Progress In Electromagnetics Research, PIER 86, 321–339, 2008

STUDY OF FOCUSING OF FIELD REFRACTED BY THE QUARTIC WOOD LENS INTO AN UNIAXIAL CRYSTAL

A. Ghaffar, M. Sajjad, and Q. A. Naqvi

Department of Electronics
Quaid-i-Azam University
45320, Islamabad, Pakistan

Abstract—A geometrical high-frequency approximation method for solving the propagation of electromagnetic wave through the quartic Wood lens into an uniaxial crystal is presented in this paper. Caustic problem of electromagnetic wave is translated into non-caustic problem by using the hybrid space. The drawback that the solution in the caustic region cannot be obtained with geometrical optics is overcome by this method known as Maslov's method. The high-frequency approximation solution that is valid around caustic region is obtained by using this method which combines the simplicity of ray and generality of the transform method. And the results are compared with those obtained by Huygens-Kirchhoff's expression.

1. INTRODUCTION

Geometrical optics (GO) approximations are often the tool of choice for modelling the propagation of waves through inhomogeneous media. Their advantages are numerous: They are faster and physically more intuitive than fully numerical techniques such as finite difference method. In large three-dimensional inhomogeneous problems where fully numerical techniques are computationally very expensive, geometrical optics approximation may be the suitable technique. Applications of geometrical optics approximation in wave propagation include in both acoustics [1] and electrodynamics [2, 3].

Unfortunately, geometrical optics encounters the problem of caustics, means it produces nonphysical wavefield at caustics. A number of theoretical techniques have been developed to circumvent this problem, including Maslov's theory [4], coherent state approximations [5], direct use of the caustic classification theorem [6], and Kirchhoff's approximation [7]. However, each of these methods has its own difficulties. According to Maslov's method, the field expression near the caustic can be constructed by using the GO information, though we must perform the integration in the spectrum domain in order to predict the field in the space domain.

In Maslov's method, the ray is expressed in terms of the six coordinates, i.e., components of wave vector (p_x, p_y, p_z) and space coordinates (x, y, z). The conventional ray expression may be considered its projection into space coordinates. The expression in the hybrid coordinates may be transformed into that of space coordinates through Fourier transform. For a review and application of the Maslov's method on different problems, the reader is referred to Maslov [8], Kravtsov [9–12], Gorman [13, 14], Ziolkowski and Dechamps [15]. Hongo and Ji [16–20], Naqvi and co-workers [21–35].

In present work, field refracted by a Wood lens into uniaxial crystal is determined analytically by using the Maslov's method. Results are compared with obtained using Kirchhoff's integral. Transmission of electromagnetic waves through a planner interface separating two media becomes very complicated when one of the medium is uniaxially anisotropic. The source of the complication can be mentioned as mode coupling. Mode coupling takes place because incident plane wave produces both ordinary and extraordinary transmitted waves. Mode coupling can be avoided only by choosing special orientations of the optical axis with respect to the interface normal or the direction of propagation of the incident plane wave. Study of fields in uniaxial, isotropic, and anisotropic material has been carried out by various authors [36–39].

2. HAMILTON'S EQUATION AND SOLUTION

Consider the distribution of permittivity given by the following quartic expression

$$\epsilon = \epsilon_c[1 - b^2(x^2 + y^2) + cb^4(x^2 + y^2)^2] \tag{1a}$$

It may be noted that constants b and c are related to focal length and are discussed later. The Hamilton's equation for the medium described by equation (1a) is given by

$$\frac{dx}{dt} = p_x, \qquad \frac{dy}{dt} = p_y, \qquad \frac{dz}{dt} = p_z \tag{1b}$$

$$\frac{dp_x}{dt} = -\frac{1}{2}\frac{\partial\epsilon}{\partial x}, \qquad \frac{dp_y}{dt} = -\frac{1}{2}\frac{\partial\epsilon}{\partial y}, \qquad \frac{dp_z}{dt} = -\frac{1}{2}\frac{\partial\epsilon}{\partial z} \tag{1c}$$

It is of interest to determine the solution of Hamilton's equation in medium defined by (1a) and value of c is very small in expression for permittivity.

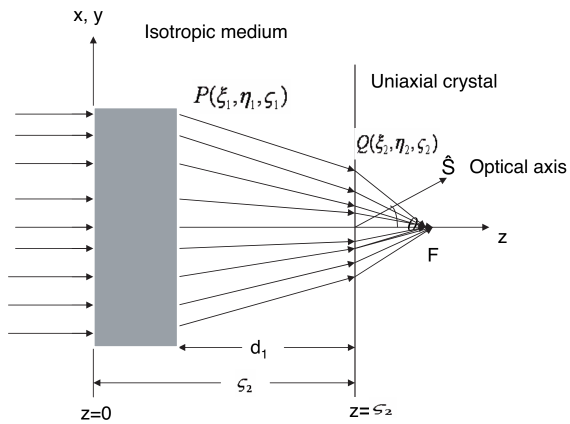

Figure 1. Geometry of refraction of field from Wood lens into uniaxial crystal.

3. DERIVATION OF THE FIELD EXPRESSION FOR QUARTIC WOOD LENS

Consider the geometry as shown in Figure 1. It contains a quartic Wood lens placed apart from a uniaxial crystal interface. The crystal has been assumed to be $LiNbO_3$ with $\mu = 1$, $\sigma = 0$, and $n^o = 2.300$ and $n^e = 2.208$, and its optical axis has been assumed to be in the z direction. Front face of Wood lens is placed at $z = 0$ while rear face is placed at $z = L$. The thickness of the lens is L. Uniaxial crystal occupies half space $z \geq z_0$. It is assumed that uniaxial crystal occupying the half space $z \geq z_0$ has principle permittivities (ϵ^o, ϵ^e), permeability μ_2. Region $L < z < z_0$ has constitutive parameters (ϵ_1, μ_1).

Electromagnetic plane wave polarized in x-direction and propagating in z-direction, is incident on a Wood lens. After passing through the Wood lens, ray is refracted through plane interface of uniaxial crystal. We may write approximate solution for Hamiltons equations as

$$\begin{aligned} x &= \xi[(1+g)\cos\psi - g\cos 3\psi] \\ y &= \eta[(1+g)\cos\psi - g\cos 3\psi] \\ z &= p_z t \\ p_x &= -\beta\xi[(1+g)\sin\psi - 3g\sin 3\psi] \\ p_y &= -\beta\eta[(1+g)\sin\psi - 3g\sin 3\psi], \\ p_z &= \sqrt{\epsilon - p_x^2 - p_y^2} \end{aligned} \tag{1d}$$

$$= \sqrt{\epsilon_c \left(1 - b^2 r_0^2 F_1 + c b^4 r_0^4 [(1+g)\cos\psi - g\cos 3\psi]^4\right)}$$

where

$$F_1 = [(1+g)\cos\psi - g\cos 3\psi]^2 + [(1+g)\sin\psi - 3g\sin 3\psi]^2$$

In above equation p_z has been assumed in the form

$$p_z = \sqrt{\epsilon_c \left(1 - b'^2 r_0^2 + c' b'^4 r_0^4\right)} \tag{2a}$$

where

$$\beta = \sqrt{\epsilon_c} b, \quad g = \frac{c' b'^2 r_0^2}{4}, \quad b' = \lambda b, \quad c' = \kappa c$$

and g is perturbation parameter, the parameters λ and κ are determined so that (1b) and (1c) give better solution. The Cartesian coordinates of refraction point at the rear face (ξ_1, η_1, ζ_1) and components of associated wave vector are given by

$$\begin{aligned} \xi_1 &= \xi[(1+g)\cos\psi - g\cos 3\psi] \\ \eta_1 &= \eta[(1+g)\cos\psi - g\cos 3\psi] \\ \zeta_1 &= L, \qquad \psi_1 = \beta t_1 \end{aligned}$$

t_1 is the arc length of the ray for region occupying the Wood lens and $r_0 = \sqrt{\xi^2 + \eta^2}$. In equation (1d), (ξ, η, ζ) are the Cartesian coordinates of refraction point of the front face of the Wood lens. Wave refracted by the Wood lens hits uniaxial crystal interface. The electromagnetic field that is incident on the plane interface is TM field. Due to this incidence, TE as ordinary wave and TM as extraordinary wave are excited inside uniaxial crystal half space. There is no coupling between TE and TM waves [36]. Ray vector of the refracted wave into uniaxial crystal may be obtained as [36–39]

$$P^{et} = n^0 p_{x1} i_x + n^0 p_{y1} i_y + p_z^e i_z \tag{2b}$$

where n^0 is ordinary refractive index of crystal and z component may be written as

$$\begin{aligned} p_z^e &= A_1 + \sqrt{B_1} \\ A_1 &= \frac{-\chi \sin\theta\cos\theta(\cos\phi n^0 p_{x1} + \sin\phi n^0 p_{y1})}{1 + \chi\cos^2\theta} \\ B_1 &= \frac{(p^0)^2(1+\chi) - (n^0)^2(p_{x1}^2 + p_{y1}^2)}{1 + \chi\cos^2\theta} - \frac{A_1^2}{\chi\cos^2\theta} \end{aligned}$$

where χ is measure of the anisotropy in the uniaxial crystal and is given by

$$\chi = \frac{(p^e)^2}{(p^0)^2} - 1$$
$$p^e = \frac{\omega}{c}(\sqrt{\epsilon^e \mu_2})$$
$$p^0 = \frac{\omega}{c}(\sqrt{\epsilon^0 \mu_2})$$

The coordinates of the ray at the front face of the uniaxial crystal are given by

$$\begin{aligned} \xi_2 &= \xi_1 + p_{x1} t_2, \qquad \eta_2 = \eta_1 + p_{y1} t_2 \\ \zeta_2 &= \zeta_1 + p_{z1} t_2, \qquad p_{x1} = p_{x0}, \\ p_{y1} &= p_{y0}, \qquad p_{z1} = \sqrt{1 - p_{x1}^2 - p_{y1}^2} \end{aligned} \tag{2c}$$

where $t_2 = \frac{\zeta_2 - \zeta_1}{p_{z1}}$ signifies the arc length of the ray after passing through the uniaxial crystal. The coordinates of the cartesian and Ray vector may be given as

$$\begin{aligned} x &= \xi_1 + p_{x1} t_2 + n^0 p_{x1} t \\ y &= \eta_1 + p_{y1} t_2 + n^0 p_{y1} t \\ z &= \zeta_1 + p_{z1} t_2 + {p_z}^e t \end{aligned}$$

where $t_2 = \frac{\zeta_2 - \zeta_1}{p_{z1}} > 0$ is the distance between the point $P(\xi_1, \eta_1, \zeta_1)$ on the rear face of the Wood lens and the point $Q(\xi_2, \eta_2, \zeta_2)$ on the front face of the uniaxial crystal and t signifies the arc length of the ray after refraction into the uniaxial crystal. The GO solution is given by [9–11]

$$u(x, y, z) = [J(t)]^{-\frac{1}{2}} \exp[-jk(\psi_0' + t)] \tag{3}$$

In the above equation,ψ_0' is the phase difference between front and rear faces of the Wood lens and can be calculated see Appendix A and

$$J(t) = \frac{U}{W} t^2 + \frac{V}{W} t + 1$$

For detailed jacobian calculations see Appendix B. Geometrical optics field contains singularity at the focal point. Our interest is to find the uniform field expression valid in focal region using Maslov's method. The uniform expression which is valid in the focal region is given by

[9, 16–18]

$$u(r) = \frac{k}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(r_0)\left[\frac{1}{D(0)}\frac{\partial((p_{x1},p_{y1},z)}{\partial(\xi,\eta_1,t)}\right]^{-\frac{1}{2}} \times \exp\Big[-jk\Psi_2(p_{x1},p_{y1},z)\Big]dp_{x1}dp_{y1} \quad (4)$$

$T(r_0)$ is the initial value at the rear face of the Wood lens. Phase $\Psi_2(p_{x1},p_{y1},z)$ has been determined in appendix C and the results are

$$\begin{aligned}\Psi_2(p_{x1},p_{y1},z) =& \Psi_0' + \beta r_0^2[(1+g)\cos\psi_1 - g\cos 3\psi_1] \\ &\times[(1+g)\sin\psi_1 - 3g\sin 3\psi_1] \\ &+\sqrt{\{1-\beta^2 r_0^2[(1+2g)\sin^2\psi_1 - 6g\sin\psi_1\sin 3\psi_1]\}}(\zeta_2-\zeta_1) \\ &-\beta r r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\cos(\delta-\phi)+(z-\zeta_2)p_z^e\end{aligned} \quad (5)$$

Quantities in square bracket of equation (4) are

$$\begin{aligned}\frac{\partial(p_{x1},p_{y1},z)}{\partial(\xi,\eta,t)} =& \beta^2 p_z^e\bigg\{\Big[(1+2g)\sin^2\psi_1 - 6g\sin 3\psi_1\sin\psi_1\Big] \\ &+\cos\psi_1\sin\psi_1\frac{\beta L}{p_{z0}^3}r_0^2\Big[b'^2\epsilon_c - 2\epsilon_c c' b'^4 r_0^2\Big] \\ &+g\,[2\cos\psi_1\sin\psi_1 - 3\sin 3\psi_1\cos\psi_1 - 9\cos 3\psi_1\sin\psi_1]\,\frac{\beta\beta'^2 L}{p_{z0}^3}r_0^2\bigg\} \\ D(0) =& p_z^e\bigg\{[(1+2g)\cos^2\psi_1 - 2g\cos\psi\cos 3\psi_1] \\ &-\sin\psi_1\cos\psi_1\frac{\beta L}{p_{z0}^3}(\epsilon_c b'^2 - 2\epsilon_c c' b'^4 r_0^2)r_0^2 \\ &-g[2\sin\psi_1\cos\psi_1 - \sin\psi_1\cos 3\psi_1 - 3\cos\psi_1\sin 3\psi_1]\frac{\beta\beta'^2 L}{p_{z0}^3}r_0^2\bigg\}\end{aligned} \quad (6)$$

Conversion of(p_{x1},p_{y1}) into (r_0,δ) yields as

$$\begin{aligned}dp_{x1}dp_{y1} =& \beta^2\bigg\{\Big[(1+2g)\sin^2\psi_1 - 6g\sin 3\psi_1\sin\psi_1\Big] \\ &+\cos\psi_1\sin\psi_1\frac{\beta L}{p_{z0}^3}r_0^2\Big[b'^2\epsilon_c - 2\epsilon_c c' b'^4 r_0^2\Big] \\ &+g\,[2\cos\psi_1\sin\psi_1 - 3\sin 3\psi_1\cos\psi_1 - 9\cos 3\psi_1\sin\psi_1] \\ &\times\frac{\beta\beta'^2 L}{p_{z0}^3}r_0^2\bigg\}r_0 dr_0 d\delta\end{aligned} \quad (7)$$

$T(r_0)$ and other related parameters are given by

$$T(r_0) = T_1 T_2 T_\alpha^{ee}, \quad T_1 = \frac{2}{1+\sqrt{\epsilon_c - \beta^2 r_0^2 + c\beta^2 b^2 r_0^4}}$$

$$T_2 = \frac{2\sqrt{\epsilon_c - \beta^2 r_0^2 \cos^2\psi_1 + c\beta^2 b^2 r_0^4 \cos^2\psi_1 \cos\theta_{i2}}}{\cos\theta_{i2} + \sqrt{\epsilon_c - \beta^2 r_0^2 \cos^2\psi_1 + c\beta^2 b^2 r_0^4 \cos^2\psi_1 \cos\theta_{t2}}}$$

$$\theta_{i2} = \tan^{-1}\frac{\sqrt{p_{x0}^2 + p_{y0}^2}}{p_{z0}} = \tan^{-1}\frac{\beta r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]}{\sqrt{\epsilon_c(1 - b'^2 r_0^2 + c' b'^4 r_0^4)}},$$

$$\theta_{t2} = \tan^{-1}\frac{\beta r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]}{\sqrt{1-\beta^2 r_0^2[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]^2}}$$

$$t_1 = \frac{L}{\sqrt{\epsilon_c(1 - b'^2 r_0^2 + c' b'^4 r_0^4)}}$$

and θ_{i2} and θ_{t2} are angles of the incidence and the refraction of the ray at the rear face of the Wood lens. The direction of the optical axis in the uniaxial crystal along the unit vector $\hat{s}$ is given by

$$\hat{s} = \sin\theta\cos\phi i_x + \sin\theta\sin\phi i_y + \cos\theta i_z$$

The transmission coefficients may be obtained [36, 37] by

$$T_\alpha^{ee} = \frac{2\mu p^2 p_{x1} p_{z1}}{\mu_1 (p^0)^2 p_{z1} A^{et} - \mu p^2 B^{et}}$$

where

$$A^{et} = \cos\theta p_{x1} - \sin(\theta+\phi) p_z^e$$

And

$$B^{et} = \sin(\theta+\phi)(p^0)^2 - \sin(\theta+\phi) p_{x1} - p_z^e p_{x1}^2 \cos\theta$$

Substituting equations (5)–(8) into equation (4) we get

$$u(r) = \frac{k}{2\pi}\int_0^a T(r_0)\,[\Gamma\Xi]^{\frac{1}{2}} \exp\Big[-jk\Psi_2(p_{x1}, p_{y1}, z)\Big] r_0 dr_0 \qquad (8)$$

where a is radius of lens and

$$\begin{aligned}\Gamma &= \Big[(1+2g)\sin^2\psi_1 - 6g\sin 3\psi_1 \sin\psi_1\Big] \\ &\quad + \cos\psi_1 \sin\psi_1 \frac{\beta L}{p_{z0}^3} r_0^2 \left[b'^2\epsilon_c - 2\epsilon_c c' b'^4 r_0^2\right] \\ &\quad + g\left[2\cos\psi_1\sin\psi_1 - 3\sin 3\psi_1\cos\psi_1 - 9\cos 3\psi_1 \sin\psi_1\right] \frac{\beta\beta'^2 L}{p_{z0}^3} r_0^2 \\ \Xi &= \beta^2\Big\{\Big[(1+2g)\sin^2\psi_1 - 6g\sin 3\psi_1 \sin\psi_1\Big] \\ &\quad + \cos\psi_1 \sin\psi_1 \frac{\beta L}{p_{z0}^3} r_0^2 \left[b'^2\epsilon_c - 2\epsilon_c c' b'^4 r_0^2\right] \\ &\quad + g\left[2\cos\psi_1\sin\psi_1 - 3\sin 3\psi_1\cos\psi_1 - 9\cos 3\psi_1 \sin\psi_1\right] \frac{\beta\beta'^2 L}{p_{z0}^3} r_0^2\Big\}\end{aligned}$$

4. COMPARISON TO THE HUYGENS-KIRCHHOFF'S EXPRESSION

To verify the validity of the uniform expression which is valid near the caustic, we compare the numerical results with those computed from the Huygens-Kirchhoff's radiation integral given by

$$u(r) = \frac{-k\beta}{2\pi}\int\int T_1 T_2 T_\alpha^{ee}\phi(\xi_2,\eta_2,\zeta_2)\frac{\exp[-jkR]}{R}\cos\gamma dS \qquad (9)$$

where ϕ is the field distribution at the front face of the uniaxial crystal and $\cos\gamma$ is the angle behind the lens.

$$\begin{aligned}R &= \sqrt{(x-\xi_2)^2 + (y-\eta_2)^2 + (z-\zeta_2)} \\ &= p_{x1}(x-\xi_2) + p_{y1}(y-\eta_2) + p_z^e(z-\zeta_2) \\ &= -p_{x1}\xi_2 - p_{y1}\eta_2 + \beta r r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1] \\ &\quad \times\cos(\delta-\phi) + p_z^e(z-\zeta_2) \\ \Phi &= \frac{1}{\sqrt{\Xi}}\exp[-jk(\psi_0' + t_2)]\end{aligned}$$

Substituting these values into equation (9) integrating with respect to angular coordinates we get result

$$\begin{aligned}u(r) &= \frac{-k}{2\pi}\int_0^a \frac{T_1 T_2 T_\alpha^{ee}}{R\sqrt{\Xi}} J_0(k\beta r r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]) \\ &\quad \exp[-jk(\Psi_0'' + (\zeta_2-\zeta_1)p_{z1} + (z-\zeta_2)p_z^e - p_{x1}\xi_2 - p_{y1}\eta_2]r_0 dr_0\end{aligned} \qquad (10)$$

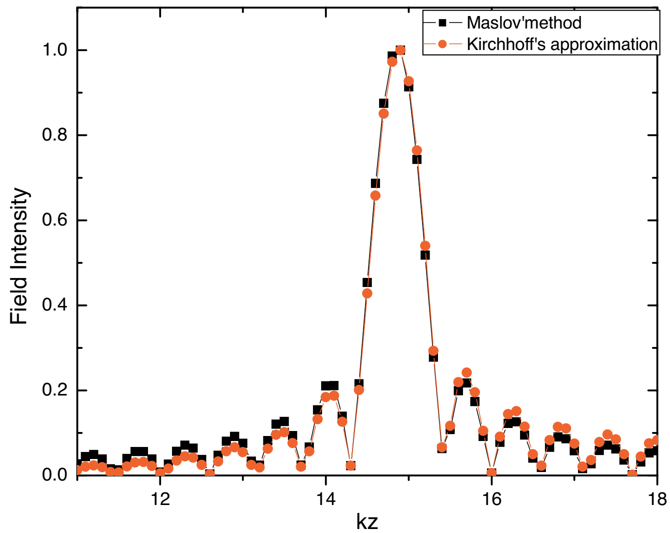

Figure 2. Comparison field Intensity at focal point of Wood lens Into uniaxial crystal by Maslov's method and Kirchhof's approximation.

5. RESULTS AND DISCUSSION

Field patterns around the caustic of an Wood lens are determined using equation (8) and (10). These integrals have been solved using Gauss Quadrature method numerically. Present discussion is extension of our previous work accepted for publication in *Central European Journal of Physics* [27]. Figure 2 deals with the field patterns, computed by Maslov's method and Huygens-Kirchhoff's integral along z-axis, around the caustic region. It is assumed that $kL = 1.8$, $\epsilon_c = 2.25$, $\beta = 0.2$, $c = 0.1$ and $\theta = 0^0$. The diameter of wood lens is $kD = 3.6$. The results are in good agreement. It is difficult to determine which method provides more precise solution, but each method give a similar order of accuracy. Figure 3 shows comparison of two situations, one deals with isotropic medium and other deals with uniaxial crystal with optical axis making angles at $\theta = 0°$. The results in this Figure 3 show that the maximum intensities are indeed the same, as expected, but the focus in the crystal is shifted towards the interface compared to the focus in the isotropic medium. The crystal can be replaced by an isotropic medium by putting $n^e = n^o = 1$. Figure 4 show comparison of field distribution by optical axis along x-axis and z-axis

Figure 5 and Figure 6 show comparison of field distribution at different orientation of optical axis, that is, at $\theta = 0°$, $\theta = 15°$, $\theta = 30°$,

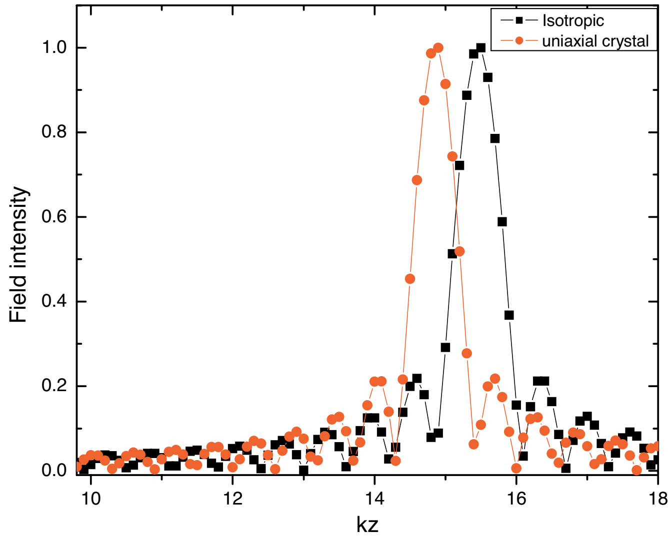

Figure 3. Comparison of Field distribution at focal point of Wood lens with isotropic medium and into uniaxial crystal.

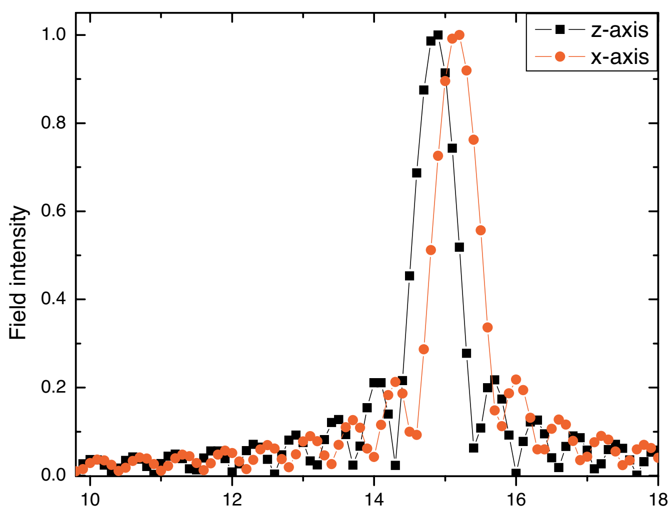

Figure 4. Comparison of field distribution at focal point with optical axis along z-axis and x-axis.

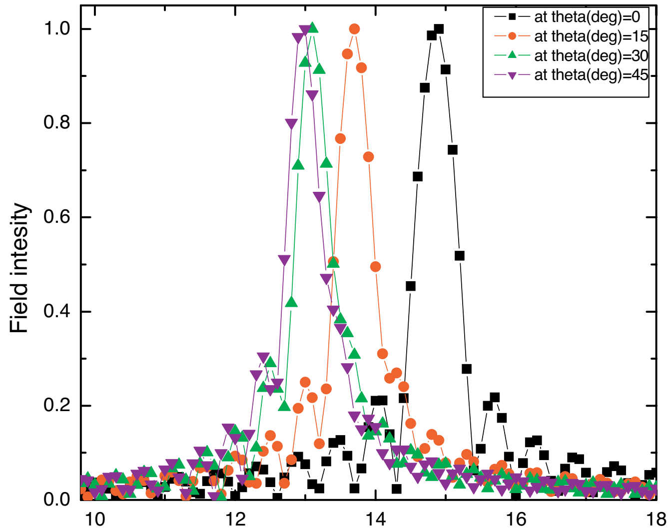

Figure 5. Comparison of field distribution of wood lens into uniaxial crystal at different orientations of optical axis.

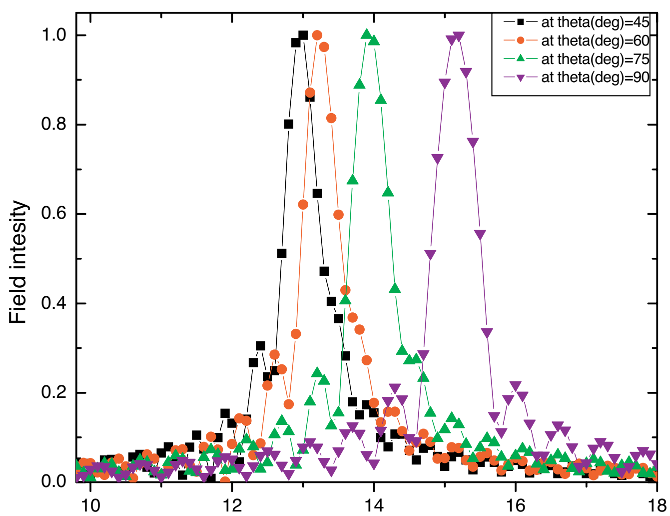

Figure 6. Comparison of field distribution of wood lens into uniaxial crystal at different orientations of optical axis.

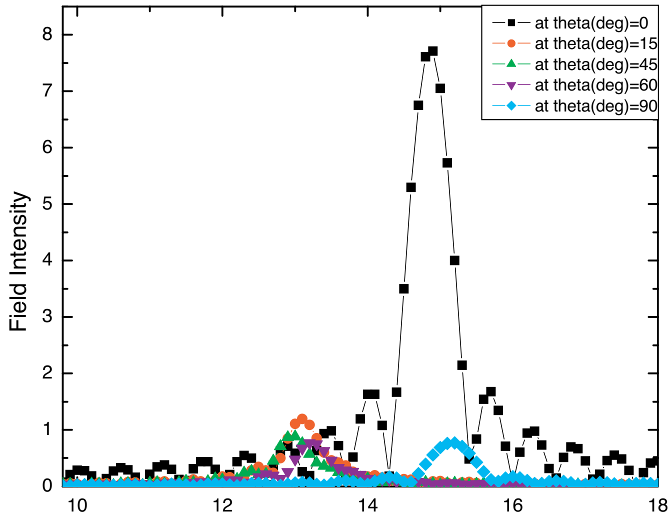

Figure 7. Variation of field intensity of Wood lens into uniaxial crystal with orientation of different angle.

$\theta = 45°$ and $\theta = 45°$, $\theta = 60°$, $\theta = 75°$, $\theta = 90°$. Figure 7 shows variation of field intensity at different orientation of optical axis that is at $\theta = 0°$, $\theta = 15°$, $\theta = 45°$, $\theta = 60°$, and $\theta = 90°$. It is shown that the focal area for a negative uniaxial crystal is displaced in the x and z directions as the angle θ is increased from $\theta = 0°$. If we continue to increase the angle θ, we will obtain a maximum displacement of the focal area when $\theta = 45°$ If the angle $\theta$is monotonically increased above $\theta = 45°$, then the displacement of the focal area will be monotonically reduced until the displacement in the x direction vanishes when θ approaches $\theta = 90°$.

Throughout the discussion, for uniaxial crystal case, we have used $LiNbO_3$, which has ordinary refractive index of $n^o = 2.208$ and extraordinary refractive index of $n^e = 2.300$. The distance between the rear face of the Wood lens and front face of uniaxial crystal $kd_1 = 7$ from the plane uniaxial interface.

APPENDIX A.

$$\begin{aligned}
\Psi_0'' &= \int_0^{t_1} \epsilon_c \left[1 - b^2(x^2+y^2) + cb^4(x^2+y^2)^2\right] dt \\
&= \int_0^{t_1} \epsilon_c \left[1 - b^2 r_0^2[(1+g)\cos\psi - g\cos 3\psi]^2\right.
\end{aligned}$$

$$
\begin{aligned}
& + cb^4 r_0^2 [(1+g)\cos\psi - g\cos 3\psi]^4 \Big] \, dt \\
= & \int_0^{t_1} \epsilon_c \Big[1 - b^2 r_0^2 [(1+2g)\cos^2\psi - 2g\cos 3\psi \cos\psi] \\
& + cb^4 r_0^2 [(1+2g)\cos^2\psi - 2g\cos 3\psi\cos\psi]^2 \Big] \, dt \\
= & \int_0^{t_1} \epsilon_c \Big[1 - b^2 r_0^2 \cos^2\psi \Big] \, dt - 2\epsilon_c g b^2 r_0^2 \int_0^{t_1} [\cos^2\psi - \cos 3\psi \cos\psi] dt \\
& + \epsilon_c c b^4 r_0^2 \int_0^{t_1} [(1+2g)\cos^2\psi - 2g\cos 3\psi \cos\psi]^2 dt \\
= & \int_0^{t_1} \epsilon_c \Big[1 - b^2 r_0^2 \cos^2\psi \Big] \, dt - \epsilon_c g b^2 r_0^2 \int_0^{t_1} [1 - \cos 4\psi] \, dt \\
& + \epsilon_c c b^4 r_0^2 \int_0^{t_1} [(1+2g)\cos^2\psi - 2g\cos 3\psi \cos\psi]^2 dt \\
= & \Psi_0 + \epsilon_c g b^2 r_0^2 t_1 \left(1 - \frac{\sin 4\psi_1}{4\beta} \right) \\
& + \epsilon_c c b^4 r_0^4 t_1 \left(\frac{3}{8} + \frac{2}{16\beta} \sin 2\psi_1 + \frac{1}{32\beta} \sin 4\psi_1 \right)
\end{aligned}
$$

where Ψ_0 is the phase difference between front and rear faces of the Wood lens.

$$
\begin{aligned}
\Psi_0 &= \int_0^{t_1} \epsilon(t) dt = \int_0^{t_1} \epsilon_c \Big[1 - b^2 r_0^2 \cos^2 \beta t \Big] dt \\
&= \epsilon_c \left(1 - \frac{b^2 r_0^2}{2} \right) t_1 - \frac{\beta r_0^2}{4} \sin 2\psi_1
\end{aligned}
$$

APPENDIX B. EXPRESSION FOR JACOBIAN

$$
\begin{aligned}
x &= \xi_1 + p_{x1} t_2 + n^0 p_{x1} t, \qquad y = \eta_1 + p_{y1} t_2 + n^0 p_{y1} t \\
z &= \zeta_1 + p_{z1} t_2 + p_z^e t
\end{aligned}
$$

$$
\begin{aligned}
D(t) &= \frac{\partial(x,y,z)}{\partial(\xi,\eta,t)} \\
&= \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} = Ut^2 + Vt + W
\end{aligned}
$$

where

$$
\begin{aligned}
U \;=\;& \left(\frac{\partial(n^o p_{x1})}{\partial\xi}\frac{\partial(n^o p_{y1})}{\partial\eta} - \frac{\partial(n^o p_{y1})}{\partial\xi}\frac{\partial(n^o p_{x1})}{\partial\eta}\right) p_z^e \\
&+ \left(\frac{\partial(n^o p_{y1})}{\partial\xi}\frac{\partial p_z^e}{\partial\eta} - \frac{\partial(n^o p_{y1})}{\partial\eta}\frac{\partial p_z^e}{\partial\xi}\right) p_{x1} \\
&+ \left(\frac{\partial(n^o p_{x1})}{\partial\eta}\frac{\partial p_z^e}{\partial\xi} - \frac{\partial(n^o p_{x1})}{\partial\xi}\frac{\partial p_z^e}{\partial\eta}\right) p_{y1} \\
V \;=\;& p_{x1}\Big(- B\eta\frac{\partial\psi_1}{\partial\xi}\frac{\partial p_z^e}{\partial\eta} + \frac{\partial p_{y1}}{\partial\xi}\frac{\partial p_z^e}{\partial\eta}t_2 + \frac{\partial p_{z1}}{\partial\eta}\frac{\partial(n^0 p_{y1})}{\partial\xi}t_2 \\
&-\frac{\partial p_{z1}}{\partial\xi}\frac{\partial(n^0 p_{y1})}{\partial\eta}t_2 - \frac{\partial p_{y1}}{\partial\eta}\frac{\partial p_z^e}{\partial\xi}t_2 + B\eta\frac{\partial\psi}{del\eta}\frac{\partial p_z^e}{\partial\xi} - A\frac{\partial p_z^e}{\partial\xi}\Big) \\
&+p_{y1}\Big(\frac{-A\partial p_z^e}{\partial\eta} + \frac{B\xi\partial\psi}{\partial\xi}\frac{\partial p_z^e}{\partial\eta} - \frac{\partial p_{x1}}{\partial\xi}\frac{\partial p_z^e}{\partial\eta}t_2 - \frac{\partial p_{z1}}{\partial\eta}\frac{\partial(n^0 p_{x1}}{\partial\xi)}t_2 \\
&+\frac{\partial p_{z1}}{\partial\xi}\frac{\partial(n^0 p_{x1})}{\partial\eta}t_2 - B\xi\frac{\partial\psi}{\partial\eta}\frac{\partial p_z^e}{\partial\xi} + \frac{\partial p_{x1}}{\partial\eta}\frac{\partial p_z^e}{\partial\xi}\Big) \\
&+p_z^e\Big(\frac{\partial(n^0 p_{y1})t_2}{\partial\eta}\frac{\partial p_{y1}}{\partial\xi} + \frac{\partial(n^0 p_{x1})t_2}{\partial\xi}\frac{\partial p_{y1}}{\partial\eta} + A\frac{\partial(n^0 p_{y1})}{\partial\eta} \\
&-B\xi\frac{\partial\psi}{\partial\xi}\frac{\partial(n^0 p_{y1})}{\partial\eta} + A\frac{\partial(n^0 p_{x1})}{\partial\xi} - B\eta\frac{\partial\psi}{\partial\eta}\frac{\partial(n^0 p_{x1})}{\partial\xi}\Big) \\
&+p_z^e\Big(\frac{\partial(n^0 p_{y1})t_2}{\partial\eta}\frac{\partial p_{y1}}{\partial\xi} + \frac{\partial(n^0 p_{x1})t_2}{\partial\xi}\frac{\partial p_{y1}}{\partial\eta} + A\frac{\partial(n^0 p_{y1})}{\partial\eta} \\
&-B\xi\frac{\partial\psi}{\partial\xi}\frac{\partial(n^0 p_{y1})}{\partial\eta} + A\frac{\partial(n^0 p_{x1})}{\partial\xi} - B\eta\frac{\partial\psi}{\partial\eta}\frac{\partial(n^0 p_{x1})}{\partial\xi} \\
&+\Big(B\eta\frac{\partial\psi}{\partial\xi}\frac{\partial(n^0 p_{x1})}{\partial\eta} - \frac{\partial p_{y1}}{\partial\xi}\frac{\partial(n^0 p_{x1})}{\partial\eta}t_2 - \frac{\partial p_{x1}}{\partial\eta}\frac{\partial(n^0 p_{y1})}{\partial\xi}t_2 \\
&+B\xi\frac{\partial\psi}{\partial\eta}\frac{\partial(n^0 p_{y1})}{\partial\xi}\Big) \\
W \;=\;& p_z^e\Big(A^2 - AB\eta\frac{\partial\psi}{\partial\eta} + A\frac{\partial p_{y1}}{\partial\eta}t_2 - AB\xi\frac{\partial\psi}{\partial\xi} - B\xi\frac{\partial\psi}{\partial\xi}\frac{\partial p_{y1}}{\partial\eta}t_2 \\
&+A\frac{\partial p_{x1}}{\partial\xi}t_2 - B\eta\frac{\partial p_{x1}}{\partial\xi}\frac{\partial\psi}{\partial\eta}t_2 + \frac{\partial p_{x1}}{\partial\xi}\frac{\partial p_{y1}}{\partial\eta}t_2^2\Big) \\
&+p_z^e\left(\frac{B\eta\partial\psi}{\partial\xi}\frac{\partial p_{x1}}{\partial\eta}t_2 - \frac{\partial p_{x1}}{\partial\eta}\frac{\partial p_{y1}}{\partial\xi}t_2^2 + B\xi\frac{\partial\psi}{\partial\eta}\frac{\partial p_{y1}}{\partial\xi}t_2\right) \\
&-Ap_{y1}\frac{\partial p_{z1}}{\partial\eta}t_2 + B\xi p_{y1}\frac{\partial\psi}{\partial\xi}\frac{\partial p_{z1}}{\partial\eta}t_2 - p_{y1}\frac{\partial p_{x1}}{\partial\xi}\frac{\partial p_{z1}}{\partial\eta}t_2^2
\end{aligned}
$$

$$+p_z^e\left(\frac{B\eta\partial\psi}{\partial\xi}\frac{\partial p_{x1}}{\partial\eta}t_2-\frac{\partial p_{x1}}{\partial\eta}\frac{\partial p_{y1}}{\partial\xi}t_2^2+B\xi\frac{\partial\psi}{\partial\eta}\frac{\partial p_{y1}}{\partial\xi}t_2\right)$$
$$-Ap_{y1}\frac{\partial p_{z1}}{\partial\eta}t_2+B\xi p_{y1}\frac{\partial\psi}{\partial\xi}\frac{\partial p_{z1}}{\partial\eta}t_2-p_{y1}\frac{\partial p_{x1}}{\partial\xi}\frac{\partial p_{z1}}{\partial\eta}t_2^2$$
$$-B\eta p_{x1}\frac{\partial\psi}{\partial\xi}\frac{\partial p_{z1}}{\partial\eta}t_2+p_{x1}\frac{\partial p_{z1}}{\partial\eta}\frac{\partial p_{y1}}{\partial\xi}t_2^2-B\eta p_{y1}\frac{\partial\psi}{\partial\eta}\frac{\partial p_{z1}}{\partial\xi}$$
$$+p_{y1}\frac{\partial p_{x1}}{\partial\eta}\frac{\partial p_{z1}}{\partial\xi}t_2^2-Ap_{x1}\frac{\partial p_{z1}}{\partial\xi}t_2+B\eta p_{x1}\frac{\partial\psi}{\partial\eta}\frac{\partial p_{z1}}{\partial\xi}t_2$$
$$-p_{x1}\frac{\partial p_{y1}}{\partial\eta}\frac{\partial p_{z1}}{\partial\xi}t_2^2$$

By using the relations

$$\frac{\partial p_{x1}}{\partial\xi}=-\beta\left(B+\xi A_2\frac{\partial\psi}{d\xi}\right),\ \frac{\partial p_{x1}}{\partial\eta}=-\beta\xi A_2\frac{\partial\psi_1}{d\eta},\qquad p_{x1}=-\beta\xi B$$
$$\frac{\partial p_{y1}}{\partial\xi}=-\beta\eta A_2\frac{\partial\psi_1}{d\xi},\qquad \frac{\partial p_{y1}}{\partial\eta}=-\beta\left(B+A_2\eta\frac{\partial\psi}{d\eta}\right),\ p_{y1}=-\beta\eta B$$
$$\frac{\partial\psi_1}{\partial\xi}=\frac{\beta\epsilon_c L}{p_{z0}^3}\left(b'^2-2c'b'^4r_0^2\right)\xi\qquad \frac{\partial\psi_1}{\partial\eta}=\frac{\beta\epsilon_c L}{p_{z0}^3}\left(b'^2-2c'b'^4r_0^2\right)\eta$$
$$p_{z0}=\sqrt{\epsilon-p_{x0}^2-p_{y0}^2}\qquad p_{z1}=\sqrt{1-p_{x1}^2-p_{y1}^2}$$

where

$$A_1=\frac{-\chi\sin\theta\cos\theta(\cos\phi n^0p_{x1}+\sin\phi n^0p_{y1})}{1+\chi\cos^2\theta},$$
$$B_1=\frac{(p^0)^2(1+\chi)-(n^0)^2(p_{x1}^2+p_{y1}^2)}{1+\chi\cos^2\theta}-\frac{A_1^2}{\chi\cos^2\theta}$$

and

$$A=[(1+g)\cos\psi_1-g\cos3\psi_1],\quad B=[(1+g)\sin\psi_1-3g\sin3\psi_1]$$
$$A_2=[(1+g)\cos\psi_1-9g\cos3\psi_1]$$
$$\frac{\partial p_z^e}{\partial\xi}=-\frac{\partial A_1}{\partial\xi}+\frac{1}{2\sqrt{B_1}}\frac{\partial B_1}{\partial\xi}$$
$$\frac{\partial p_z^e}{\partial\eta}=\frac{\partial A_1}{\partial\eta}+\frac{1}{2\sqrt{B_1}}\frac{\partial B_1}{\partial\eta}$$

Jacobian of coordinates transformation may be derived as

$$J(t)=\frac{D(t)}{D(0)}=\frac{U}{W}t^2+\frac{V}{W}t+1$$

APPENDIX C. DERIVATION OF THE PHASE

$$\begin{aligned}
&\Psi_2(p_{x1}, p_{y1}, z)\\
&= \Psi_0' + t_2 + t - p_{x1} x(p_{x1}, p_{y1}, t) - p_{y1} y(p_{x1}, p_{y1}, t) + p_{x1} x + p_{y1} y\\
&= \Psi_0'' + t_2 + t - p_{x1}(\xi_2 + n^o p_{x1} t) - p_{y1}(\eta_2 + n^o p_{y1} t) + p_{x1} x + p_{y1} y\\
&= \Psi_0' + t_2 + t - p_{x1}(\xi_1 + p_{x1} t_2 + n^o p_{x1} t)\\
&\quad - p_{y1}(\eta_1 + p_{y1} t_2 + n^o p_{y1} t) + p_{x1} x + p_{y1} y\\
&= \Psi_0' + (\zeta_2 - \zeta_1) p_{z1} + p_{x1}(x - \xi_1) + p_{y1}(y + \eta_1) + (z - \zeta_2) p_z^e\\
&= \Psi_0' + \beta r_0^2 [(1+g)\cos\psi_1 - g\cos 3\psi_1][(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\\
&\quad + \sqrt{\{1 - \beta^2 r_0^2 [(1+2g)\sin^2\psi_1 - 6g\sin\psi_1 \sin 3\psi_1]\}}(\zeta_2 - \zeta_1)\\
&\quad - \beta r r_0 [(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\cos(\delta - \phi) + (z - \zeta_2) p_z^e
\end{aligned}$$

where

$$\begin{aligned}
\xi &= r_0[(1+g)\cos\psi_1 - g\cos 3\psi_1]\cos\delta\\
\eta &= r_0[(1+g)\cos\psi_1 - g\cos 3\psi_1]\sin\delta\\
p_{x1} &= -\beta r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\cos\delta\\
p_{y1} &= -\beta r_0[(1+g)\sin\psi_1 - 3g\sin 3\psi_1]\sin\delta\\
x &= r\cos\phi\\
y &= r\sin\phi
\end{aligned}$$

have been used.

REFERENCES

1. Keller, J. B. and J. S. Papadakis, *Wave Propagation and Underwater Acoustics*, Springer-Verlag, Berlin, 1977.
2. Kline, M. and I. Kay, *Electromagnetic Theory and Geometrical Optics*, Interscience, New York, NY, 1965.
3. Budden, K., *The Propagation of Radio Waves*, Cambridge University Press, Cambridge, U.K., 1985.
4. Maslov, V. and M. Fedoriuk, *Semiclassical Approximation in Quantum Mechanics*, Reidel Publications Co., Dordrecht, Holland, 1981.
5. Klauder, J., *Ann. Phys.*, 180–108, N.Y., 1987.
6. Kravtsov, Y. A. and Y. I. Orlov, *Caustics, Catastrophes, and Wave Fields*, 2nd edition, Springer-Verlag, Berlin, 1998.

7. Zhu, T., "A ray-Kirchhoff method for body-wave calculations in inhomogeneous media: Theory," *Geophys. J.*, Vol. 92, 181–193, 1988.
8. Maslov, V. P., *Perturbation Theory and Asymptotic Method*, Moskov., Gos. Univ., Moscow, 1965 (in Russian). (Translated into Japanese by Ouchi et al., Iwanami, Tokyo, 1976.)
9. Kravtsov, Y. A., "Two new methods in the theory of wave propagation in inhomogeneous media (review)," *Sov. Phys. Acoust.*, Vol. 14, No. 1, 1–17, 1968.
10. Kravtsov, Y. A. and Y. I. Orlov, *Caustics, Catastrophes, and Wave Fields*, 306, Springer-Verlag, Berlin, 1993, 2nd edition, 1997.
11. Kravtsov, Y. A. and Y. I. Orlov, *Geometrical Optics in Inhomogeneous Media*, 312, Springer-Verlag, Berlin, 1990.
12. Kravtsov, Y. A. and Y. I. Orlov, "Geometrical optics in engineering physics," *Alpha Science*, International Ltd., UK, 2005. 1990.
13. Gorman, A. D., S. P. Anderson, and R. B. Mohindra, "On caustic related to several common indices of refraction," *Radio Sci.*, Vol. 21, 434–436, 1986.
14. Gorman, A. D., "Vector field near caustics," *J. Math. Phys.*, Vol. 26, 1404–1407, 1985.
15. Ziolkowski, R. W. and G. A. Deschamps, "Asymptotic evaluation of high frequency field near a caustic: An introduction to Maslov's method," *Radio Sci.*, Vol. 19, 1001–1025, 1984.
16. Hongo, K., Y. Ji, and E. Nakajimi, "High-frequency expression for the field in the caustic region of a reflector using Maslov's method," *Radio Sci.*, Vol. 21, No. 6, 911–919, 1986.
17. Hongo, K. and Y. Ji, "High-frequency expression for the field in the caustic region of a cylindrical reflector using Maslov's method," *Radio Sci.*, Vol. 22, No. 3, 357–366, 1987.
18. Hongo, K. and Y. Ji, "Study of the field around the focal region of spherical reflector antenna by Maslov's method," *IEEE Trans.Antennas Propagat.*, Vol. 36, 592–598, May 1988.
19. Ji, Y. and K. Hongo, "Field in the focal region of a dielectric spherical by Maslov's method," *J. Opt. Soc. Am. A*, Vol. 8, 1721–1728, 1991.
20. Ji, Y. and K. Hongo, "Analysis of electromagnetic waves refracted by a spherical dielectric interface by Maslov's method," *J. Opt. Soc. Am. A*, Vol. 8, 541–548, 1991.
21. Ghaffar, A., Q. A. Naqvi, and K. Hongo, "Analysis of the fields in three dimensional Cassegrain system," *Progress In*

Electromagnetics Research, PIER 72, 215–240, 2007.

22. Ghaffar, A., A. Hussain, Q. A. Naqvi, and K. Hongo, "Radiation characteristics of an inhomogeneous slab," *J. of Electromagnetic. Waves and Appl.*, Vol. 22, No. 2, 301–312, 2008.
23. Ghaffar, A., Q. A. Naqvi, and K. Hongo, "Study of focusing of field refracted by a cylindrical plano-convex lens into a uniaxial crystal by using Maslov's method," *J. of Electromagnetic. Waves and Appl.*, Vol. 22, 665–679, 2008.
24. Ghaffar, A., Q. A. Naqvi, and K. Hongo, "Focal region fields of three dimensional Gregorian system," *Optics Communications*, Vol. 281, 1343–1353, 2008.
25. Ghaffar, A. and Q. A. Naqvi, "Focusing of electromagnetic plane wave into uniaxial crystal by a three dimensional plano convex lens," *Progress In Electromagnetics Research*, PIER 83, 25–42, 2008.
26. Ghaffar, A. and Q. A. Naqvi, "Study of focusing of field refracted by an inhomogeneous slab into uniaxial crystal by using Maslov's method," Accepted for publication in *Journal of Modern Optics.*
27. Ghaffar, A. and Q. A. Naqvi, "Focusing of electromagnetic plane wave into uniaxial crystal by a Wood lens," Accepted for publication in *Central European Journal of Physics.*
28. Ashraf, M. R., A. Ghaffar, and Q. A. Naqvi, "Fields in the focal space of symmetrical hyperbolic focusing lens," *J. of Electromagn. Waves and Appl.*, Vol. 22, 815–828, 2008.
29. Fiaz, M. A., A. Ghaffar, and Q. A. Naqvi, "High-frequency expressions for the field in the caustic region of a PEMC cylinderical reflector using Maslov's method," *J. of Electromagn. Waves and Appl.*, Vol. 22, 385–397, 2008.
30. Fiaz, M. A., A. Ghaffar, and Q. A. Naqvi, "High-frequency expressions for the field in the caustic region of a PEMC Gregorian system using Maslov's method," *Progress In Electromagnetics Research*, PIER 81, 135–148, 2008.
31. Aziz, A., Q. A. Naqvi, and K. Hongo, "Analysis of the fields in two dimensional Cassegrain system," *Progress In Electromagnetics Research*, PIER 71, 227–241, 2007.
32. Aziz, A., A. Ghaffar, Q. A. Naqvi, and K. Hongo, "Analysis of the fields in two dimensional Gregorian system," *J. of Electromagn. Waves and Appl.*, Vol. 22, No. 1, 85–97, 2008.
33. Faryad, M. and Q. A. Naqvi, "Highfrequency expressions for the field in the caustic region of a cylindrical reflector placed in chiral medium," *Progress In Electromagnetics Research*, PIER 76, 153–

182, 2007.

34. Faryad, M. and Q. A. Naqvi, "Cylindrical reflector in chiral medium supporting simultaneously positive phase velocity and negative phase velocity," *J. of Electromagn. Waves and Appl.*, Vol. 22, 563–572, 2008.
35. Hussain, A., Q. A. Naqvi, and K. Hongo, "Radiation characteristics of the Wood lens using Maslov's method," *Progress In Electromagnetics Research*, PIER 73, 107–129, 2007.
36. Stamnes, J. J., *Waves in Focal Regions*, Adam Hilger, Bristolr Boston, 1986.
37. Stamnes, J. J. and G. C. Sherman, "Radiation of electromagnetic fields in uniaxially anisotropic media," *J. Opt. Soc. Am.*, Vol. 66, No. 8, 780–788, 1976.
38. Stamnes, J. J. and D. Jiang, "Focusing of electromagne waves into a uniaxial crystal," *Opt. Comm.*, Vol. 150, 251–262, 1998.
39. Jiang, D. and J. J. Stamnes, "Numerical and asymptotic results for focusing of two-dimensional electromagnetic waves in uniaxial crystals," *Opt. Comm.*, Vol. 163, 55–71, 1999.