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COMPARISON OF THE COULOMBIAN AND AMPERIAN CURRENT MODELS FOR CALCULATING THE MAGNETIC FIELD PRODUCED BY RADIALLY MAGNETIZED ARC-SHAPED PERMANENT MAGNETS

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Abstract—This paper presents some improved analytical expressions of the magnetic field produced by arc-shaped permanent magnets whose polarization is radial with the amperian current model. First, we show that the radial component of the magnetic field produced by a ring permanent magnet whose polarization is radial can be expressed in terms of elliptic integrals. Such an expression is useful for optimization purposes. We also present a semi-analytical expression of the axial component produced by the same configuration. For this component, we discuss the terms that are difficult to integrate analytically and compare our expression with the one established by Furlani [1]. In the second part of this paper, we use the amperian current model for calculating the magnetic field produced by a tile permanent magnet radially magnetized. This method was in fact still employed by Furlani for calculating the magnetic field produced by radially polarized cylinders. We show that it is possible to obtain a fully analytical expression of the radial component based on elliptic integrals. In addition, we show that the amperian current model allows us to obtain a fully analytical expression of the azimuthal component. All the expressions determined in this paper are compared with the ones established by Furlani [1] or in previous works carried out by the authors.

1. INTRODUCTION

More and more analytical approaches were proposed by many authors [1–3] for calculating the magnetic field created by arc-shaped permanent magnets whose polarizations can be radial [4–7] or axial [8–10]. There are mainly four analytical methods that are used for

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calculating the three magnetic field components produced by tile permanent magnets or ring permanent magnets radially magnetized. These four methods derived directly from the Maxwell's equations and the properties of the magnetic field shape. Consequently, some authors used the coulombian model or the magnetic potential [11–14] while other authors used the amperian current model and the Biot-Savart Law [15–18] for calculating the magnetic field created by currents or permanent magnets. However, these analytical methods are not suitable for the study of iron-core structures. This is why some alternative methods are based on series expansions [19], or finite-element methods are used for the design of permanent-magnet topologies [20].

The coulombian model implies the existence of magnetic charges located on the faces of a magnet and inside it. The amperian current model implies the determination of the vector potential produced by fictitious currents flowing around or inside a magnet. The two approaches are equivalent. The problem is thus to guess what model is the most appropriate for calculating the three components of the magnetic field produced by permanent magnets. It does not seem to be more difficult to use the amperian current model rather than the coulombian model for calculating the magnetic field created by parallelepiped magnets. Indeed, the coulombian model applied to a parallelepiped magnet whose polarization is directed along the normal direction implies two times the calculation of two surface integrals whereas the amperian current model implies four times the calculation of two surface integrals.

However, the two analytical methods lead to a fully analytical expression of the magnetic field [21]. This is why this configuration is often used by authors for optimization purposes: it has a very low computational cost [22–26]. However, as shown in [5], this method can only be used in the near-field if parallelepiped magnets are used for representing arc-shaped permanent magnets. It is emphasized here that numerous approaches based on series expansions have been proposed [27–29] while semi-analytical approaches have led to compute the magnetic field in all points in space [30]. Moreover, other analytical methods allow the determination of the force between coil currents [31, 32].

For arc-shaped permanent magnets, it seems to be more difficult to guess what model is the most appropriate. For arc-shaped permanent magnets whose polarization is radial, some new analytical expressions are derived from the vector potential, not only for ring but also for tile permanent magnets. These expressions are compared with the ones established by Furlani [1], Babic [4], Rakotoarison [30] and Ravaud [7].

Moreover, we discuss the validity and the computational cost of all these expressions. A Mathematica file is available online for showing the validity of the expressions obtained and the equivalence between the amperian current and couloumbian models [33].

The first part of this paper deals with the calculation of the magnetic field created by ring permanent magnets radially magnetized, and the second part of this paper deals with the magnetic field produced by tile permanent magnets radially magnetized. By using the amperian current model, we show how some expressions can be significantly improved and why some expressions cannot still be expressed entirely in terms of special functions.

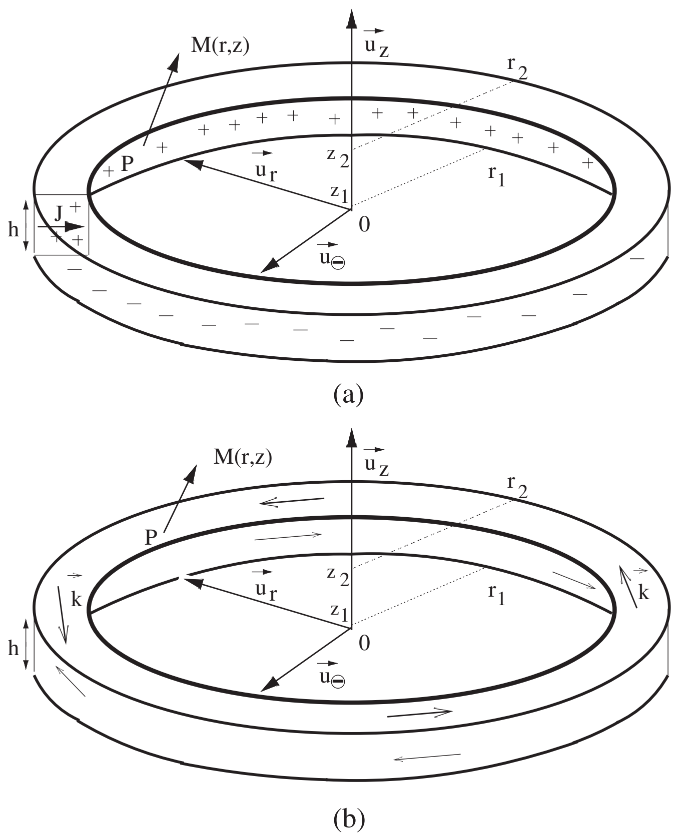

Figure 1. Representation of a ring permanent magnet radially magnetized: (a) coulombian model, (b) amperian current model.

2. RING PERMANENT MAGNETS WHOSE POLARIZATION IS RADIAL

2.1. Notation and Geometry

In this first part, we study the magnetic field produced by ring permanent magnets whose polarization is radial. For this purpose, let us consider the geometry shown in Figure 1. Figure 1(a) represents a ring permanent magnet radially magnetized according to the coulombian model. The ring inner radius is r_1, and its outer one is r_2. Its height is $h = z_2 - z_1$. Its magnetic polarization $\vec{J}$ is directed towards 0. Some fictitious charge surface densities are located on the inner and outer faces of the ring, and a charge volume density appears inside the ring. Consequently, as mentioned in [6], up to three integrals must be determined for obtaining an analytical expression of the three magnetic field components created by this ring permanent magnet. The contribution of the charge surface densities can be expressed in terms of elliptic integrals of the first, second and third kind [4] but the contribution of the charge volume density can be expressed in an analytical part based on elliptic functions and a numerical part [6]. Consequently, the coulombian model applied to this configuration does not seem to be the best model for obtaining a fully analytical expression of the magnetic field.

In this paper, we use the amperian current model for calculating the magnetic field produced by a ring permanent magnet radially magnetized, as shown in Figure 1(b). The magnetic polarization $\vec{J}$ is transformed into a surface density currents $\vec{k}$ flowing on the upper and lower faces of the ring. Such a model has been employed for example by Lang [9] in the case of axially magnetized ring permanent magnets. The ratio between $\vec{J}$ and $\vec{k}$ is expressed as follows:

$$\left|\vec{k}\right| = \frac{\left|\vec{J}\right|}{\mu_0} \tag{1}$$

By using the amperian current model in the case of radially magnetized ring permanent magnets, the expression of the radial component can be improved significantly because it can be expressed only in terms of elliptic integrals, and no further numerical integrations are required. Consequently, the time necessary to calculate this component is lower with the amperian current model than the coulombian model. However, the two models are equivalent and give the same results as it is shown in the next section. We present now the new magnetic field expressions determined with the amperian current model.

The analytical expression of the radial component can be determined by using either the vector potential or directly the Biot-Savart law. Indeed, by using the analogy between the coulombian model and amperian current model, the expression of the radial component can be expressed in terms of elliptic integrals. By denoting $\vec{r}$ the observation point and $\vec{r'}$ a point located on the charge distribution on the magnet, it is useful to define the reciprocal distance $\left|\vec{r}-\vec{r'}\right|$ as follows:

$$\frac{1}{\left|\vec{r}-\vec{r'}\right|}=\frac{1}{\sqrt{r^2+\tilde{r}^2-2r\tilde{r}\cos(\tilde{\theta})+(z-\tilde{z})^2}} \tag{2}$$

By using the amperian current model and as stated in [1], the potential vector created by a ring permanent magnet whose polarization is radial can be expressed as follows:

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\left(\int_{S_{up}}\frac{\vec{k}(\vec{r'})}{\left|\vec{r}-\vec{r'}\right|}d\tilde{S}_{up}-\int_{S_{down}}\frac{\vec{k}(\vec{r'})}{\left|\vec{r}-\vec{r'}\right|}d\tilde{S}_{down}\right) \tag{3}$$

where S_{up} is the upper surface of the ring ($z=z_2$), and S_{down} is the lower surface of the ring ($z=z_1$). Moreover, it is noted that in this analogy, the current volume density $\vec{k}_v$ is 0 because the polarization is supposed to be perfectly radial.

$$\vec{k}_v=\frac{\vec{\nabla}\wedge\vec{J}}{\mu_0}=\vec{0} \tag{4}$$

Thus, the magnetic field can be obtained by the following expressions:

$$\vec{H}(\vec{r})=\frac{1}{\mu_0}\vec{\nabla}\wedge\vec{A}(\vec{r}) \tag{5}$$

As there is a symmetry according θ in our configuration, the azimuthal component $H_\theta(r,z)$ equals zero, and the magnetic field depends only on r and z, that is:

$$\vec{H}(\vec{r})=\vec{H}(r,z)=H_r(r,z)\vec{u}_r+H_z(r,z)\vec{u}_z \tag{6}$$

For the rest of this paper, we use the operator $\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}[\bullet]$ for the case of ring permanent magnets whose polarization is radial. However, according to the convention chosen for calculating the three magnetic components, this operator can be replaced by $\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k+1}[\bullet]$. The modulus of the magnetic field created is always the same, but the sign of the magnetic components depends directly on the way that we define the axes. Such a way of writing the magnetic field expressions has also been adopted by Furlani [1] and

Babic [4]. The only influence of the choice of $(-1)^{i+k}[\bullet]$, $(-1)^{i+k+1}[\bullet]$ or $(-1)^{i+j+k}[\bullet]$ and $(-1)^{i+j+k+1}[\bullet]$ is the change in sign of the magnetic field component expressions.

2.2. Radial Component Determined with the Amperian Current Model

The radial component $H_r(r,z)$ can be determined by calculating the projection of $\vec{H}(\vec{r})$ along $\vec{u}_r$:

$$H_r(r,z) = \vec{H}(\vec{r}) \bullet \vec{u}_r = \left(\frac{1}{\mu_0}\vec{\nabla} \wedge \vec{A}(\vec{r})\right) \bullet \vec{u}_r \tag{7}$$

The interest of using the amperian current model for this configuration lies in the fact that only two integrals must be calculated. Consequently, we obtain an analytical expression of the radial field component based on elliptic integrals. By using Mathematica, the arguments of the elliptic integrals used are defined as follows:

$$\begin{aligned} \phi_{\mathbf{1}}^{+,-} &= \frac{(b+2e)x}{bx \pm \sqrt{2}\sqrt{xe^2(x-c)}} \\ \phi_{\mathbf{2}} &= i \sinh^{-1}\left(\sqrt{\frac{-1}{b+2e}}\sqrt{b-2e\cos(\tilde{\theta})}\right) \\ \phi_{\mathbf{3}} &= \frac{b+2e}{b-2e} \end{aligned} \tag{8}$$

It is noted that we use mainly the elliptic integrals of the second and third kind for the calculation of the radial component. These special functions have also been used in [5, 6]. However, no further numerical integrations are required here. In short, the radial component $H_r(r,z)$ can be expressed as follows:

$$H_r(r,z) = \frac{J}{4\pi\mu_0}\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}\left(g(i,k,2\pi) - g(i,k,0)\right) \tag{9}$$

with

$$g\left(i,k,\tilde{\theta}\right) =$$

$$2(z-z_k) f\left(r^2+(z-z_k)^2, r^2+r_i^2+(z-z_k)^2, rr_i, -r^2-2(z-z_k)^2, \tilde{\theta}\right) \tag{10}$$

where

$$\begin{aligned} f\left(a,b,e,c,x,\tilde{\theta}\right) = & \eta\left(2\xi_1\left(2ce^2+\xi_2\right)\right)\mathbf{F}^*\left[\phi_{\mathbf{2}},\phi_{\mathbf{3}}\right] \\ & +\eta\left(-e^2(c-x)\left(bx\sqrt{2}+2\xi_1\right)\right)\mathbf{\Pi}^*\left[\phi_1^+,\phi_2,\phi_3\right] \\ & +\eta\left(e^2(c-x)\left(bx\sqrt{2}-2\xi_1\right)\right)\mathbf{\Pi}^*\left[\phi_1^-,\phi_2,\phi_3\right] \\ & -2\eta ax\left(xe^2-ce^2\sqrt{2}+b\xi_1\right)\mathbf{\Pi}^*\left[\phi_{\mathbf{1}}^+,\phi_{\mathbf{2}},\phi_{\mathbf{3}}\right] \\ & -2\eta ax\left(-xe^2+ce^2\sqrt{2}+b\xi_1\right)\mathbf{\Pi}^*\left[\phi_{\mathbf{1}}^-,\phi_{\mathbf{2}},\phi_{\mathbf{3}}\right] \end{aligned} \quad (11)$$

where $\mathbf{F}^*[\mathbf{x},\mathbf{y}]$ and $\mathbf{\Pi}^*[\mathbf{x},\mathbf{y},\mathbf{z}]$ are the elliptic integrals of the second and third kind that have been used in previous papers [5, 6].

In addition, the parameters ξ_1,ξ_2,η are defined as follows:

$$\begin{aligned} \xi_1 &= \sqrt{e^2x(x-c)} \\ \xi_2 &= x(b^2-2e^2) \\ \eta &= \frac{i\sqrt{\frac{-e^2\sin(\tilde{\theta})^2}{(b-2e)^2}}\csc\left(\tilde{\theta}\right)}{2\sqrt{\frac{-1}{b+2e}}x\xi_1(2ce^2+\xi_2)} \end{aligned} \quad (12)$$

Therefore, by using the amperian current model applied to a ring permanent magnet radially magnetized, it is possible to obtain a fully analytical expression of the radial field component. This result is of great importance because it can be used for optimizing the radial field in ironless loudspeakers [34, 35].

2.3. Illustration of the Analogy between the Amperian Current and Coulombian Models for the Radial Component Expression $H_r(r,z)$

A comparison has been performed for verifying the accuracy of the radial component expression with the expressions determined in [6] and [7]. It is noted that this expression can also be compared to the expression determined by Furlani by using $\theta_s(2)-\theta_s(1)=2\pi$ and by neglecting the current densities flowing in the axial direction (see [1]). We take the following dimensions: $r_1=0.025\,\mathrm{m}$, $r_2=0.028\,\mathrm{m}$, $h=z_2-z_1=0.003\,\mathrm{m}$, $J=1\,\mathrm{T}$, $z=0.0035\,\mathrm{m}$. We represent in Figure 2 the radial component $H_r(r,z=0.0035\,\mathrm{m})$ versus the axial displacement z. Figure 2 clearly shows that the analytical expression determined in this paper is accurate and can be used instead of the one established in [7] because it is easier to use for optimization purposes. In fact, we

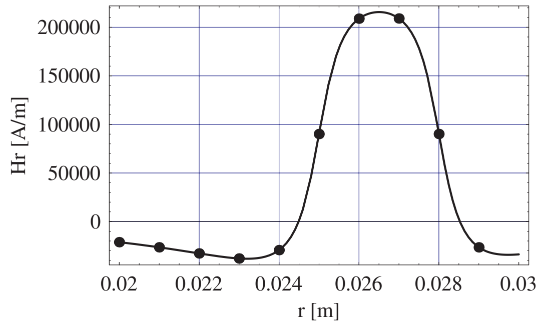

Figure 2. Representation of the radial component $H_r(r, z = 0.0035)$ versus the radial displacement r with the following values: $r_1 = 0.025\,\mathrm{m}$, $r_2 = 0.028\,\mathrm{m}$, $h = z_2 - z_1 = 0.003\,\mathrm{m}$, $J = 1\,\mathrm{T}$; (thick line = amperian current model), (points = coulombian model).

can see that the choice of the model used is very important because it seems to be more difficult to obtain a fully analytical expression of the radial component $H_r(r, z)$ in the coulombian approach. On the other hand, this does not imply that the amperian current model is always more appropriate than the coulombian model, as shown in [36] in which radial currents in massive disks generate a magnetic field that can be deducted more easily from the coulombian model. In fact, more generally, for cylindrical geometries, the authors feel that the analysis of the geometry studied must receive particular attention according to the choice of the model used.

2.4. Axial Component $H_z(r, z)$

The axial component $H_z(r, z)$ can be determined by calculating the projection of $\vec{H}(\vec{r})$ along $\vec{u}_z$:

$$H_z(r, z) = \vec{H}\,(\vec{r}\,) \bullet \vec{u}_z = \left(\frac{1}{\mu_0}\vec{\nabla} \wedge \vec{A}(\vec{r})\right) \bullet \vec{u}_z \tag{13}$$

This axial component $H_z(r, z)$ is thus given by:

$$H_z(r, z) = \frac{J}{4\pi\mu_0}\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}\left(I_1(r, z) + I_2(r, z)\right) \tag{14}$$

with

$$I_1(r, z) = \frac{-4r_i}{\sqrt{(r - r_i)^2 + (z - z_k)^2}}\mathbf{K}^*\left[\frac{-4rr_i}{(r - r_i)^2 + (z - z_k)^2}\right] \tag{15}$$

where $\mathbf{K}^*[x]$ is the complete elliptic integral of the first kind [5].

$$I_2(r,z)=\int_0^{2\pi}\log\left[r_i-r\cos(\tilde{\theta})+\sqrt{r^2+r_i^2+(z-z_k)^2-2rr_i\cos(\tilde{\theta})}\right]d\tilde{\theta} \tag{16}$$

It is emphasized here that this expression is given in a more compact form than the one determined in a previous paper by the authors [7]. Consequently, it seems to be more judicious to use the amperian current model for calculating the axial component of the magnetic field produced by a ring permanent magnet radially magnetized rather than the coulombian model. However, there is still a part of this expression that can be determined by using a numerical integration.

2.5. Illustration of the Analogy between the Amperian Current and Coulombian Models for the Radial Component Expression $H_z(r,z)$

A comparison has been performed for verifying the accuracy of the axial component expression with the expressions determined in [6] and [7]. It is noted that this expression can also be compared to the expression determined by Furlani by using $\theta_s(2)-\theta_s(1)=2\pi$. We take the following dimensions: $r_1=0.025\,\text{m}$, $r_2=0.028\,\text{m}$, $h=z_2-z_1=0.003\,\text{m}$, $J=1\,\text{T}$, $r=0.024\,\text{m}$. We represent in Figure 3 the axial component $H_z(r=0.024\,\text{m},z)$ versus the axial displacement z. Figure 3 clearly shows that the analytical expression determined in this paper is accurate and can be used instead of the one established in [7] because it has a lower computational cost, and it is given in a more compact form.

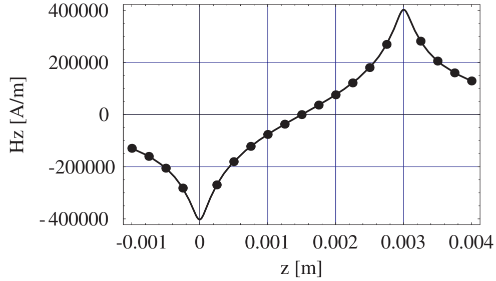

Figure 3. Representation of the axial component H_z $(r=0.0249\,\text{m},z)$ versus the axial displacement z with the following values: $r_1=0.025\,\text{m}$, $r_2=0.028\,\text{m}$, $h=z_2-z_1=0.003\,\text{m}$, $J=1\,\text{T}$; (thick line = amperian current model), (points = coulombian model).

3. TILE PERMANENT MAGNETS WHOSE POLARIZATION IS RADIAL

3.1. Notation and Geometry

In this second part, we study the magnetic field produced by tile permanent magnets whose polarization is radial. For this purpose, let us consider the geometry shown in Figure 4. Figure 4(a) represents a tile permanent magnet radially magnetized according to the coulombian model. The ring inner radius is r_1, and its outer one is r_2. Its height is $h = z_2 - z_1$. Its magnetic polarization $\vec{J}$ is directed towards 0. Its angular width is $\theta_2 - \theta_1$. Some fictitious charge surface densities are located on the inner and outer faces of the ring, and a charge volume density appears inside the ring. Figure 4(b) represents a tile permanent magnet radially magnetized according to the amperian current model. As stated in the first section of this paper, we consider the current surface densities flowing around the tile permanent magnet.

In addition, we define the following operator for the rest of this paper:

$$\wp_{\mathbf{i},\mathbf{j},\mathbf{k}}\left[\bullet\right] = \frac{J}{4\pi\mu_0}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k}\left[\bullet\right] \tag{17}$$

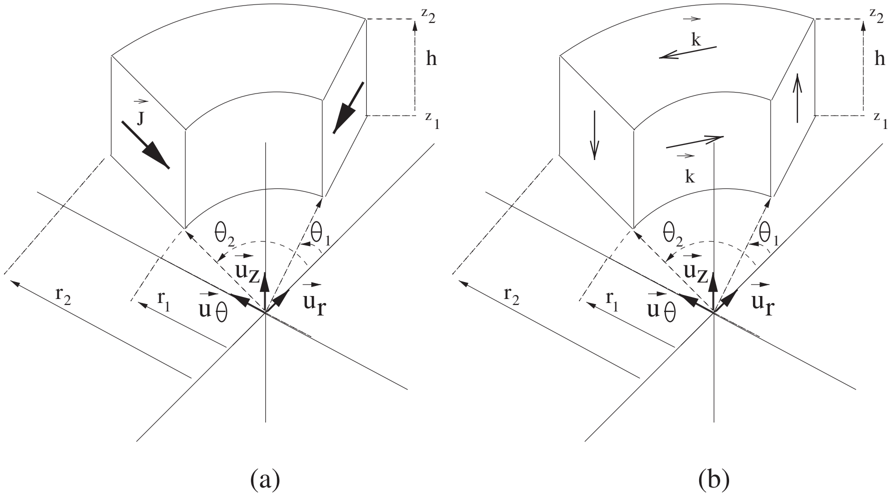

Figure 4. Representation of a tile permanent magnet radially magnetized: (a) coulombian model, (b) amperian current model.

3.2. Radial Component $H_r(r,\theta,z)$

We use the same equation as Furlani [1] for calculating the radial component of the magnetic field produced by a tile permanent magnet whose polarization is radial.

$$
\begin{aligned}
H_r(r,\theta,z) &= \frac{J}{4\pi\mu_0}\int_{z_1}^{z_2}\int_{r_1}^{r_2}\frac{\tilde{r}\sin\left(\theta-\theta_1\right)}{\left|\vec{\tilde{r}}-\vec{\tilde{r}'}\left(\theta_1,\tilde{z}\right)\right|^3}d\tilde{r}d\tilde{z}\\
&\quad -\frac{J}{4\pi\mu_0}\int_{z_1}^{z_2}\int_{r_1}^{r_2}\frac{\tilde{r}\sin\left(\theta-\theta_2\right)}{\left|\vec{\tilde{r}}-\vec{\tilde{r}'}\left(\theta_2,\tilde{z}\right)\right|^3}d\tilde{r}d\tilde{z}\\
&\quad +\frac{J}{4\pi\mu_0}\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\frac{\left(z-z_2\right)\cos\left(\theta-\tilde{\theta}\right)}{\left|\vec{\tilde{r}}-\vec{\tilde{r}'}\left(\tilde{\theta},z_2\right)\right|^3}\tilde{r}d\tilde{\theta}d\tilde{r}\\
&\quad -\frac{J}{4\pi\mu_0}\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\frac{\left(z-z_1\right)\cos\left(\theta-\tilde{\theta}\right)}{\left|\vec{\tilde{r}}-\vec{\tilde{r}'}\left(\tilde{\theta},z_1\right)\right|^3}\tilde{r}d\tilde{\theta}d\tilde{r}
\end{aligned}
\tag{18}
$$

where

$$
\frac{1}{\left|\vec{\tilde{r}}-\vec{\tilde{r}'}(\alpha,\beta)\right|}=\frac{1}{\sqrt{r^2+\tilde{r}^2-2r\tilde{r}\cos(\theta-\alpha)+(z-\beta)^2}}
\tag{19}
$$

It is noted that Eq. (18) can be decomposed in two parts. The contribution of the charges located on the curved planes (inner and outer faces of the ring) can be expressed in terms of elliptic integrals by using (9) in which $\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}\left(g(i,k,2\pi)-g(i,k,0)\right)$ is transformed into $\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}\left(g(i,k,\theta-\theta_2)-g(i,k,\theta-\theta_1)\right)$. Thus, the radial field component $H_r(r,\theta,z)$ can be decomposed as follows:

$$
\begin{aligned}
H_r(r,\theta,z) &= \frac{J}{4\pi\mu_0}\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}\left(g(i,k,\theta-\theta_2)-g(i,k,\theta-\theta_1)\right)\\
&\quad +\frac{J}{4\pi\mu_0}\int_{z_1}^{z_2}\int_{r_1}^{r_2}\frac{\tilde{r}\sin(\theta-\theta_1)}{\left|\vec{r}-\vec{r'}\right|^3}d\tilde{r}d\tilde{z}\\
&\quad -\frac{J}{4\pi\mu_0}\int_{z_1}^{z_2}\int_{r_1}^{r_2}\frac{\tilde{r}\sin(\theta-\theta_2)}{\left|\vec{r}-\vec{r'}\right|^3}d\tilde{r}d\tilde{z}
\end{aligned}
\tag{20}
$$

The last $\tilde{r}$ and $\tilde{z}$ integrations can be performed analytically, and we obtain the following expression for the radial field component:

$$H_r(r,\theta,z) = \wp_{\mathbf{i,j,k}}\left[g(i,k,\theta-\theta_j) + k_{i,j,k} + l_{i,j,k}\right] \tag{21}$$

$$k_{i,j,k} = \sin(\theta-\theta_j)\log\left[Y_{i,j,k}\right] \tag{22}$$

$$l_{i,j,k} = -\cos(\theta-\theta_j)\arctan\left[X_{i,j,k}\right] \tag{23}$$

$$Y_{i,j,k} = (z-z_k) + \sqrt{r^2+r_i^2+(z-z_k)^2-2rr_i\cos(\theta-\theta_j)}$$

$$X_{i,j,k} = \frac{(z-z_k)(r_i-r\cos(\theta-\theta_j))}{r\sin(\theta-\theta_j)\sqrt{r^2+r_i^2+(z-z_k)^2-2rr_i\cos(\theta-\theta_j)}} \tag{24}$$

As a conclusion, it is possible to obtain an analytical expression of the radial field component produced by a tile permanent magnet radially magnetized. This result is interesting for the optimization of radial fields in electrical machines.

3.3. Illustration of the Analogy between the Amperian Current and Coulombian Models for the Radial Component Expression $H_r(r,z)$ in the Case of a Tile Permanent Magnet

The expression of the radial field component produced by a tile permanent magnet radially magnetized is compared with the expression obtained by Furlani [1]. We take the following dimensions: $r_1 = 0.025\,\text{m}$, $r_2 = 0.028\,\text{m}$, $h = z_2 - z_1 = 0.003\,\text{m}$, $J = 1\,\text{T}$, $r = 0.02\,\text{m}$. We represent in Figure 5 H_r $(r = 0.024\,\text{m}, z)$ versus the angular displacement θ.

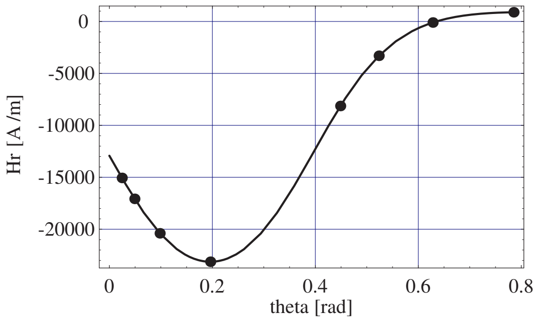

Figure 5. Representation of the radial component H_r $(r = 0.02\,\text{m}, \theta, z = 0.0035\,\text{m}))$ versus the angular displacement θ with the following dimensions: $r_1 = 0.025\,\text{m}$, $r_2 = 0.028\,\text{m}$, $h = z_2 - z_1 = 0.003\,\text{m}$, $J = 1\,\text{T}$, $r = 0.02\,\text{m}$; (thick line = this work), (Points = Furlani's work [1]).

3.4. Azimuthal Component $H_\theta(r, \theta, z)$

Let us now consider the azimuthal component $H_\theta(r, \theta, z)$ of the magnetic field produced by a tile permanent magnet radially magnetized. We use the same equation as Furlani [1] for calculating this component. Therefore, by using the amperian current model, the azimuthal component is expressed as follows:

$$H_\theta(r,\theta,z) = \frac{J}{4\pi\mu_0}\sum_{j=1}^{2}(-1)^j\left(\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\frac{(z-\tilde{z})\sin\left(\theta-\tilde{\theta}\right)}{\left|\vec{r}-\vec{r'}_j\left(\tilde{r},\tilde{\theta}\right)\right|^3}\right)\tilde{r}d\tilde{r}d\tilde{\theta}$$
$$+\frac{J}{4\pi\mu_0}\sum_{j=1}^{2}(-1)^j\left(\int_{r_1}^{r_2}\int_{z_1}^{z_2}\frac{r-\tilde{r}\cos(\theta-\theta_j)}{\left|\vec{r}-\vec{r'}\left(\tilde{r},\tilde{z}\right)\right|^3}\right)d\tilde{r}d\tilde{z} \quad (25)$$

where

$$\frac{1}{\left|\vec{r}-\vec{r'}_j(\alpha,\beta)\right|} = \frac{1}{\sqrt{r^2+\alpha^2-2r\alpha\cos(\theta-\beta)+(z-z_j)^2}}$$
$$\frac{1}{\left|\vec{r}-\vec{r'}(\alpha,\beta)\right|} = \frac{1}{\sqrt{r^2+\alpha^2-2r\alpha\cos(\theta-\theta_j)+(z-\beta)^2}} \quad (26)$$

We find:

$$H_\theta(r,\theta,z) = \wp_{\mathbf{i,j,k}}\left[-\cos(\theta-\theta_j)\log\left[Y_{i,j,k}\right]-\sin(\theta-\theta_j)\arctan\left[X_{i,j,k}\right]\right]$$
$$+\wp_{\mathbf{i,j,k}}\left[\frac{(z-z_k)}{r}\left(r_i-r\cos(\theta-\theta_j)+\xi_{i,j,k}\right)\right] \quad (27)$$

where

$$\xi_{i,j,k} = \sqrt{r^2+r_i^2+(z-z_k)^2-2rr_i\cos(\theta-\theta_j)} \quad (28)$$

Therefore, it is possible to obtain a fully analytical expression of the azimuthal field component by using the amperian current model.

3.5. Illustration of the Analogy between the Amperian Current and Coulombian Models for the Azimuthal Component Expression $H_\theta(r, z)$ in the Case of a Tile Permanent Magnet

The expression of the azimuthal field component produced by a tile permanent magnet radially magnetized is compared with the expression obtained by Furlani [1]. We take the following dimensions: $r_1 = 0.025\,\text{m}$, $r_2 = 0.028\,\text{m}$, $h = z_2 - z_1 = 0.003\,\text{m}$, $J = 1\,\text{T}$,

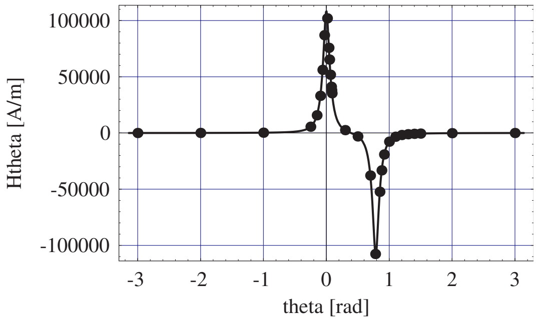

Figure 6. Representation of the azimuthal component H_θ ($r = 0.024, \theta, z = 0.002$)) versus the angular displacement θ with the following dimensions: $r_1 = 0.025\,\mathrm{m}$, $r_2 = 0.028\,\mathrm{m}$, $h = z_2 - z_1 = 0.003\,\mathrm{m}$, $J = 1\,\mathrm{T}$, $r = 0.024\,\mathrm{m}$, $\theta_2 - \theta_1 = \frac{\pi}{4}$ rad; (thick line = this work), (Points = Furlani's work [1]).

$r = 0.024\,\mathrm{m}$, $\theta_2 - \theta_1 = \frac{\pi}{4}$. We represent in Figure 7 the axial component H_z ($r = 0.024, \theta, z = 0.002$) versus the angular displacement θ.

Figure 6 shows that the fully analytical expression obtained in this work for the azimuthal component gives the same azimuthal field as the one established by Furlani [1]. However, the expression we give in this paper does not use any special functions or numerical integrals. Consequently, such an expression has a lower computational cost and is useful for optimization purposes.

3.6. Axial Component $H_z(r, \theta, z)$

The axial component $H_z(r, \theta, z)$ is also determined with the same integral formulation of Furlani [1].

$$\begin{aligned} H_z(r,\theta,z) &= \frac{J}{4\pi\mu_0}\sum_{j=1}^{2}(-1)^j\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\frac{r\tilde{r}\cos\left(\theta-\tilde{\theta}\right)}{\left|\vec{r}-\vec{\tilde{r}'}(\alpha,\gamma)\right|^3}d\tilde{r}d\tilde{\theta} \\ &\quad -\frac{J}{4\pi\mu_0}\sum_{j=1}^{2}(-1)^j\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\frac{\tilde{r}^2}{\left|\vec{r}-\vec{\tilde{r}'}(\alpha,\gamma)\right|^3}d\tilde{r}d\tilde{\theta} \end{aligned} \quad (29)$$

with

$$\frac{1}{\left|\vec{r}-\vec{\tilde{r}'}(\alpha,\gamma)\right|} = \frac{1}{\sqrt{r^2+\gamma^2-2r\gamma\cos(\theta-\alpha)+(z-z_j)^2}} \quad (30)$$

After the two integrations, we obtain a compact form of the axial component:

$$H_z(r,\theta,z) =$$
$$\wp_{\mathbf{i,j,k}} \left[\frac{-2r_i}{(r-r_i)^2+(z-z_k)^2} \mathbf{F}^* \left[\frac{\theta-\theta_j}{2}, -\frac{4rr_i}{(r-r_i)^2+(z-z_k)^2} \right] \right]$$
$$+\frac{J}{4\pi\mu_0} \int_{\theta_1}^{\theta_2} \log \left[r_i - r\cos\left(\theta-\tilde{\theta}\right) + \xi_{\tilde{\theta}} \right] d\tilde{\theta} \tag{31}$$

with

$$\xi_{\tilde{\theta}} = \sqrt{r^2 + r_i^2 + (z-z_k)^2 - 2rr_i \cos\left(\theta-\tilde{\theta}\right)} \tag{32}$$

where $\mathbf{F}^*[\phi, m]$ gives the elliptic integral of the first kind [5]. This expression has also been improved compared to the one determined in [6] because it is presented in a more compact form.

3.7. Illustration of the Analogy between the Amperian Current and Coulombian Models for the Axial Component Expression $H_z(r, z)$ in the Case of a Tile Permanent Magnet

The expression of the axial field component produced by a tile permanent magnet radially magnetized is compared with the expression obtained by Furlani [1]. We take the following dimensions: $r_1 = 0.025\,\mathrm{m}$, $r_2 = 0.028\,\mathrm{m}$, $h = z_2 - z_1 = 0.003\,\mathrm{m}$, $J = 1\,\mathrm{T}$, $r = 0.015\,\mathrm{m}$, $\theta_2 - \theta_1 = \frac{\pi}{4}$. We represent in Figure 7 the axial component H_z $(r = 0.015, z = 0.002\,\mathrm{m})$ versus the angular displacement θ.

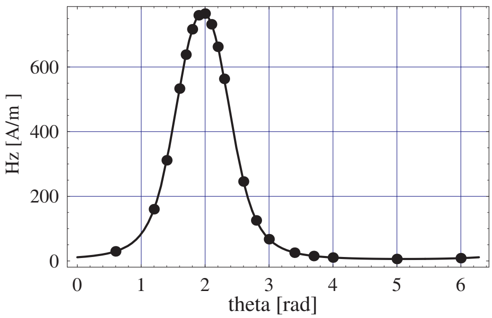

Figure 7. Representation of the axial component H_z $(r = 0.015, \theta, z = 0.002\,\mathrm{m}))$ versus the angular displacement θ with the following dimensions: $r_1 = 0.025\,\mathrm{m}$, $r_2 = 0.028\,\mathrm{m}$, $h = z_2 - z_1 = 0.003\,\mathrm{m}$, $J = 1\,\mathrm{T}$, $r = 0.015\,\mathrm{m}$, $\theta_2 - \theta_1 = \frac{\pi}{4}$; (thick line = this work), (Points = Furlani's work [1]).

Figure 7 shows that the reduced expression obtained in this work for the axial component gives the same axial field as the one established by Furlani [1].

4. CONCLUSION

This paper has presented some improved analytical expressions of the magnetic field produced by ring and tile permanent magnets radially magnetized. The radial field is given in terms of elliptic integrals, and no further numerical integrations are required. Consequently, the time necessary for calculating this radial field is very low and allows us to easily carry out parametric studies. In the case of tile permanent magnets radially magnetized, the amperian current model leads to a fully analytical expression of the azimuthal component, as in the case of the coulombian model. However, only two integrals must be determined with the amperian current model, and we think that this model is more appropriate than the coulombian model for the study of this azimuthal field. The axial field produced by a tile permanent magnet or a ring permanent magnet can be expressed as in an analytical part based on elliptic integrals and a numerical part. This expression has been improved compared to the one established by Furlani, but we have not succeeded in analytically integrating the last term. Nevertheless, we have presented the expression of the axial field in a compact form, and this expression seems to be easier to use than the one determined by the authors in a previous paper with the coulombian model [7]. The amperian current model seems to be more appropriate than the coulombian model for calculating the magnetic field produced by radially magnetized arc-shaped permanent magnets.

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