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FRACTIONAL RECTANGULAR IMPEDANCE WAVEGUIDE

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Abstract—Fractional rectangular impedance waveguide has been studied using fractional curl operator. Behavior of field inside the fractional rectangular impedance waveguide has been studied with respect to the original impedance of walls of the guide as well as fractional parameter. Analysis of the impedance of the walls as well as power distribution over the cross sectional plane of fractional impedance rectangular waveguide has been given. It has been found that fractional curl can be used to control the power distribution pattern over the cross sectional plane.

1. INTRODUCTION

Applications of fractional calculus in physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology and electrochemistry have made it an exciting new mathematical tool for solution of diverse problems in the field of science and engineering. It is the branch of mathematics that deals with operators having non-integer and/or complex order, e.g., fractional derivative and fractional integral. Fractional derivatives/integrals are mathematical operators involving differentiation/integration of arbitrary (non-integer) real or complex orders such as $d^\alpha f(x)/dx^\alpha$, where α can be taken to be a non integer real or even complex number [1]. In a sense, these operators effectively behave as the so-called intermediate cases between the integer-order differentiation and integration.

While bringing the tools of fractional calculus and electromagnetic theory together, Engheta [2–9] has explored and developed the subject of fractional paradigm in electromagnetic theory. It is the area in which fractional operators are used to model electromagnetic solutions.

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Curl operation is the basis of Maxwell equations and hence the electromagnetics. Fractionalization of the curl operator by Engheta has led us to novel solutions, interpretable as "fractional solutions", for certain electromagnetic problems.

In electromagnetics, principle of duality stats that if $(\mathbf{E}, \eta\mathbf{H})$ is one set of solutions (original solutions) to Maxwell equations, then other set of solutions (dual to the original solutions) is $(\eta\mathbf{H}, -\mathbf{E})$, where η is the impedance of the medium. The solutions which may be regarded as intermediate step between the original and dual to the original solutions can be obtained using the following relations [2]

$$\begin{aligned} \mathbf{E}_{\mathrm{fd}} &= \frac{1}{(ik)^\alpha}(\nabla\times)^\alpha \mathbf{E} \\ \eta\mathbf{H}_{\mathrm{fd}} &= \frac{1}{(ik)^\alpha}(\nabla\times)^\alpha \eta\mathbf{H} \end{aligned}$$

where $(\nabla\times)^\alpha$ means fractional curl operator and $k = \omega\sqrt{\mu\epsilon}$ is the wave number of the medium. It may be noted that fd means fractional dual solutions. It is obvious from above set of equations that

$$\begin{aligned} \alpha = 0 \Rightarrow \quad (\mathbf{E}_{\mathrm{fd}}, \eta\mathbf{H}_{\mathrm{fd}}) &= (\mathbf{E}, \eta\mathbf{H}) \\ \alpha = 1 \Rightarrow \quad (\mathbf{E}_{\mathrm{fd}}, \eta\mathbf{H}_{\mathrm{fd}}) &= (\eta\mathbf{H}, -\mathbf{E}) \end{aligned}$$

Hence $(\mathbf{E}, \eta\mathbf{H})$ and $(\eta\mathbf{H}, -\mathbf{E})$ are two sets of solutions to Maxwell equations. The solutions which may be regarded as intermediate step between the above two sets of solutions may be obtained by varying parameter α between zero and one.

Various investigations have been made in exploring the role of fractional duality. Naqvi et al. [10–15] derived the fractional dual solutions for different unbounded homogeneous media. These include dielectric, chiral and meta materials as the medium of propagation. Veliev et al. [17–21] addressed the problems of reflection and diffraction from the fractional surface boundaries. Hussain and Naqvi has used the concept of fractional dual solutions to transmission lines and waveguides which may be termed as fractional transmission lines and fractional waveguides [22–26]. Some other coworkers of Naqvi have also contributed towards fractional waveguides [30, 31]. In this work we have analyzed the fractional solutions of a rectangular waveguide with impedance walls. This is an extension of the work given in [30] which is for the perfect electric conductor (PEC) rectangular plates waveguide. The waveguide has been termed as fractional rectangular impedance waveguide.

General theory of rectangular waveguides has been given in Section 2, electric and magnetic fields inside the waveguide with impedance walls are derived in Section 3 while fractional solutions are

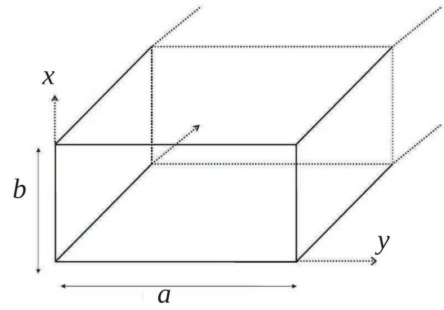


Figure 1. Geometry of rectangular waveguide.

derived in Section 4. The results have been discussed in Section 5 and the paper is concluded in Section 6.

2. GENERAL THEORY OF RECTANGULAR WAVEGUIDE

Consider a waveguide having rectangular cross section of size $b \times a$ in xy-plane while z-axis is the axis of propagation as shown in Figure 1. The guide has been considered as infinitely long along z-axis and filled with a dielectric medium of permittivity ϵ and permeability μ. As a general recipe, we solve Helmholtz equation for the axial components and the transverse components are derived from the the axial ones using Maxwell equations. Let us consider a case of transverse magnetic mode TM^z ($H_z = 0$) propagating through the guide. Axial component $E_z(x, y)$ can be written as a product of two independent functions as

$$E_z(x, y) = X(x)Y(y)$$

General solutions for $X(x)$ and $Y(y)$ can be written as

$$X(x) = A_m \cos(k_x x) + B_m \sin(k_x x) \quad \text{(1a)}$$

$$Y(y) = A_n \cos(k_y y) + B_n \sin(k_y y) \quad \text{(1b)}$$

where A_i and B_i, $(i = m, n)$ are constants which can be found from the boundary conditions while k_x and k_y are the components of the wave number $k = \omega\sqrt{\mu\epsilon}$ in the direction of x-axis and y-axis respectively. Once axial component of electric field is found as $E_z = X(x)Y(y)$, we can write the other field components using Maxwell curl equations as

$$E_z(x, y) = X(x)Y(y) \quad \text{(2a)}$$

$$E_x(x, y) = -\frac{1}{k_c^2}\left(i\beta\frac{\partial E_z(x, y)}{\partial x}\right) = -\frac{i\beta}{k_c^2}Y(y)X'(x) \quad \text{(2b)}$$

$$E_y(x,y) = -\frac{1}{k_c^2}\left(i\beta\frac{\partial E_z(x,y)}{\partial y}\right) = -\frac{i\beta}{k_c^2}X(x)Y'(y) \tag{2c}$$

$$H_x(x,y) = -\frac{1}{k_c^2}\left(-\frac{ik}{\eta}\frac{\partial E_z(x,y)}{\partial y}\right) = \frac{1}{k_c^2}\frac{ik}{\eta}X(x)Y'(y) \tag{2d}$$

$$H_y(x,y) = -\frac{1}{k_c^2}\left(\frac{ik}{\eta}\frac{\partial E_z(x,y)}{\partial x}\right) = -\frac{1}{k_c^2}\frac{ik}{\eta}Y(y)X'(x) \tag{2e}$$

Here β is the propagation constant, $k_c = \sqrt{k_x^2 + k_y^2}$ and $\eta = \sqrt{\mu/\epsilon}$ is the impedance of the medium inside the guide. Prime means derivative w.r.t. the argument.

3. RECTANGULAR WAVEGUIDE WITH IMPEDANCE WALLS

Let walls of the waveguide shown in Figure 1 has impedance Z_w. Fields inside the guide must satisfy the impedance boundary conditions as given below

$$E_z = -Z_w H_y \quad \text{at} \quad x = 0 \tag{3a}$$

$$E_z = Z_w H_y \quad \text{at} \quad x = a \tag{3b}$$

$$E_z = Z_w H_x \quad \text{at} \quad y = 0 \tag{3c}$$

$$E_z = -Z_w H_x \quad \text{at} \quad y = b \tag{3d}$$

Using these boundary conditions and the general solution given in Equation (2), we can write the field solution as

$$\begin{aligned} E_x(x,y) = \frac{A_{mn}}{2}\left(\frac{\beta k_x}{k_c^2}\right)&[\{-(1-F_xF_y)\,iS_{x-y} + (F_x - F_y)\,C_{x-y}\} \\ &+ \{(1+F_xF_y)\,iS_{x+y} - (F_x+F_y)\,C_{x+y}\}] \end{aligned} \tag{4a}$$

$$\begin{aligned} E_y(x,y) = \frac{A_{mn}}{2}\left(\frac{\beta k_y}{k_c^2}\right)&[\{(1-F_xF_y)\,iS_{x-y} - (F_x - F_y)\,C_{x-y}\} \\ &+ \{(1+F_xF_y)\,iS_{x+y} - (F_x+F_y)\,C_{x+y}\}] \end{aligned} \tag{4b}$$

$$\begin{aligned} E_z(x,y) = \frac{A_{mn}}{2}&[\{(1-F_xF_y)\,C_{x-y} - (F_x - F_y)\,iS_{x-y}\} \\ &- \{(1+F_xF_y)C_{x+y} - (F_x+F_y)iS_{x+y}\}] \end{aligned} \tag{4c}$$

$$\begin{aligned} \eta H_x(x,y) = \frac{A_{mn}}{2}\left(\frac{k k_y}{k_c^2}\right)&[\{(1-F_xF_y)\,iS_{x-y} - (F_x - F_y)\,C_{x-y}\} \\ &+ \{(1+F_xF_y)iS_{x+y} - (F_x+F_y)C_{x+y}\}] \end{aligned} \tag{4d}$$

$$\eta H_y(x,y) = \frac{A_{mn}}{2}\left(\frac{kk_x}{k_c^2}\right)[\{(1-F_xF_y)iS_{x-y}-(F_x-F_y)C_{x-y}\} \\ -\{(1+F_xF_y)iS_{x+y}-(F_x+F_y)C_{x+y}\}] \quad (4e)$$

where A_{mn} are constants that depend upon initial conditions. Other parameters are as given below

$$F_x = z_w\left(\frac{kk_x}{k_c^2}\right), \quad F_y = z_w\left(\frac{kk_y}{k_c^2}\right), \quad z_w = \frac{Z_w}{\eta}$$
$$C_{x-y} = \cos(k_x x - k_y y), \quad C_{x+y} = \cos(k_x x + k_y y)$$
$$S_{x-y} = \sin(k_x x - k_y y), \quad S_{x+y} = \sin(k_x x + k_y y)$$

while dispersion relations for the possible values of $\frac{k_x}{k_c}$ and $\frac{k_y}{k_c}$ can be written as

$$i\tan(k_x a) = \frac{2F_x}{1+F_x^2}, \quad i\tan(k_y b) = \frac{2F_y}{1+F_y^2}$$

4. FRACTIONAL RECTANGULAR IMPEDANCE WAVEGUIDE

Fields given in Equation (4) can be written in terms of four independent plane waves. Re-introducing the z-dependance ($e^{i\beta z}$), electric and magnetic fields of the four plane waves can be written as

$$\mathbf{E}_1 = \frac{A_{mn}}{4k_c^2}B_1\left(-\beta k_x\hat{\mathbf{x}}-\beta k_y\hat{\mathbf{y}}-k_c^2\hat{\mathbf{z}}\right)\exp\{i(-k_x x-k_y y+\beta z)\} \quad (5a)$$
$$\mathbf{E}_2 = \frac{A_{mn}}{4k_c^2}B_2\left(\beta k_x\hat{\mathbf{x}}-\beta k_y\hat{\mathbf{y}}+k_c^2\hat{\mathbf{z}}\right)\exp\{i(-k_x x+k_y y+\beta z)\} \quad (5b)$$
$$\mathbf{E}_3 = \frac{A_{mn}}{4k_c^2}B_3\left(-\beta k_x\hat{\mathbf{x}}+\beta k_y\hat{\mathbf{y}}-k_c^2\hat{\mathbf{z}}\right)\exp\{i(k_x x-k_y y+\beta z)\} \quad (5c)$$
$$\mathbf{E}_4 = \frac{A_{mn}}{4k_c^2}B_4\left(\beta k_x\hat{\mathbf{x}}+\beta k_y\hat{\mathbf{y}}-k_c^2\hat{\mathbf{z}}\right)\exp\{i(k_x x+k_y y+\beta z)\} \quad (5d)$$
$$\eta\mathbf{H}_1 = \frac{kA_{mn}}{4k_c^2}B_1(-k_y\hat{\mathbf{x}}+k_x\hat{\mathbf{y}})\exp\{i(-k_x x-k_y y+\beta z)\} \quad (5e)$$
$$\eta\mathbf{H}_2 = \frac{kA_{mn}}{4k_c^2}B_2(-k_y\hat{\mathbf{x}}-k_x\hat{\mathbf{y}})\exp\{i(-k_x x+k_y y+\beta z)\} \quad (5f)$$
$$\eta\mathbf{H}_3 = \frac{kA_{mn}}{4k_c^2}B_3(k_y\hat{\mathbf{x}}+k_x\hat{\mathbf{y}})\exp\{i(k_x x-k_y y+\beta z)\} \quad (5g)$$
$$\eta\mathbf{H}_4 = \frac{kA_{mn}}{4k_c^2}B_4(k_y\hat{\mathbf{x}}-k_x\hat{\mathbf{y}})\exp\{i(k_x x+k_y y+\beta z)\} \quad (5h)$$

where

$$B_1 = 1 + F_x F_y + F_x + F_y \tag{6a}$$
$$B_2 = 1 - F_x F_y + F_x - F_y \tag{6b}$$
$$B_3 = 1 - F_x F_y - F_x + F_y \tag{6c}$$
$$B_4 = 1 + F_x F_y - F_x - F_y \tag{6d}$$

As a general scheme [2], for a plane wave propagating in an arbitrary direction $\hat{\mathbf{k}}_l$, fractional dual electric and magnetic fields can be written as [10–35]

$$\mathbf{E}_{l\text{fd}} = [(a_{lj})^\alpha P_{lj} \mathbf{A}_{lj}]\, e^{i(\mathbf{k}_l \cdot \mathbf{r})}, \quad l = 1, 2, 3, 4, \quad j = 1, 2, 3 \tag{7a}$$
$$\eta \mathbf{H}_{l\text{fd}} = \hat{\mathbf{k}}_l \times \mathbf{E}_{l\text{fd}} \tag{7b}$$

where P_{lj} are the coefficients of expansion. a_{lj} and A_{lj} are the eigen values and eigen vectors of the cross product operator $(\hat{\mathbf{k}}_l \times)$ for the direction vectors $\hat{\mathbf{k}}_l$ which are given by

$$\hat{\mathbf{k}}_1 = \frac{1}{k} (-k_x \hat{\mathbf{x}} - k_y \hat{\mathbf{y}} + \beta \hat{\mathbf{z}}) \tag{8a}$$
$$\hat{\mathbf{k}}_2 = \frac{1}{k} (-k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}} + \beta \hat{\mathbf{z}}) \tag{8b}$$
$$\hat{\mathbf{k}}_3 = \frac{1}{k} (k_x \hat{\mathbf{x}} - k_y \hat{\mathbf{y}} + \beta \hat{\mathbf{z}}) \tag{8c}$$
$$\hat{\mathbf{k}}_4 = \frac{1}{k} (k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}} + \beta \hat{\mathbf{z}}) \tag{8d}$$

Hence fractional dual electric and magnetic fields corresponding to the four plane waves can be written as

$$\begin{aligned} \mathbf{E}_{1\text{fd}} = & \frac{A_{mn}}{4k_c^2} B_1 \exp(i\beta z) \\ & \left[\cos\left(\frac{\alpha\pi}{2}\right) \left\{-\beta k_x \hat{\mathbf{x}} - \beta k_y \hat{\mathbf{y}} - k_c^2 \hat{\mathbf{z}}\right\} + \sin\left(\frac{\alpha\pi}{2}\right) \left\{k k_y \hat{\mathbf{x}} - k k_x \hat{\mathbf{y}}\right\}\right] \\ & \exp\left[-i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[-i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \end{aligned} \tag{9a}$$

$$\begin{aligned} \mathbf{E}_{2\text{fd}} = & \frac{A_{mn}}{4k_c^2} B_2 \exp(i\beta z) \\ & \left[\cos\left(\frac{\alpha\pi}{2}\right) \left\{\beta k_x \hat{\mathbf{x}} - \beta k_y \hat{\mathbf{y}} + k_c^2 \hat{\mathbf{z}}\right\} - \sin\left(\frac{\alpha\pi}{2}\right) \left\{k k_y \hat{\mathbf{x}} + k k_x \hat{\mathbf{y}}\right\}\right] \\ & \exp\left[-i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \end{aligned} \tag{9b}$$

$$\mathbf{E}_{3\text{fd}} = \frac{A_{mn}}{4k_c^2} B_3 \exp(i\beta z)$$
$$\left[\cos\left(\frac{\alpha\pi}{2}\right)\left\{-\beta k_x \hat{\mathbf{x}} + \beta k_y \hat{\mathbf{y}} - k_c^2 \hat{\mathbf{z}}\right\} + \sin\left(\frac{\alpha\pi}{2}\right)\left\{k k_y \hat{\mathbf{x}} + k k_x \hat{\mathbf{y}}\right\}\right]$$
$$\exp\left[i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[-i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \tag{9c}$$

$$\mathbf{E}_{4\text{fd}} = \frac{A_{mn}}{4k_c^2} B_4 \exp(i\beta z)$$
$$\left[\cos\left(\frac{\alpha\pi}{2}\right)\left\{\beta k_x \hat{\mathbf{x}} + \beta k_y \hat{\mathbf{y}} - k_c^2 \hat{\mathbf{z}}\right\} - \sin\left(\frac{\alpha\pi}{2}\right)\left\{k k_y \hat{\mathbf{x}} - k k_x \hat{\mathbf{y}}\right\}\right]$$
$$\exp\left[i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \tag{9d}$$

$$\eta\mathbf{H}_{1\text{fd}} = \frac{A_{mn}}{4k_c^2} B_1 \exp(i\beta z)$$
$$\left[\sin\left(\frac{\alpha\pi}{2}\right)\left\{-\beta k_x \hat{\mathbf{x}} - \beta k_y \hat{\mathbf{y}} - k_c^2 \hat{\mathbf{z}}\right\} - \cos\left(\frac{\alpha\pi}{2}\right)\left\{k k_y \hat{\mathbf{x}} - k k_x \hat{\mathbf{y}}\right\}\right]$$
$$\exp\left[-i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[-i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \tag{9e}$$

$$\eta\mathbf{H}_{2\text{fd}} = \frac{A_{mn}}{4k_c^2} B_2 \exp(i\beta z)$$
$$\left[-\sin\left(\frac{\alpha\pi}{2}\right)\left\{\beta k_x \hat{\mathbf{x}} - \beta k_y \hat{\mathbf{y}} + k_c^2 \hat{\mathbf{z}}\right\} - \cos\left(\frac{\alpha\pi}{2}\right)\left\{k k_y \hat{\mathbf{x}} + k k_x \hat{\mathbf{y}}\right\}\right]$$
$$\exp\left[-i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \tag{9f}$$

$$\eta\mathbf{H}_{3\text{fd}} = \frac{A_{mn}}{4k_c^2} B_3 \exp(i\beta z)$$
$$\left[\sin\left(\frac{\alpha\pi}{2}\right)\left\{\beta k_x \hat{\mathbf{x}} - \beta k_y \hat{\mathbf{y}} + k_c^2) \hat{\mathbf{z}}\right\} + \cos\left(\frac{\alpha\pi}{2}\right)\left\{k k_y \hat{\mathbf{x}} + k k_x \hat{\mathbf{y}}\right\}\right]$$
$$\exp\left[i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[-i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \tag{9g}$$

$$\eta\mathbf{H}_{4\text{fd}} = \frac{A_{mn}}{4k_c^2} B_4 \exp(i\beta z)$$
$$\left[\sin\left(\frac{\alpha\pi}{2}\right)\left\{\beta k_x \hat{\mathbf{x}} + \beta k_y \hat{\mathbf{y}} - k_c^2) \hat{\mathbf{z}}\right\} + \cos\left(\frac{\alpha\pi}{2}\right)\left\{k k_y \hat{\mathbf{x}} - k k_x \hat{\mathbf{y}}\right\}\right]$$
$$\exp\left[i\left(k_x x + \frac{\alpha\pi}{2}\right)\right] \exp\left[i\left(k_y y + \frac{\alpha\pi}{2}\right)\right] \tag{9h}$$

Fractional dual solutions of the total electric and magnetic field inside the guide can be written as

$$\mathbf{E}_{\text{fd}} = \mathbf{E}_{1\text{fd}} + \mathbf{E}_{2\text{fd}} + \mathbf{E}_{3\text{fd}} + \mathbf{E}_{4\text{fd}}$$
$$\eta\mathbf{H}_{\text{fd}} = \eta\mathbf{H}_{1\text{fd}} + \eta\mathbf{H}_{2\text{fd}} + \eta\mathbf{H}_{3\text{fd}} + \eta\mathbf{H}_{4\text{fd}}$$

which give

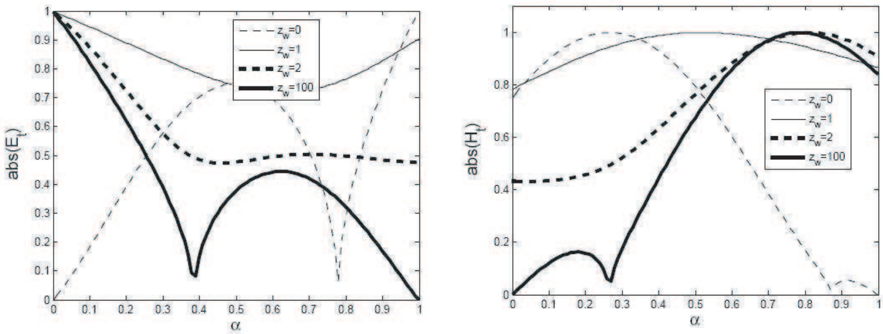
$$\begin{aligned}
\mathbf{E}_{\mathrm{fdx}} &= \frac{A_{mn}}{k_c^2}\exp(i\beta z)\left\{-\beta k_x C_\alpha - k k_y S_\alpha\right\} \\
&\quad [(C_{y+\alpha}F_x - iS_{y+\alpha})(C_{x+\alpha} - F_y i S_{x+\alpha})] && \text{(10a)} \\
\mathbf{E}_{\mathrm{fdy}} &= \frac{A_{mn}}{k_c^2}\exp(i\beta z)\left\{-\beta k_y C_\alpha + k k_x S_\alpha\right\} \\
&\quad [(C_{y+\alpha} - F_x i S_{y+\alpha})(C_{x+\alpha}F_y - iS_{x+\alpha})] && \text{(10b)} \\
\mathbf{E}_{\mathrm{fdz}} &= -A_{mn}\exp(i\beta z)C_\alpha \\
&\quad [(C_{y+\alpha}F_x - iS_{y+\alpha})(C_{x+\alpha}F_y - iS_{x+\alpha})] && \text{(10b)} \\
\eta\mathbf{H}_{\mathrm{fdx}} &= \frac{A_{mn}}{k_c^2}\exp(i\beta z)\left\{-\beta k_x S_\alpha + k k_y C_\alpha\right\} \\
&\quad [(C_{y+\alpha} - F_x i S_{y+\alpha})(C_{x+\alpha}F_y - iS_{x+\alpha})] && \text{(10c)} \\
\eta\mathbf{H}_{\mathrm{fdy}} &= \frac{A_{mn}}{k_c^2}\exp(i\beta z)\left\{-\beta k_y S_\alpha - k k_x C_\alpha\right\} \\
&\quad [(C_{y+\alpha}F_x - iS_{y+\alpha})(C_{x+\alpha} - F_y i S_{x+\alpha})] && \text{(10d)} \\
\eta\mathbf{H}_{\mathrm{fdz}} &= -A_{mn}S_\alpha\exp(i\beta z) \\
&\quad [(C_{y+\alpha} - F_x i S_{y+\alpha})(C_{x+\alpha} - F_y i S_{x+\alpha})] && \text{(10e)}
\end{aligned}$$

where

$$C_{x+\alpha} = \cos\left(k_x x + \frac{\alpha\pi}{2}\right), \quad C_{y+\alpha} = \cos\left(k_y y + \frac{\alpha\pi}{2}\right)$$
$$S_{x+\alpha} = \sin\left(k_x x + \frac{\alpha\pi}{2}\right), \quad S_{y+\alpha} = \sin\left(k_y y + \frac{\alpha\pi}{2}\right)$$

It may be noted from Equation (10) that fractional dual fields satisfy the principle of duality for the limiting values of α, i.e., for $\alpha = 0$, $(\mathbf{E}_{\mathrm{fd}}, \eta\mathbf{H}_{\mathrm{fd}})$ represents the original solution and for $\alpha = 1$, $(\mathbf{E}_{\mathrm{fd}}, \eta\mathbf{H}_{\mathrm{fd}})$ represents dual to the original solution. For the range $0 < \alpha < 1$, $(\mathbf{E}_{\mathrm{fd}}, \eta\mathbf{H}_{\mathrm{fd}})$ are the intermediate step between the original and dual to the original solutions and hence may be called as the fractional dual solutions. Further from Equation (10), we see that for $\alpha = 0$, $E_z \neq 0$ and $H_z = 0$ which shows the transverse magnetic mode, while for $\alpha = 1$, $E_z = 0$ and $H_z \neq 0$ which shows the transverse electric mode.

In order to validate the dependance on impedance of the walls (i.e., $z_w = Z_w/\eta$), tangential electric and magnetic fields at the wall at $x = 0$ of the fractional rectangular impedance waveguide have been plotted versus α for different values of the original impedance of walls, i.e., ($z_w = 0, 1, 2, 100$) as shown in Figure 2. Simulation data is for the mode which propagate through the guide at an angle $\phi_z = \pi/6$ with z-axis in the yz-plane and $\phi_x = \pi/4$ with x-axis in



(a) (b)

Figure 2. Plots of tangential fractional dual fields, (a) electric field, (b) magnetic field at $x = 0$.

the xy-plane. Hence the simulation parameters $(k_x/k_c, k_y/k_c, \beta/k_c) = (\cos(\pi/4), \sin(\pi/4), \cot(\pi/6))$ may be used. Normalized impedance of the guide walls $z_w = 10$ has been used. Figure 2(a) shows the plots for tangential electric fields at an observation point $((k_x x,\ k_y y,\ \beta z) = (0,\ \pi/4,\ \pi/4))$ and the corresponding magnetic fields are shown in Figure 2(b). It can be seen from the figures that tangential electric field is zero only at $((\alpha, z_w) = (0, 0)$ or $(\alpha, z_w) = (1, 100))$, i.e., PEC walls while tangential magnetic field is zero at $((\alpha, z_w) = (1, 0)$ or $(\alpha, z_w) = (0, 100))$, i.e., PMC walls case.

5. RESULTS AND DISCUSSION

5.1. Field Distribution

In order to study the behavior of field lines inside the fractional rectangular impedance waveguide, the field plots are given in the transverse xy-plane for different values of fractional parameter, i.e., $(\alpha = 0, 0.5, 1)$ and original impedance of walls of the guide, i.e., $(z_w = 0, 1, 10)$ as shown in Figure 3. Solid lines show the electric field while magnetic field is shown by the dashed lines. From these figures, it can be seen that electric field lines are perpendicular and magnetic field lines are parallel to the guide plates when the walls meet the conditions of PEC, i.e, $((\alpha, z_w) = (0, 0)$ or at $(\alpha, z_w) = (1, 10))$ while magnetic field lines are perpendicular and electric field lines are parallel to the guide walls when the walls meet the conditions of PMC, i.e., $((\alpha, z_w) = (0, 10)$ or at $(\alpha, z_w) = (1, 0))$. This is also in accordance with [31].

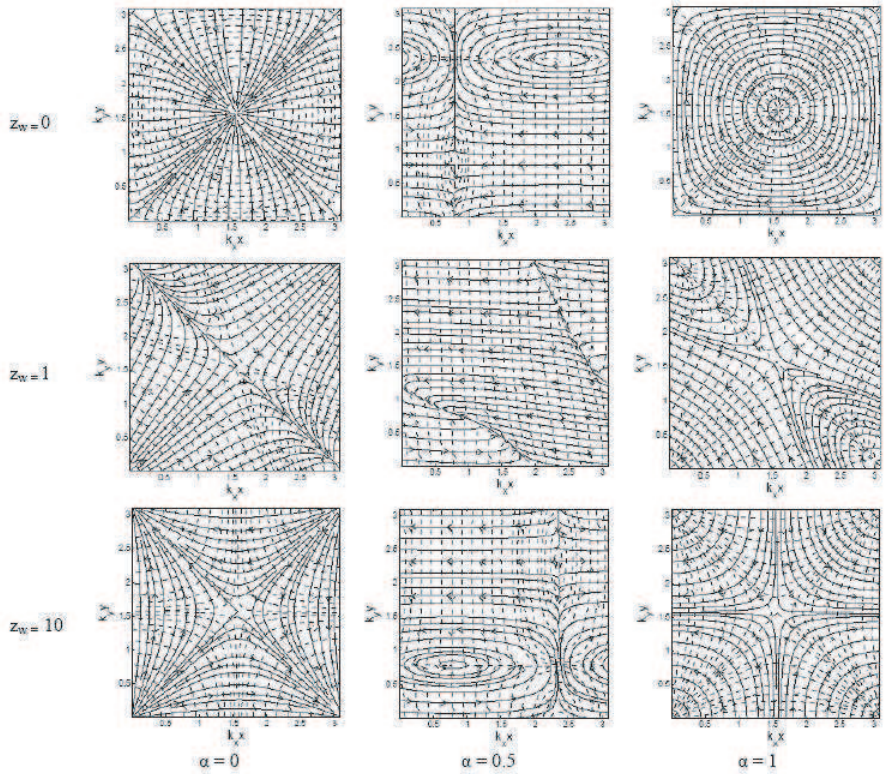


Figure 3. Field lines; solid lines show electric field and dashed lines show the magnetic field.

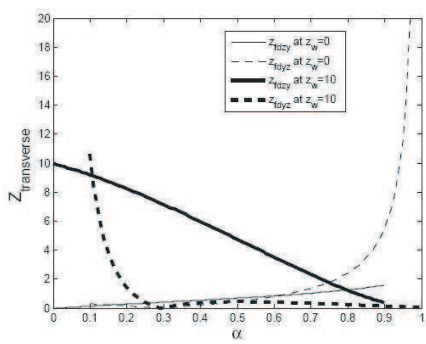


Figure 4. Transverse impedance of guide wall versus α.

5.2. Transverse Impedance

Transverse impedance of walls of the fractional rectangular waveguide can be found using ratio of the electric and magnetic field components transverse to the walls of the waveguide. For a wall at $x = 0$, impedance can be defined by using ratio of the y and z components of the electric and magnetic fields as

$$\underline{\underline{z}}_x = \begin{cases} [z_{\mathrm{fdyz}}\hat{\mathbf{y}}\hat{\mathbf{z}} + z_{\mathrm{fdzy}}\hat{\mathbf{z}}\hat{\mathbf{y}}] & 0 < \alpha < 1 \\ \frac{k_x}{k_y} z_w & \alpha = 0 \\ \frac{k_y}{k_x} \frac{1}{z_w} & \alpha = 1 \end{cases} \tag{11}$$

where

$$z_{\mathrm{fdyz}} = \left[\frac{-kk_x}{k_c^2} + \frac{\beta k_y}{k_c^2} \cot\left(\frac{\alpha\pi}{2}\right)\right] \frac{F_y - i\tan\left(\alpha\frac{\pi}{2}\right)}{1 - iF_y\tan\left(\alpha\frac{\pi}{2}\right)} \tag{11a}$$

$$z_{\mathrm{fdzy}} = \left[\frac{kk_x}{k_c^2} + \frac{\beta k_y}{k_c^2} \tan\left(\frac{\alpha\pi}{2}\right)\right]^{-1} \frac{F_y - i\tan\left(\alpha\frac{\pi}{2}\right)}{1 - iF_y\tan\left(\frac{\alpha\pi}{2}\right)} \tag{11b}$$

The absolute values of the transverse impedance z_{fdyz} and z_{fdzy} have been plotted versus α for different values of fractional parameter α in the range of $0 \leq \alpha \leq 1$ as in Figure 4. The plots are given for the original impedance of the wall as $z_w = 0$ and $z_w = 10$. Solid lines are for the impedance components z_{fdzy} while dashed lines show the impedance component z_{fdyz}. It may be noted that dashed lines do not exist at $\alpha = 0$ because the original solution is for the mode TM^z in which $H_z = 0$ while solid lines do not exist at $\alpha = 1$ because at this value only transverse electric mode exist, i.e, $E_z = 0$. It can be seen from the plot that at $\alpha = 0$, transverse impedance is equal to the original impedance of the wall, i.e., $(z_w = 0, 10)$ and for $\alpha = 1$, the transverse impedance becomes equal to the admittance of the original wall of the guide, i.e., $(z_w = \infty, 0.1)$. Between these values of α, transverse impedance is an-isotropic as shown in Figure 4. This behavior is also according to the published results [19, 26].

5.3. Power Transferred through a Cross Section

The time averaged power density at any point of the transverse plane (i.e., xy-plane) of the fractional rectangular impedance waveguide can be obtained using the Poynting vector theorem as

$$\mathrm{P_{av}}(x, y, z) = \frac{1}{2}\mathrm{Re}[\mathrm{E_{fdx}H^*_{fdy}} - \mathrm{E_{fdy}H^*_{fdx}}] \tag{12}$$

where H^*_{fdy} shows the complex conjugate of H_{fdy} and so on. Contour plots for the power density given by Equation (12) have been plotted for different values of the fractional parameter as shown in Figure 5. The variation in the power distribution at the transverse plane may be noted. Time averaged power density at the center point of the cross

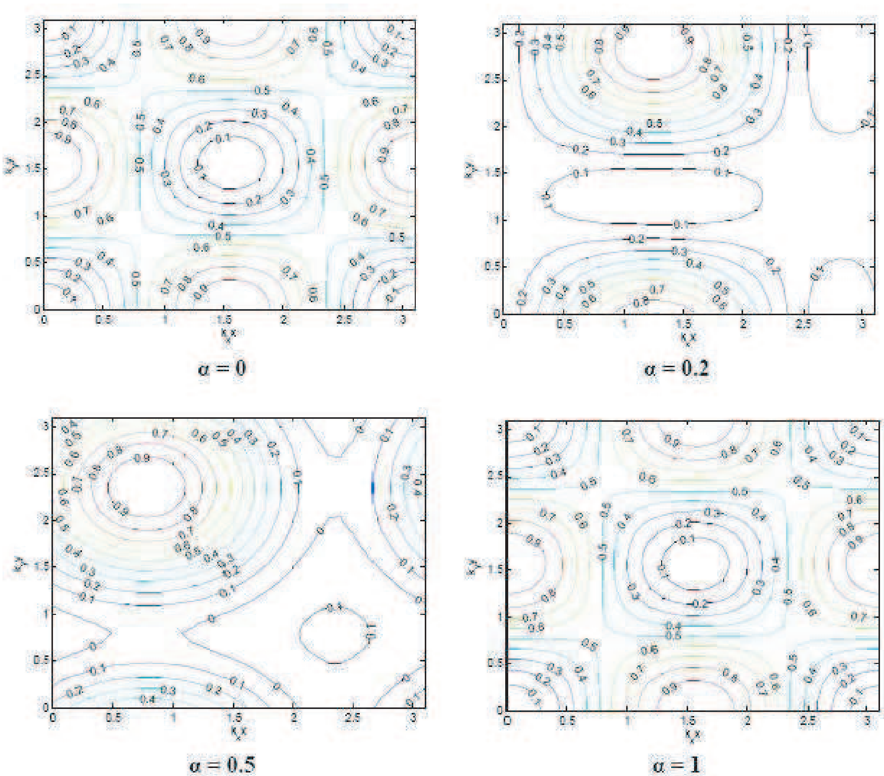


Figure 5. Time averaged power distribution over the cross section for different values of α.

sectional face has been plotted in Figure 6 which shows the relative maxima at $\alpha = 0.5$. This shows that one may use the fractional curl operator to control the pattern of the transmitted power through the waveguide. The average power density at the cross sectional plane can be obtained by integrating the local power density given in Equation (13) over the whole cross section as

$$\mathrm{P}(z) = \int_0^\pi \int_0^\pi \frac{1}{2}\mathrm{Re}[E_{\mathrm{fdx}} H^*_{\mathrm{fdy}} - E_{\mathrm{fdy}} H^*_{\mathrm{fdx}}] d(k_x x) d(k_y y) \qquad (13)$$

This power density has been plotted for the entire range of α as in Figure 7 which shows that the average power density through the cross sectional plane remains fairly constant for the whole range of the fractional parameter.

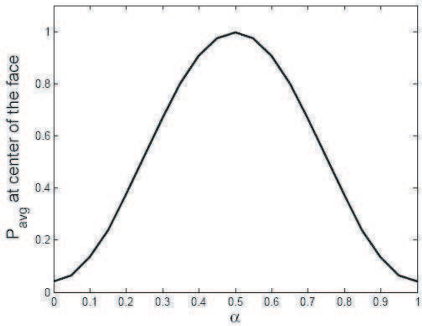


Figure 6. Time averaged power density at the center of the transverse plane, i.e., at $(k_x x, k_y y) = (\pi/2, \pi/2)$.

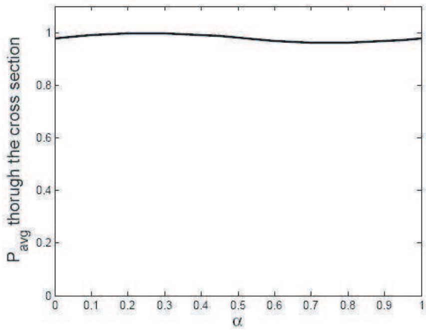


Figure 7. Average power density at the transverse plane.

6. CONCLUSION

Fractional rectangular impedance waveguide is the waveguide whose walls have an-isotropic impedance and the propagation mode is a hybrid mode. When original model is a rectangular waveguide with impedance walls for transverse magnetic modes then dual to the original solution is a rectangular waveguide with admittance walls for transverse electric modes and vice versa. For a special case of PEC walls, i.e., $z_w = 0$, this is a generalized model which can represent the solution of a rectangular waveguide whose walls are intermediate step of perfect electric conductor (PEC) and perfect magnetic conductor (PMC) and the propagating mode is also intermediate step of transverse magnetic mode and transverse electric mode. Field lines patterns show that electric field lines in the transverse plane are perpendicular to the guide plates while magnetic field lines are parallel to the plates at $(\alpha, z_w) = (0, 0),\ (1, 10))$ which simulate the conditions of PEC walls. An other observation from the field lines plots is that electric field lines in the transverse plane are parallel to the guide plates while magnetic field lines are perpendicular to the plates at $(\alpha, z_w) = (0, 10),\ (1, 0))$ which simulate the conditions of PMC walls. It has been seen that the relative power density distributions at the the cross sectional plane changes with varying α. For example the relative power density distribution at center of the cross sectional plane is maximum at $\alpha = 0.5$. However, the average power density at the cross sectional plane remains fairly constant for all values of α between 0 and 1. Hence, it is concluded that fractional curl may be used to control the power distribution pattern over the cross section of the guide.

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