) \equiv E_z^0(x,y)$, is an incident field scattered by a strip located at the plane $y = 0$ and infinite along the axis z. The width of the strip is $2a$. The incident field $U_0(x,y)$ is coming from the half space $y > 0$ and can be described by the following expression:

$$U_0(x,y) = e^{-\imath k\left(x\alpha_0 + y\sqrt{1-\alpha_0^2}\right)} \tag{6}$$

where $\alpha_0 = \cos\theta_0$, θ_0 is the incidence angle. Here the time dependence is assumed to be $e^{-\imath\omega t}$ and deprecated throughout the paper. The scattered field is denoted by the function $U_r(x,y)$, then the total field $U(x,y)$ is a sum of the incident and scattered fields, i.e.,

$$U(x,y) = U_0(x,y) + U_r(x,y) \tag{7}$$

The solution $U(x,y)$ should satisfy the following conditions:

- the Helmholtz equation everywhere outside the strip:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0;$$

- the radiation condition [19] at infinity:

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial U_r}{\partial r}-\imath U_r\right)=0,\quad r=\sqrt{x^2+y^2};$$

- the Meixner's condition [19] on the edges of the strip;
- FBC on the strip surface:

$$D^{\nu}_{ky}[U_0(x,y)+U_r(x,y)]|_{y=0}=0,\quad -a<x<a \tag{8}$$

Here for convenience the fractional derivative is applied with respect to the non-dimensional variable ky. Following [1, 20] we can write the representation of the $U_r(x,y)$ through fractional Green's function (FGF) G^{ν} :

$$U_r(x,y)\equiv\int_{-a}^{a}f^{1-\nu}(x')G^{\nu}(x-x',y)dx' \tag{9}$$

where FGF G^{ν} is defined in the two-dimensional case as follows

$$G^{\nu}(x-x',y)=-\frac{\imath}{4}D^{\nu}_{ky}H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right) \tag{10}$$

Here $H_0^{(1)}(x)$ is the Hankel function of the first kind. The function $f^{1-\nu}(x)$ that describes the potential density in the integral (9), is the discontinuity of the fractional derivative of $U(x,y)$ on the plane $y=0$:

$$f^{1-\nu}(x)=D^{1-\nu}_{ky}U(x,y)|_{y=+0}-D^{1-\nu}_{ky}U(x,y)|_{y=-0},\quad x\in(-a,a) \tag{11}$$

It will be shown later that electrical and magnetic surface currents can be derived from the function $f^{1-\nu}(x)$.

The scattered field expression (9) is a particular case of more general representations obtained in the papers [20, 21], where for the fractional derivative of the function $U(\vec{r})$ that satisfies the Helmholtz equation was derived as

$$\begin{aligned}D^{\beta}_{ky}U(\vec{r}) \;=\; \oint_{S_0}&[D^{\nu}_{ky_0}G(\vec{r},\vec{r_0})\nabla_0 D^{\beta-\nu}_{ky_0}U(\vec{r_0})\\ &-D^{\beta-\nu}_{ky_0}U(\vec{r_0})\nabla_0 D^{\nu}_{ky_0}G(\vec{r},\vec{r_0})]ds_0\end{aligned} \tag{12}$$

Here $0\le\nu\le1$, $0\le\beta\le1$, $\vec{r}$ is outside S_0, and $G(\vec{r},\vec{r_0})$ is the Green's function of free space. If $\beta=0$ in (12) we obtain

$$\begin{aligned}U(\vec{r}) \;=\; \oint_{S_0}&[D^{\nu}_{ky_0}G(\vec{r},\vec{r_0})\nabla_0 D^{-\nu}_{ky_0}U(\vec{r_0})\\ &-D^{-\nu}_{ky_0}U(\vec{r_0})\nabla_0 D^{\nu}_{ky_0}G(\vec{r},\vec{r_0})]ds_0\end{aligned} \tag{13}$$

It follows from (13) that if the function $U(\vec{r})$ satisfies FBC (8) then the presentation (9) is valid for the surface S_0 that represents a contour of the strip. The "fractional potential" in (9) can be considered as intermediate potential between the simple layer potential ($\nu = 0$, $f^{1-\nu}(x)|_{\nu=0} = f^1(x) = f'(x)$) and the double layer potential ($\nu = 1$, $f^{1-\nu}(x)|_{\nu=1} = f^0(x) = f(x)$). In particular cases when the fractional order (FO) $\nu = 0$ or $\nu = 1$, the above presentations (9), (11) can be reduced to the form that usually used for the boundary conditions of the Dirichlet and Newmann type [18, 19]:

$$U(x,y) = \begin{cases} -\dfrac{\imath}{4}\displaystyle\int_{-a}^{a} f^{(1)}(x')H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right)dx', & \nu = 0 \\ \dfrac{\imath}{4}\displaystyle\int_{-a}^{a} f(x')\dfrac{\partial}{\partial y}H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right)dx', & \nu = 1 \end{cases} \tag{14}$$

$$f^{1-\nu}(x) = \begin{cases} \left.\dfrac{\partial U(x,y)}{\partial y}\right|_{y=+0} - \left.\dfrac{\partial U(x,y)}{\partial y}\right|_{y=-0}, & \nu = 0 \\ U(x,+0) - U(x,-0), & \nu = 1 \end{cases} \tag{15}$$

3. SOLUTION TO THE PROBLEM

In order to find the function $f^{1-\nu}(x)$ we satisfy the total field to FBC (8):

$$\lim_{y\to 0} D_{ky}^{y} \int_{-a}^{a} f^{1-\nu}(x')G^{\nu}(x-x',y)dx' = -\lim_{y\to 0} D_{ky}^{y} U_0(x,y) \tag{16}$$

The fractional differential-integral equation (16) is the main equation of the boundary value problem, but it is more convenient to reduce the equation to the pure integral equation (IE) using the Fourier transform (FT) of the function $f^{1-\nu}(x)$. In order to derive the integral equation we assume that $f^{1-\nu}(x)$ is zero outside the interval $[-a, a]$. Then for the function $f^{1-\nu}(x)$ and its FT $F^{1-\nu}(\alpha)$ we have the following expressions

$$\begin{aligned} \widetilde{f}^{1-\nu}(\xi) &= \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} F^{1-\nu}(\alpha)e^{\imath\epsilon\alpha\xi}d\alpha, \quad \widetilde{f}^{1-\nu}(\xi) = af^{1-\nu}(a\xi) \\ F^{1-\nu}(\alpha) &= \int_{-1}^{1} \widetilde{f}^{1-\nu}(\xi)e^{-\imath\epsilon\alpha\xi}d\alpha \end{aligned} \tag{17}$$

Here in (17) the dimensionless coordinate $\xi = \frac{x}{a}$ is introduced, and the frequency parameter is $\epsilon = ka$. We use the spectral representation of the Hankel function $H_0^{(1)}(k\sqrt{(x-x')^2+y^2})$ [19]

$$H_0^{(1)}(k\sqrt{(x-x')^2+y^2}) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{\imath k\left((x-x')\alpha+|y|\sqrt{1-\alpha^2}\right)}\frac{d\alpha}{\sqrt{1-\alpha^2}} \tag{18}$$

where the branch of the multi-valued function $\sqrt{1-\alpha^2}$ is chosen so that the radiation conditions are satisfied ($\mathrm{Im}\sqrt{1-\alpha^2}>0$).

Then using (18) we obtain the expression for the FGF $G^\nu(x-x',y)$

$$G^\nu(x-x',y) = -\frac{\imath e^{\pm\imath\pi\nu/2}}{4\pi}\int_{-\infty}^{\infty}\frac{e^{\imath k((x-x')\alpha+|y|\sqrt{1-\alpha^2})}}{(1-\alpha^2)^{(1-\nu)/2}}d\alpha \tag{19}$$

where upper sign in the exponential function is used for $y>0$, and the lower one is used for $y<0$. Using the formula (19) we can derive the presentation for the scattered field (9)

$$U_r = E_z^r(x,y) = -\frac{\imath e^{\pm\imath\pi\nu/2}}{4\pi}\int_{-\infty}^{\infty}F^{1-\nu}(\alpha)\frac{e^{\imath k(x\alpha+|y|\sqrt{1-\alpha^2})}}{(1-\alpha^2)^{(1-\nu)/2}}d\alpha \tag{20}$$

Substituting the expression (17) and (19) into the fractional differential-integral equation (16) we obtain the dual integral equations (DIE) for the Fourier transform $F^{1-\nu}(\alpha)$

$$\begin{cases}\displaystyle\int_{-\infty}^{\infty}F^{1-\nu}(\alpha)\frac{e^{\imath\epsilon\alpha\xi}}{(1-\alpha^2)^{(1-\nu)/2}}d\alpha = -4\pi\frac{e^{\imath\pi(1-\nu)/2}e^{-\imath\epsilon\alpha_0\xi}}{(1-\alpha_0^2)^{-\nu/2}}, & \xi\in[-1,1] \\ \displaystyle\int_{-\infty}^{\infty}F^{1-\nu}(\alpha)e^{\imath\epsilon\alpha\xi}d\alpha = 0, & |\xi|>1\end{cases} \tag{21}$$

The simplified versions of the system of DIE (21) if $\nu=0$ and $\nu=1$ were studied in [19, 22–24], where various methods were proposed to solve the problem. It can be noted that the system of DIE (21) for $\nu=0$ describes the problem of diffraction of E-polarized wave on a plane PEC infinitely thin strip, while for $\nu=1$ the system describes the diffraction problem on a PMC strip.

4. DUAL INTEGRAL EQUATIONS

Usage the system of DIE (21) is more general approach and it includes some particular cases described by DIE obtained in previous studies. To solve these DIE we propose the generalized method based on the previous work published in [24].

Before describing the approach we consider one interesting case when the fractional order $\nu = 0.5$. $F^{1-\nu}(\alpha)$ from the system (21) is the Fourier transform of the function

$$\widetilde{f}^{0.5}(\xi) = -2\imath\epsilon(1-\alpha_0^2)^{1/4}e^{-\imath\epsilon\alpha_0\xi+\imath\pi/4} \tag{22}$$

and $F^{1-\nu}(\alpha)$ equals to

$$F^{0.5}(\alpha) = -4\imath(1-\alpha_0^2)^{1/4}e^{\imath\pi/4}\frac{\sin\epsilon(\alpha+\alpha_0)}{\alpha+\alpha_0} \tag{23}$$

It means that for the fractional order $\nu = 0.5$ the system of DIE has an analytical solution in the form (22), (23).

Now we build a solution of (21) in the general case of $0 < \nu < 1$. The function $\widetilde{f}^{1-\nu}(\xi)$ must satisfy the edge conditions for $\xi \to \pm 1$. For special cases $\nu = 0$ and $\nu = 1$ the edge conditions have the form [19, 24]

$$\widetilde{f}^{1-\nu}(\xi) = \begin{cases} O\left((1-\xi^2)^{-1/2}\right), & \nu = 0 \\ O\left((1-\xi^2)^{1/2}\right), & \nu = 1 \end{cases}, \quad \xi \to \pm 1 \tag{24}$$

The equations (24) are well-known Meixner's edge conditions in diffraction problems [19]. In general case we assume that $\widetilde{f}^{1-\nu}(\xi)$ satisfy the edge conditions in the following form [24]

$$\widetilde{f}^{1-\nu}(\xi) = O\left((1-\xi^2)^{\nu-1/2}\right), \quad \xi \to \pm 1 \tag{25}$$

In order to satisfy the conditions (25) we use the Gegenbauer polynomials series representation for $\widetilde{f}^{1-\nu}(\xi)$:

$$\widetilde{f}^{1-\nu}(\xi) = (1-\xi^2)^{\nu-1/2}\sum_{n=0}^{\infty} f_n^\nu \frac{1}{\nu}C_n^\nu(\xi) \tag{26}$$

Here f_n^ν are unknown coefficients, $C_n^\nu(\xi)$ are Gegenbauer polynomials, which in special cases $\nu = 0$ or $\nu = 1$ are expressed as [27]

$$\begin{aligned} \lim_{\nu\to 0}\frac{C_n^\nu(\xi)}{\nu} &= \begin{cases} \dfrac{2}{n}T_n(\xi), & n \neq 0 \\ 1, & n = 0 \end{cases} \\ \lim_{\nu\to 1}\frac{C_n^\nu(\xi)}{\nu} &= C_n^1(\xi) = U_n(\xi) \end{aligned} \tag{27}$$

where $T_n(\xi)$ and $U_n(\xi)$ are Chebyshev polynomials of the first and second type, respectively. We may say that Gegenbauer polynomials are the intermediate polynomials between Chebyshev polynomials of the first and second kind.

Using Equation (26) we can obtain the presentation for $F^{1-\nu}(\alpha)$

$$F^{1-\nu}(\alpha) = \frac{2\pi}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} (-\imath)^{\nu} x_n^{\nu} \frac{J_{n+\nu}(\epsilon\alpha)}{(2\epsilon\alpha)^{\nu}} \tag{28}$$

where $J_{n+\nu}(x)$ are Bessel functions, and $x_n^{\nu} = \frac{\Gamma(n+2\nu)}{\Gamma(n+1)} f_n^{\nu}$.

The first equation in (21) multiplied by $e^{-\imath\epsilon\beta\xi}$ and integrated by ξ from -1 to 1 can be rewritten in the more convenient form as follows

$$\begin{aligned} &\int_{-\infty}^{\infty} F^{1-\nu}(\alpha) \frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{\nu-1/2} d\alpha \\ &= -4\pi e^{\imath\pi(1-\nu)/2}(1-\alpha_0^2)^{\nu/2} \frac{\sin\epsilon(\beta+\alpha_0)}{\beta+\alpha_0} \end{aligned} \tag{29}$$

Substituting (28) into (29) and taking into account the properties of discontinuous integrals of Weber-Shafheitlin [25] and the formula [19, 26]

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{J_{n+\nu}(\epsilon\alpha)}{\alpha^{\nu}} \frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta} d\alpha = \frac{J_{n+\nu}(\epsilon\beta)}{\beta^{\nu}} \tag{30}$$

we can show that homogenous equation in the system (21) is satisfied and the coefficients f_n^{ν} can be found as a solution of infinite system of linear algebraic equations (ISLAE)

$$\sum_{n=0}^{\infty} (-\imath)^n x_n^{\nu} C_{kn}^{\nu} = \gamma_k^{\nu}, \ k = 0, 1, 2, \dots \tag{31}$$

with the following matrix elements

$$C_{kn}^{\nu} = \int_{-\infty}^{\infty} J_{n+\nu}(\epsilon\alpha) J_{k+\nu}(\epsilon\alpha)(1-\alpha^2)^{\nu-1/2} \frac{d\alpha}{\alpha^{2\nu}} \tag{32}$$

and

$$\gamma_k^{\nu} = -2\Gamma(\nu+1)(2\epsilon)^{\nu} \imath^{1-\nu}(1-\alpha_0^2)^{\nu/2} \frac{J_{k+\nu}(\epsilon\alpha_0)}{\alpha_0} \tag{33}$$

Since coefficients f_n^{ν} are found, the potential density function $\widetilde{f}^{1-\nu}(\xi)$ and $F^{1-\nu}(\alpha)$ can be calculated using the formulas (26) and (28), respectively.

Now we will show that the ISLAE (31) can be reduced to ISLAE of the Fredholm type of the second kind. For this purpose let's introduce the function $\delta_\nu(\alpha)$

$$\begin{aligned}\delta_\nu(\alpha) &\equiv |\alpha|^{2\nu-1}e^{\imath\pi(\nu-1/2)}\left[\left(1-\frac{1}{\alpha^2}\right)^{\nu-1/2}-1\right]\\ \delta_\nu(-\alpha) &= \delta_\nu(\alpha)\\ \delta_\nu(\alpha)|_{\nu=1/2} &= 0\end{aligned}\tag{34}$$

For $\alpha \to \infty$ the function $\delta_\nu(\alpha)$ behavior is described by the following expression:

$$\delta_\nu(\alpha) = O\left((\nu-1/2)\alpha^{2\nu-3}\right), \quad \alpha\to\infty \tag{35}$$

It follows from (34) that

$$\frac{(1-\alpha^2)^{\nu-1/2}}{\alpha^{2\nu}} = \frac{\delta_\nu(\alpha)}{\alpha^{2\nu}} + \frac{e^{\imath\pi(\nu-1/2)}}{|\alpha|} \tag{36}$$

Taking the expression (36) into account the matrix elements C^ν_{kn} can be represented as the sum

$$C^\nu_{kn} = C^{1\nu}_{kn} + C^{2\nu}_{kn} \tag{37}$$

where

$$\begin{aligned}C^{1\nu}_{kn} &= e^{\imath\pi(\nu-1/2)}\left[1+(-1)^{k+n}\right]\int_0^\infty J_{k+\nu}(\epsilon\alpha)J_{n+\nu}(\epsilon\alpha)\frac{d\alpha}{\alpha}\\ C^{2\nu}_{kn} &= \left[1+(-1)^{k+n}\right]\int_0^\infty J_{k+\nu}(\epsilon\alpha)J_{n+\nu}(\epsilon\alpha)\delta_\nu(\alpha)\frac{d\alpha}{\alpha^{2\nu}}\end{aligned}\tag{38}$$

The integral in the expression for the coefficient $C^{1\nu}_{kn}$ is evaluated analytically [26]:

$$C^{1\nu}_{kn} = e^{\imath\pi(\nu-1/2)}\frac{1}{k+\nu}\delta_{kn} \tag{39}$$

where δ_{kn} is the Kronnecker symbol.

Finally, we obtain the ISLAE:

$$x^\nu_k + \sum_{n=0}^{\infty} x^\nu_n \widetilde{C}^{2\nu}_{kn} = \widetilde{\gamma}^\nu_k,\ \ k=0,1,2,\ldots \tag{40}$$

where

$$\begin{aligned}\widetilde{C}^{2\nu}_{kn} &= (-1)^{n-k}\left[1+(-1)^{k+n}\right](k+\nu)e^{-\imath\pi(\nu-1/2)}C^{2\nu}_{kn}\\ \widetilde{\gamma}^\nu_k &= \imath^k(k+\nu)e^{-\imath\pi(\nu-1/2)}\gamma^\nu_k\end{aligned}\tag{41}$$

It can be shown that

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}|\widetilde{C}_{kn}^{2\nu}|^2<\infty, \quad \sum_{k=0}^{\infty}|\widetilde{\gamma}_k^{\nu}|^2<\infty \tag{42}$$

It means that ISLAE (40) is SLAE of the Fredholm type of second kind and the unknown coefficients f_n^{ν} can be calculated with any given accuracy using the reduction method of solving infinite SLAEs [24].

It can be shown that in a particular case $\nu = 0.5$ ISLAE (40) has a more simple form and can be solved analytically. Indeed, the coefficient $\widetilde{C}_{kn}^{2\nu}|_{\nu=0.5} = 0$ in (40). Then the unknown coefficients $f_n^{0.5}$ can be obtained in explicit analytical form

$$f_n^{0.5}=x_n^{0.5}=\widetilde{\gamma}_n^{0.5}=-\imath\sqrt{\frac{\pi\epsilon}{2\alpha_0}}e^{\imath\pi/4}(-\imath)^n\frac{(2n+1)}{(1-\alpha_0^2)^{-1/4}}J_{n+0.5}(\epsilon\alpha_0) \tag{43}$$

Substituting the coefficients $f_n^{0.5}$ into (26) and (28) we get

$$\widetilde{f}^{1-\nu}(\xi)|_{\nu=0.5} = 2\sum_{n=0}^{\infty}f_n^{0.5}C_n^{0.5}(\xi) \tag{44}$$

$$F^{1-\nu}(\alpha)|_{\nu=0.5} = 4\sqrt{\pi}\sum_{n=0}^{\infty}(-\imath)^n f_n^{0.5}\frac{J_{n+0.5}(\epsilon\alpha)}{\sqrt{2\epsilon\alpha}} \tag{45}$$

Using formulas [27]

$$2\epsilon e^{-\imath\epsilon\alpha_0\xi}=\sqrt{\frac{2\pi\epsilon}{\alpha_0}}\sum_{n=0}^{\infty}(-\imath)^n(2n+1)J_{n+0.5}(\epsilon\alpha_0)C_n^{0.5}(\xi) \tag{46}$$

and [26]

$$\sum_{n=0}^{\infty}(-1)^n(n+0.5)J_{n+0.5}(\epsilon\alpha)J_{n+0.5}(\epsilon\alpha_0)=\frac{\sqrt{\alpha\alpha_0}\sin\epsilon(\alpha+\alpha_0)}{\pi(\alpha+\alpha_0)} \tag{47}$$

we finally obtain the same expression for $\widetilde{f}^{0.5}(\xi)$ and $F^{0.5}(\alpha)$ as in (22), (23).

5. PHYSICAL CHARACTERISTICS OF THE SCATTERED FIELD

In this section we present the expressions for the radiation pattern (RP), monostatic and bistatic radar cross sections (RCS), and densities of the surface currents. These expressions will be used to analyze the electromagnetic characteristics of the scattered field.

Let's derive the expression for the field $E_z^r(x,y)$ in the far-zone $kr \to \infty$. In the cylindrical coordinate system (r,ϕ), $x = r\cos\phi, y = r\sin\phi$ the scattered field (20) is

$$E_z^r(r,\phi) = \frac{\imath}{4\pi}(\pm\imath)^\nu \int_{-\infty}^{\infty} F^{1-\nu}(\cos\beta) e^{\imath kr\cos(\phi\pm\beta)} \sin^\nu \beta d\beta, \tag{48}$$

where the upper sign is chosen for the values $\phi \in [0,\pi]$, and the lower sign for $\phi \in [\pi, 2\pi]$. If $kr \to \infty$ we can use the method of stationary phase to derive the expression for $E_z^r(x,y)$ as follows

$$E_z^r(r,\phi) \approx A(kr)\Phi^\nu(\phi), \quad kr \to \infty, \tag{49}$$

where

$$\begin{aligned} A(kr) &= \sqrt{\frac{2}{\pi kr}} e^{\imath kr - \imath\pi/4}, \\ \Phi^\nu(\phi) &= -\frac{\imath}{4}(\pm\imath)^\nu F^{1-\nu}(\cos\phi)\sin^\nu\phi \end{aligned} \tag{50}$$

The function $\Phi^\nu(\phi)$ denotes the radiation pattern (RP) of the scattered field that can be expressed via the coefficients f_n^ν

$$\Phi^\nu(\phi) = \frac{\pi\imath(\pm\imath)^\nu}{2\Gamma(\nu+1)} \tan^\nu\phi \sum_{n=0}^{\infty} (-\imath)^n x_n^\nu \frac{J_{n+\nu}(\epsilon\cos\phi)}{(2\epsilon)^\nu} \tag{51}$$

Using the physical optics (PO) approximation when $\epsilon = ka \to \infty$ we can derive a simple expression for the function $\Phi^\nu(\phi)$. Substituting the formula

$$\lim_{\epsilon\to\infty} \frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta} = \pi\delta(\alpha-\beta) \tag{52}$$

in the integral equation (29) the expressions for $F^{1-\nu}(\beta)$ and $\Phi^\nu(\phi)$ can be obtained in the following form

$$F^{1-\nu}(\beta) \approx -4\imath^\nu \frac{(1-\alpha_0^2)^{(1-\nu)/2}}{(1-\beta^2)^{1/2-\nu}} \frac{\sin\epsilon(\beta+\alpha_0)}{\beta+\alpha_0} \tag{53}$$

$$\Phi^\nu(\phi) \approx (\mp 1)^\nu \sin\phi \left(\frac{\sin\theta_0}{\sin\phi}\right)^\nu \frac{\sin\epsilon(\cos\phi+\cos\theta_0)}{(\cos\phi+\cos\theta_0)} \tag{54}$$

In the particular case $\nu = 0.5$ and for all values of $\epsilon = ka$ we have the exact analytical expression:

$$\Phi^{0.5}(\phi) = (\mp 1)^{1/2} \sqrt{\sin\phi\sin\theta_0} \frac{\sin\epsilon(\cos\phi+\cos\theta_0)}{(\cos\phi+\cos\theta_0)} \tag{55}$$

The formula for the bi-static RCS [30] $\frac{\sigma_{2d}(bistatic)}{\lambda}$ is derived from the expression for RP $\Phi^\nu(\phi)$ using the PO approximation

$$\begin{aligned}\frac{\sigma_{2d}(bistatic)}{\lambda} &= \frac{2}{\pi}|\Phi^\nu(\phi)|^2 \\ &= \frac{2}{\pi}\sin^2\phi\left(\frac{\sin\theta_0}{\sin\phi}\right)^{2\nu}\left[\frac{\sin\epsilon(\cos\phi+\cos\theta_0)}{(\cos\phi+\cos\theta_0)}\right]^2, \ \epsilon = ka \to \infty \quad (56)\end{aligned}$$

The formula for the monostatic RCS can be defined from the bistatic RCS expression using the observation angle $\phi = \theta_0$

$$\sigma_{2d}(monostatic) = \frac{2}{\pi}\sin^2\theta_0\left[\frac{\sin\epsilon(2\cos\theta_0)}{(2\cos\theta_0)}\right]^2, \ \epsilon = ka \to \infty \quad (57)$$

The surface currents are defined as the discontinuity of the field components on the strip. For the E-polarization case the electric currents have only z-components, $\vec{j}^{\nu(e)} = \vec{z}j^{\nu(e)}$, and the magnetic currents have only x-components, $\vec{j}^{\nu(m)} = \vec{x}j^{\nu(m)}$.

$$\begin{aligned} j^{\nu(e)} &= -(H_x(x,+0) - H_x(x,-0)), \\ j^{\nu(m)} &= -(E_z(x,+0) - E_z(x,-0)) \end{aligned} \quad (58)$$

$$\begin{aligned} j^{\nu(e)} &= \frac{\imath}{2\pi}\cos\left(\frac{\pi\nu}{2}\right)B_\nu(x) = \begin{cases} f^0(x), & \nu = 0, \\ 0, & \nu = 1 \end{cases} \\ j^{\nu(m)} &= -\frac{1}{2\pi}\sin\left(\frac{\pi\nu}{2}\right)A_\nu(x) = \begin{cases} f^{(1)}(x), & \nu = 1, \\ 0, & \nu = 0 \end{cases} \end{aligned} \quad (59)$$

where

$$\begin{aligned} A_\nu(x) &= \int_{-\infty}^{\infty} F^{1-\nu}(\alpha)e^{\imath k\alpha x}(1-\alpha^2)^{(\nu-1)/2}d\alpha, \\ B_\nu(x) &= \int_{-\infty}^{\infty} F^{1-\nu}(\alpha)e^{\imath k\alpha x}(1-\alpha^2)^{\nu/2}d\alpha \end{aligned} \quad (60)$$

It is obvious from (59) that there are only electric ($\nu = 0$) or magnetic ($\nu = 1$) currents for the limit cases of the fractional order $\nu = 0$ or $\nu = 1$, however there are both electric and magnetic surface currents for intermediate values $0 < \nu < 1$. This fact is the result of using fractional Green's function.

The ratio of two currents

$$\frac{j_x^{\nu(m)}(x)}{j_z^{\nu(e)}(x)} = \imath\tan\left(\frac{\pi\nu}{2}\right)\frac{A_\nu(x)}{B_\nu(x)}, \quad x \in (-a, a) \quad (61)$$

In the PO approximation ($\epsilon \to \infty$) the integrals $A_\nu(x)$, $B_\nu(x)$ can be expressed as

$$A_\nu(x) \approx -4\pi\imath^{1+\nu} e^{-\imath k\alpha_0 x}, B_\nu(x) \approx -4\pi\imath^{1+\nu}(1-\alpha_0^2)^{1/2} e^{-\imath k\alpha_0 x} \quad (62)$$

and the ratio (61)

$$\frac{j_x^{\nu(m)}(x)}{j_z^{\nu(e)}(x)} \approx \frac{1}{\imath \sin\theta_0} \tan\left(\frac{\pi\nu}{2}\right) = \eta_\nu \quad (63)$$

It is well known that the value of the impedance in the impedance boundary conditions can be expressed as the ratio of the surface current components [18] $\frac{j_x^{\nu(m)}(x)}{j_z^{\nu(e)}(x)}$.

For the value $\nu = 0.5$ the equation (63) has an explicit form for all values of ϵ:

$$\left.\frac{j_x^{\nu(m)}(x)}{j_z^{\nu(e)}(x)}\right|_{\nu=0.5} = \frac{1}{\imath \sin\theta_0} \quad (64)$$

The relation for the electric currents (63) proves the fact that the fractional boundary conditions are similar to the impedance boundary conditions and in the PO approximation the ratio of the surface currents is the same as for the impedance strip. The fractional boundary conditions result in the presence of both the electric and magnetic surface currents on the strip and the ratio of the currents is equal to the impedance in the PO approximation. In general case of arbitrary value of ϵ the impedance can be introduced as the ratio (61) that can be calculated numerically by solving the diffraction problem.

6. NUMERICAL RESULTS

We used the reduction method to solve the ISLAE (31) numerically and calculated the values of the coefficients f_n^ν. The physical characteristics such as the monostatic RCS, the bistatic RCS and the fractional potential density $\widetilde{f}^{1-\nu}(\xi)$ have been analyzed using the formulas (26), (51).

Figures 1–3 show the comparison of the monostatic RCS for different values of the fractional order ν and frequency parameter ϵ. The results for $\nu = 0$ and $\nu = 1$ are in perfect agreement with the results obtained in [28–30]. The results for $\nu = 0.5$ obtained numerically by solving ISLAE are in a good agreement with the results obtained using the analytical formulas.

All the curves for the monostatic RCS for all values of ν have similar behavior and have the same value for the incident angle

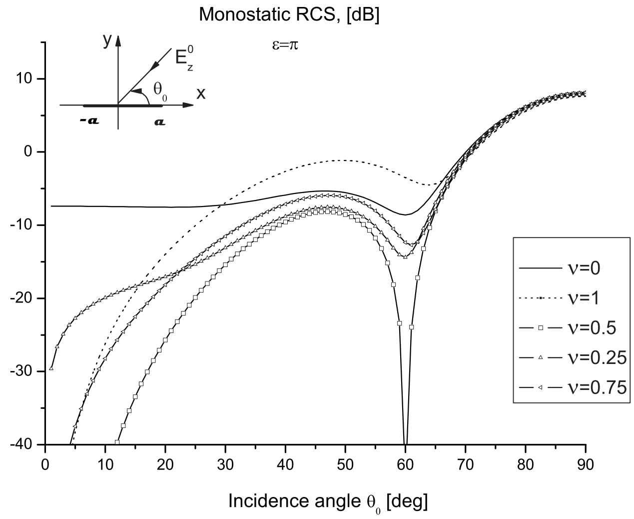

Figure 1. Monostatic RCS as a function of the incidence angle for the frequency parameter $\epsilon = \pi$ and different values of fractional order ν.

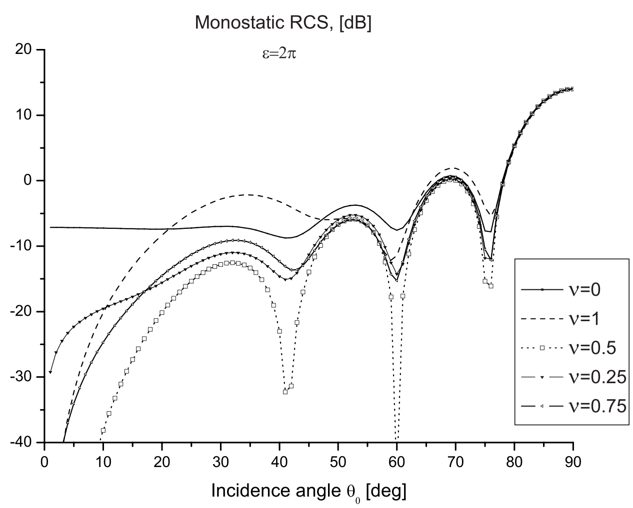

Figure 2. Monostatic RCS as a function of the incidence angle for the frequency parameter $\epsilon = 2\pi$ and different values of fractional order ν.

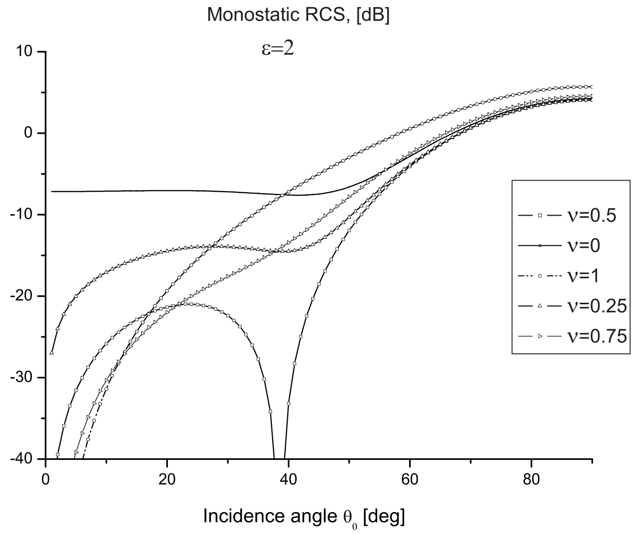

Figure 3. Monostatic RCS as a function of the incidence angle for the frequency parameter $\epsilon = 2$ and different values of fractional order ν.

$\theta_0 = 90°$. All the curves have minimums for certain values of θ_0, however there are resonances observed for $\nu = 0.5$ near these angles.

It should be noted that for small values ϵ (see Fig. 3 for $\epsilon = 2$) the monostatic RCS for $\nu = 0.5$ has a specific resonance that is not observed for the other values of ν. Using the equation (23) for $\nu = 0.5$ the resonance angles θ_r can be found from the following formula

$$\begin{aligned} 2\epsilon \cos\theta_r &= \pi n, \quad n = 1, 2, \ldots \\ \cos\theta_r &= \frac{\pi n}{2\epsilon}, \quad |\cos\theta_r| < 1 \end{aligned} \tag{65}$$

It means that resonances start to exist from the value $\epsilon = \pi/2$ that corresponds to $2a = \lambda/2$. In this case the FBC (8) describe the impedance boundary with the value of the impedance $\eta_{0.5} = -\imath\frac{1}{\sin\theta_r}$. For example, for the value $\epsilon = 2$ (see Fig. 3) the resonance angle is $\theta_r = \arccos(\pi/4) \approx 38°$ corresponded to the impedance $\eta_{0.5} \approx -1.61\imath$.

Figures 4 and 5 present the bistatic RCS for the value $\epsilon = 2\pi$ and the incident angles $\theta_0 = 90°$ and $\theta_0 = 60°$.

Figures 6 and 7 show the fractional potential densities $\widetilde{f}^{1-\nu}(\xi)$. The currents for values $\nu = 0$ and $\nu = 1$ correspond to the electric current $j_z^{0(e)}$ on PEC and the magnetic current $j_x^{1(m)}$ on PMC, respectively. It is interesting to note that the fractional density for

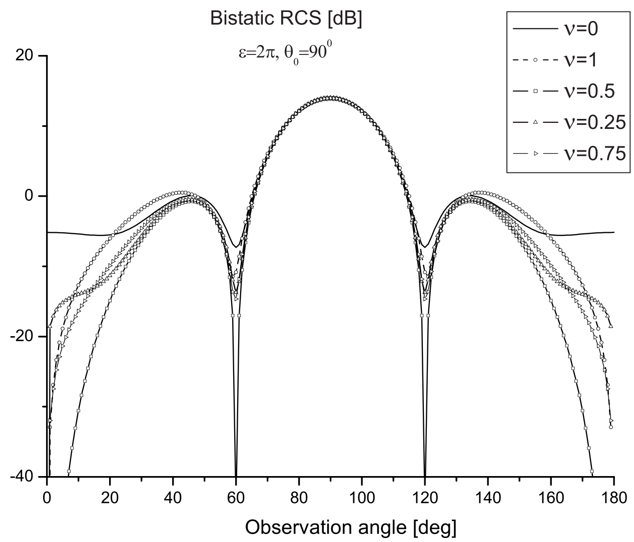

Figure 4. Bistatic RCS as a function of the observation angle for the frequency parameter $\epsilon = 2\pi$, incident angle $\theta_0 = 90°$ and different values of fractional order ν.

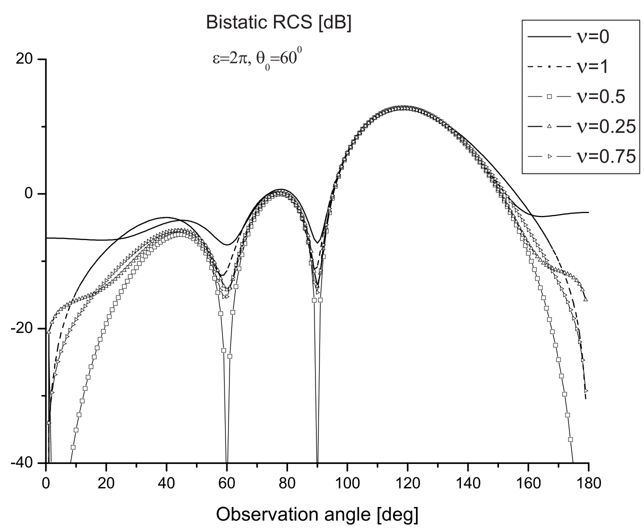

Figure 5. Bistatic RCS as a function of the observation angle for the frequency parameter $\epsilon = 2\pi$, incident angle $\theta_0 = 60°$ and different values of fractional order ν.

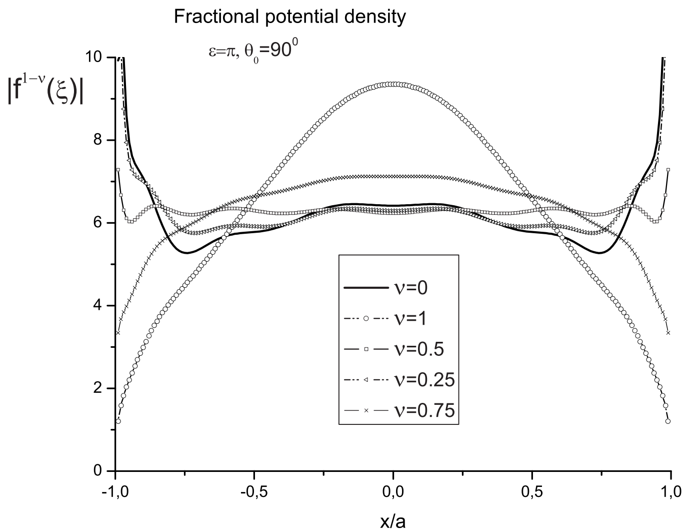

Figure 6. Fractional potential density $\widetilde{f}^{1-\nu}(\xi)$ as a function of coordinate for the frequency parameter $\epsilon = 2\pi$, incident angle $\theta_0 = 90°$ and different values of fractional order ν.

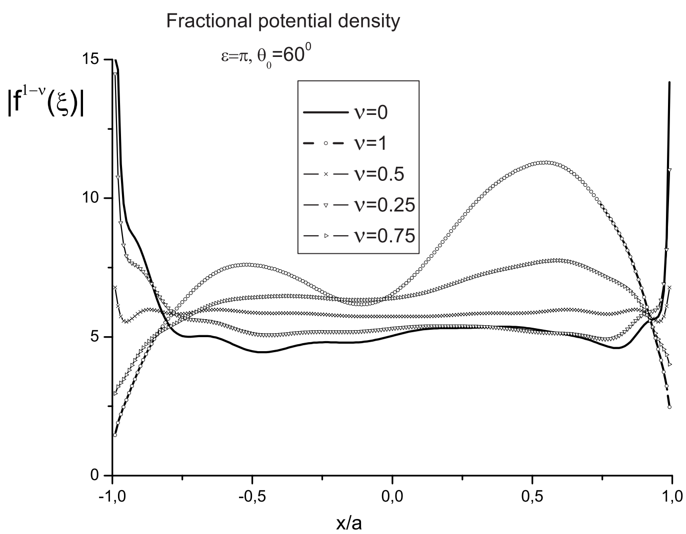

Figure 7. Fractional potential density $\widetilde{f}^{1-\nu}(\xi)$ as a function of coordinate for the frequency parameter $\epsilon = 2\pi$, incident angle $\theta_0 = 60°$ and different values of fractional order ν.

intermediate case $\nu = 0.5$ has no singularity as it is described by the formula (22).

7. CONCLUSION

In this study new FBC have been introduced. FBC are characterized by the value of the fractional order ν between 0 and 1. The problem of diffraction of plane wave by a strip with FBC has been formulated and has been reduced to dual integral equations (DIE). To solve DIE known method has been extended. For the limit cases $\nu = 0$ and $\nu = 1$ FBC describe PEC and PMC, respectively. Fractional boundary can be treated as impedance boundary with impedance of special kind. The relation between fractional order and impedance is shown. Like impedance boundary fractional boundary supports both electric and magnetic currents. Specific properties of the scattered field by such boundary are analyzed and numerical results for monostatic RCS, bistatic RCS and fractional potential densities are presented. One special case of the FBC where $\nu = 0.5$ has interesting features. Indeed, this intermediate case allows to obtain a solution in explicit form and estimates new features for radar cross sections. We believe that FBC can be a useful technique for the description of solutions to the diffraction problems for specific boundaries in terms of the fractional order.

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