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THE CLASS OF ELECTROMAGNETIC P-MEDIA AND ITS GENERALIZATION

I. V. Lindell

School of Electrical Engineering
Aalto University, Box 13000, Aalto 00076, Finland

L. Bergamin

KB & P GmbH, Fliederweg 10, Bern 3007, Switzerland

A. Favaro

Department of Physics, Imperial College London
Prince Consort Road, SW7 2AZ, UK

Abstract—Applying four-dimensional differential-form formalism, a novel class of electromagnetic media, labeled as that of P-media, is introduced in terms of a simple rule. It is shown that it is not possible to define the medium by expressing **D** and **B** in terms of **E** and **H**, whilst using 3D Gibbsian vectors and dyadics. Moreover, the basic properties of P-media are shown to be complementary to those of the previously known Q-media, which are defined in a somewhat similar manner. It is demonstrated that, for plane waves in a P-medium, there is no restriction to the wave one-form (corresponding to the **k**-vector). Importantly, the uniaxial P-medium half space also leads to another realization of the recently studied DB boundary conditions. Finally, a generalization of the class of P-media is briefly discussed. It is shown that the dispersion equation of a plane wave in the generalized P-medium is decomposed into two conditions, each of which corresponds to a certain polarization condition. This occurrence resembles the behavior of the generalized Q-medium.

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Corresponding author: Ismo Veikko Lindell (ismo.lindell@aalto.fi).

1. INTRODUCTION

The most general linear electromagnetic medium (bi-anisotropic medium) can be expressed in terms of four medium dyadics in the three-dimensional Gibbsian vector representation as [1–3]

$$\begin{pmatrix} \mathbf{D}_g \\ \mathbf{B}_g \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}}_g & \overline{\overline{\xi}}_g \\ \overline{\overline{\zeta}}_g & \overline{\overline{\mu}}_g \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_g \\ \mathbf{H}_g \end{pmatrix}, \tag{1}$$

whence the maximum number of free parameters is $4 \times 9 = 36$. The Gibbsian quantities are denoted by the subscript $()_g$ to distinguish them from quantities involving differential forms. In this study the parameter dyadics are assumed to be constant (no dependence on space or time).

The Gibbsian vector and dyadic algebra is inherently limited to three-dimensional representation while the differential-form formalism is best applied in four-dimensional form. In fact, the Maxwell equations can be compactly expressed as [4–6]

$$\mathbf{d} \wedge \mathbf{\Psi} = \boldsymbol{\gamma}, \qquad \mathbf{d} \wedge \mathbf{\Phi} = 0, \tag{2}$$

where the electromagnetic two-forms $\mathbf{\Psi}, \mathbf{\Phi}$, elements of the space $\mathbb{F}_2$, can be interpreted in terms of 3D (spatial) two-forms $\mathbf{D}, \mathbf{B}$ and one-forms $\mathbf{E}$, $\mathbf{H}$ as

$$\mathbf{\Phi} = \mathbf{B} + \mathbf{E} \wedge \boldsymbol{\varepsilon}_4, \qquad \mathbf{\Psi} = \mathbf{D} - \mathbf{H} \wedge \boldsymbol{\varepsilon}_4, \tag{3}$$

and $\boldsymbol{\varepsilon}_4$ equals the temporal basis-one-form. Definitions and operational rules for differential forms, multivectors and dyadics as applied in this study have been summarized in the Appendices of [7, 8] and, more extensively, in the book [6].

The constitutive Equation (1) can be represented by a medium dyadic $\overline{\overline{\mathrm{M}}} \in \mathbb{F}_2\mathbb{E}_2$ mapping two-forms to two-forms as

$$\mathbf{\Psi} = \overline{\overline{\mathrm{M}}}|\mathbf{\Phi}, \tag{4}$$

or, in terms of a modified medium dyadic $\overline{\overline{\mathrm{M}}}_g \in \mathbb{E}_2\mathbb{E}_2$ mapping two-forms to bivectors,

$$\mathbf{e}_N\lfloor\mathbf{\Psi} = \overline{\overline{\mathrm{M}}}_g|\mathbf{\Phi}. \tag{5}$$

Here, $\mathbf{e}_N = \mathbf{e}_{1234} \in \mathbb{E}_4$ denotes the quadrivector in the basis of vectors $\mathbf{e}_i \in \mathbb{E}_1$ and $\lfloor$ denotes the contraction operation. The reciprocal basis one-forms $\boldsymbol{\varepsilon}_j \in \mathbb{F}_1$ satisfying $\mathbf{e}_i|\boldsymbol{\varepsilon}_j = \delta_{ij}$ define the basis four-form $\boldsymbol{\varepsilon}_N = \boldsymbol{\varepsilon}_{1234}$.

Four-dimensional formalism allows simple definition of important classes of electromagnetic media [8]. For example, if the modified medium dyadic $\overline{\overline{\mathsf{M}}}_g$ can be expressed in terms of some dyadic $\overline{\overline{\mathsf{Q}}} \in \mathbb{E}_1\mathbb{E}_1$ mapping one-forms to vectors as [6, 9]

$$\overline{\overline{\mathsf{M}}}_g = \frac{1}{2}\overline{\overline{\mathsf{Q}}} {}^{\wedge}_{\wedge} \overline{\overline{\mathsf{Q}}} = \overline{\overline{\mathsf{Q}}}^{(2)}, \tag{6}$$

it is called a Q-medium. Because Q-media have the property of being non-birefringent to propagating waves, they can be conceived as generalizations of isotropic media (in the Gibbsian sense) [6].

A more general medium class was called that of generalized Q-media and defined by medium dyadics of the form

$$\overline{\overline{\mathsf{M}}}_g = \overline{\overline{\mathsf{Q}}}^{(2)} + \mathbf{AB}, \tag{7}$$

where $\mathbf{A}$, $\mathbf{B} \in \mathbb{E}_2$ are two bivectors. Such a medium was shown to generalize a medium in which any fields can be decomposed in noncoupled TE and TM components [10].

2. THE P-MEDIUM

Let us consider a class of media which is similar to that of Q-media in definition although not in properties. We assume that the medium dyadic $\overline{\overline{\mathsf{M}}}$ can be expressed in terms of some dyadic $\overline{\overline{\mathsf{P}}} \in \mathbb{F}_1\mathbb{E}_1$ in the form

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{P}}}^{(2)}. \tag{8}$$

While it is known that any dyadic $\overline{\overline{\mathsf{M}}}$ mapping two-forms to two-forms in 3D space can be represented in the form (8) [6] p. 132, this is not the case in 4D, and the representation (8) defines a certain restricted class of electromagnetic media which will be called that of P-media for brevity.

To have an idea of a medium so defined, let us expand $\overline{\overline{\mathsf{P}}}$ in its spatial and temporal components as

$$\overline{\overline{\mathsf{P}}} = \overline{\overline{\mathsf{P}}}_s + \boldsymbol{\pi}\mathbf{e}_4 + \boldsymbol{\varepsilon}_4\mathbf{p} + p\boldsymbol{\varepsilon}_4\mathbf{e}_4, \tag{9}$$

where the dyadic $\overline{\overline{\mathsf{P}}}_s$, vector $\mathbf{p}$ and one-form $\boldsymbol{\pi}$ are 3D spatial quantities, i.e., they are orthogonal to the temporal vector $\mathbf{e}_4$ or one-form $\boldsymbol{\varepsilon}_4$ as

$$\mathbf{e}_4|\overline{\overline{\mathsf{P}}}_s = \overline{\overline{\mathsf{P}}}_s|\boldsymbol{\varepsilon}_4 = 0, \quad \mathbf{p}|\boldsymbol{\varepsilon}_4 = 0, \quad \boldsymbol{\pi}|\mathbf{e}_4 = 0, \tag{10}$$

while p is a scalar. Inserting in (8) we obtain

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{P}}}_s^{(2)} + \overline{\overline{\mathsf{P}}}_s {}^{\wedge}_{\wedge} (\boldsymbol{\varepsilon}_4\mathbf{p} + \boldsymbol{\pi}\mathbf{e}_4) + \left(p\overline{\overline{\mathsf{P}}}_s - \boldsymbol{\pi}\mathbf{p}\right) {}^{\wedge}_{\wedge} \boldsymbol{\varepsilon}_4\mathbf{e}_4. \tag{11}$$

Comparing with the 3D expansion [6]

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\alpha}} + \overline{\overline{\epsilon}}' \wedge \mathbf{e}_4 + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mu}}^{-1} + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\beta}} \wedge \mathbf{e}_4, \tag{12}$$

in terms of which the medium Equation (4) can be expressed as

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\alpha}} & \overline{\overline{\epsilon}}' \\ \overline{\overline{\mu}}^{-1} & \overline{\overline{\beta}} \end{pmatrix} | \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}, \tag{13}$$

the 3D medium dyadics can be identified as

$$\overline{\overline{\alpha}} = \overline{\overline{\mathrm{P}}}{}_s^{(2)} \in \mathbb{F}_2\mathbb{E}_2, \tag{14}$$

$$\overline{\overline{\epsilon}}' = -\boldsymbol{\pi} \wedge \overline{\overline{\mathrm{P}}}_s \in \mathbb{F}_2\mathbb{E}_1, \tag{15}$$

$$\overline{\overline{\mu}}^{-1} = -\overline{\overline{\mathrm{P}}}_s \wedge \mathbf{p} \in \mathbb{F}_1\mathbb{E}_2, \tag{16}$$

$$\overline{\overline{\beta}} = \boldsymbol{\pi}\mathbf{p} - p = P_s \in \mathbb{F}_1\mathbb{E}_1. \tag{17}$$

One can easily derive the following relation between the four parameter dyadics,

$$3\overline{\overline{\beta}}{}^{(3)} = p^2 \left(\overline{\overline{\alpha}} {}^\wedge_\wedge \overline{\overline{\beta}} + \overline{\overline{\epsilon}}' {}^\wedge_\wedge \overline{\overline{\mu}}^{-1} \right). \tag{18}$$

Taking the trace operation on both sides of (18), an equivalent scalar condition is obtained.

The Gibbsian counterparts of the medium dyadics (14)–(17) can be formed applying the rules in the Appendix and the spatial metric dyadic $\overline{\overline{\mathrm{G}}}_s = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$. First, we define the Gibbsian vectors

$$\boldsymbol{\pi}_g = \overline{\overline{\mathrm{G}}}_s|\boldsymbol{\pi}, \qquad \mathbf{p}_g = \mathbf{p}, \tag{19}$$

and Gibbsian dyadics

$$\overline{\overline{\mathrm{P}}}_{sg} = \overline{\overline{\mathrm{G}}}_s|\overline{\overline{\mathrm{P}}}_s, \qquad \overline{\overline{\mathrm{P}}}{}_{sg}^{(2)} = \overline{\overline{\mathrm{G}}}_s{}^{(2)}|\overline{\overline{\mathrm{P}}}{}_s^{(2)}. \tag{20}$$

The medium dyadics take the form

$$\overline{\overline{\alpha}}_g = \overline{\overline{\mathrm{P}}}{}_{sg}^{(2)}, \tag{21}$$

$$\overline{\overline{\epsilon}}'_g = -\mathbf{e}_{123}\lfloor\left(\boldsymbol{\pi} \wedge \overline{\overline{\mathrm{P}}}_s\right) = -\boldsymbol{\pi}_g \times \overline{\overline{\mathrm{P}}}_{sg}, \tag{22}$$

$$\overline{\overline{\mu}}{}_g^{-1} = -\overline{\overline{\mathrm{G}}}_s|\left(\overline{\overline{\mathrm{P}}}_s \wedge \mathbf{p}\right)\rfloor\boldsymbol{\varepsilon}_{123}|\overline{\overline{\mathrm{G}}}_s = -\overline{\overline{\mathrm{P}}}_{sg} \times \mathbf{p}, \tag{23}$$

$$\overline{\overline{\beta}}_g = \boldsymbol{\pi}_g\mathbf{p} - p\overline{\overline{\mathrm{P}}}_{sg}. \tag{24}$$

These expressions show that, since $\overline{\overline{\mu}}{}_g^{-1}$ is an antisymmetric dyadic, there does not exist a dyadic $\overline{\overline{\mu}}_g$. Thus, it is not possible to express the P-medium in terms of the "engineering" form (1). Also, it appears that the magnetoelectric parameter $\overline{\overline{\beta}}$ is essential for the P-medium. In fact, $\overline{\overline{\beta}} = 0, p \neq 0$ implies $p\overline{\overline{\mathrm{P}}} = (\boldsymbol{\pi} + p\boldsymbol{\varepsilon}_4)(\mathbf{p} + p\mathbf{e}_4)$ whence $\overline{\overline{\mathrm{M}}} = 0$.

One can observe a certain similarity between the 3D dyadic expressions (14)–(17) and those of the Q-medium given in [9], Eqs. (5.66)–(5.69). Actually, any P-medium dyadic $\overline{\overline{\mathrm{M}}}_P$ can be transformed to a Q-medium dyadic $\overline{\overline{\mathrm{M}}}_Q$ by means of Hodge duality [11] as

$$\overline{\overline{\mathrm{M}}}_Q = \boldsymbol{\varepsilon}_N \lfloor \overline{\overline{\mathrm{G}}}^{(2)} | \overline{\overline{\mathrm{M}}}_P, \tag{25}$$

where $\overline{\overline{\mathrm{G}}} = \overline{\overline{\mathrm{G}}}_s - \mathbf{e}_4\mathbf{e}_4$ is the Minkowskian metric dyadic. In fact, we immediately obtain

$$\overline{\overline{\mathrm{M}}}_Q = \boldsymbol{\varepsilon}_N \lfloor \overline{\overline{\mathrm{G}}}^{(2)} | \overline{\overline{\mathrm{P}}}^{(2)} = \boldsymbol{\varepsilon}_N \lfloor \overline{\overline{\mathrm{Q}}}^{(2)}, \qquad \overline{\overline{\mathrm{Q}}} = \overline{\overline{\mathrm{G}}} | \overline{\overline{\mathrm{P}}}. \tag{26}$$

The 3D dyadic expressions for the Q-medium are related to those of (14)–(17) as

$$\begin{aligned}
\overline{\overline{\alpha}}_Q &= -\boldsymbol{\varepsilon}_{123} \lfloor \overline{\overline{\mathrm{G}}}_s | \left(\overline{\overline{\mathrm{P}}}_s \wedge \mathbf{p} \right) = \boldsymbol{\varepsilon}_{123} \lfloor \overline{\overline{\mathrm{G}}}_s | \overline{\overline{\mu}}_P^{-1}, && (27)\\
\overline{\overline{\epsilon}}'_Q &= -\boldsymbol{\varepsilon}_{123} \lfloor \overline{\overline{\mathrm{G}}}_s | \left(p\overline{\overline{\mathrm{P}}}_s - \boldsymbol{\pi}\mathbf{p} \right) = \boldsymbol{\varepsilon}_{123} \lfloor \overline{\overline{\mathrm{G}}}_s | \overline{\overline{\beta}}_P, && (28)\\
\overline{\overline{\mu}}_Q^{-1} &= -\boldsymbol{\varepsilon}_{123} \lfloor \left(\overline{\overline{\mathrm{G}}}_s | \overline{\overline{\mathrm{P}}}_s \right)^{(2)} = -\boldsymbol{\varepsilon}_{123} \lfloor \overline{\overline{\mathrm{G}}}_s^{(2)} | \overline{\overline{\alpha}}_P, && (29)\\
\overline{\overline{\beta}}_Q &= \boldsymbol{\varepsilon}_{123} \lfloor \overline{\overline{\mathrm{G}}}_s^{(2)} | \left(\boldsymbol{\pi} \wedge \overline{\overline{\mathrm{P}}}_s \right) = -\boldsymbol{\varepsilon}_{123} \lfloor \overline{\overline{\mathrm{G}}}_s^{(2)} | \overline{\overline{\epsilon}}'_P. && (30)
\end{aligned}$$

These expressions show us that $\overline{\overline{\mu}}^{-1}$ and $\overline{\overline{\epsilon}}'$ are respectively mapped to $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$, and conversely, in the Hodge duality.

3. DECOMPOSITION OF MEDIUM DYADIC

3.1. Hehl-Obukhov Decomposition

The medium dyadic $\overline{\overline{\mathrm{M}}}$ can be decomposed as based on the eigenproblem

$$\overline{\overline{\mathrm{I}}}^{(4)} \lfloor\lfloor \overline{\overline{\mathrm{M}}} = \mathbf{e}_N \boldsymbol{\varepsilon}_N \lfloor\lfloor \overline{\overline{\mathrm{M}}} = \lambda \overline{\overline{\mathrm{M}}}^T, \tag{31}$$

whose eigenvalues are $\lambda_\pm = \pm 1$ so that any medium dyadic can be expressed in terms of the corresponding eigendyadics $\overline{\overline{\mathrm{M}}}_\pm$ as

$$\overline{\overline{\mathrm{M}}} = \overline{\overline{\mathrm{M}}}_+ + \overline{\overline{\mathrm{M}}}_-, \qquad \overline{\overline{\mathrm{M}}}_\pm = \frac{1}{2} \left(\overline{\overline{\mathrm{M}}} \pm \boldsymbol{\varepsilon}_N \mathbf{e}_N \lfloor\lfloor \overline{\overline{\mathrm{M}}}^T \right). \tag{32}$$

This means that the four 3D medium dyadics can also be expanded as $\overline{\overline{\alpha}} = \overline{\overline{\alpha}}_+ + \overline{\overline{\alpha}}_-$ etc.

A related expansion for the medium dyadic was introduced by Hehl and Obukhov, and has the form [5]

$$\overline{\overline{\mathrm{M}}} = \overline{\overline{\mathrm{M}}}_{pr} + \overline{\overline{\mathrm{M}}}_{sk} + \overline{\overline{\mathrm{M}}}_{ax}, \tag{33}$$

where the components $\overline{\overline{\rm M}}_{pr}$, $\overline{\overline{\rm M}}_{sk}$ and $\overline{\overline{\rm M}}_{ax}$ are respectively called as the principal, skewon and axion parts of $\overline{\overline{\rm M}}$. They have the following connection to the eigendyadics:

$$\overline{\overline{\rm M}}_+ = \overline{\overline{\rm M}}_{pr} + \overline{\overline{\rm M}}_{ax}, \qquad \overline{\overline{\rm M}}_- = \overline{\overline{\rm M}}_{sk}. \tag{34}$$

The axion part is of the form $\overline{\overline{\rm M}}_{ax} = \alpha\overline{\overline{\rm I}}^{(2)T}$ while the other two parts are trace free, $\mathrm{tr}\overline{\overline{\rm M}}_{pr} = \mathrm{tr}\overline{\overline{\rm M}}_{sk} = 0$. Let us briefly summarize the properties of the three parts of the medium dyadic $\overline{\overline{\rm M}}$.

- The axion part can be extracted as

$$\overline{\overline{\rm M}}_{ax} = \frac{1}{6}\mathrm{tr}\overline{\overline{\rm M}}\,\overline{\overline{\rm I}}^{(2)T}, \tag{35}$$

  since $\mathrm{tr}\overline{\overline{\rm I}}^{(2)T} = 6$. Obviously, any medium consisting only of its axion part, also called the PEMC medium [12], is a special case of a P-medium.

- The skewon part is obtained as $\overline{\overline{\rm M}}_{sk} = \overline{\overline{\rm M}}_-$ from (32). It can be expressed in the form [13]

$$\overline{\overline{\rm M}}_{sk} = \left(\overline{\overline{\rm I}}\,{}^\wedge_\wedge\overline{\overline{\rm B}}_o\right)^T, \qquad \mathrm{tr}\overline{\overline{\rm B}}_o = 0, \tag{36}$$

  in terms of some trace-free dyadic $\overline{\overline{\rm B}}_o \in \mathbb{E}_1\mathbb{F}_1$. Conversely, the dyadic $\overline{\overline{\rm B}}_o$ can be obtained from $\overline{\overline{\rm M}}_{sk}$ as

$$\overline{\overline{\rm B}}_o = \frac{1}{2}\left(\overline{\overline{\rm M}}_{sk}\lfloor\lfloor\overline{\overline{\rm I}}\right)^T. \tag{37}$$

- Finally, the principal part satisfies

$$\overline{\overline{\rm M}}_{pr}\lfloor\lfloor\overline{\overline{\rm I}} = 0, \tag{38}$$

  which implies $\mathrm{tr}\overline{\overline{\rm M}}_{pr} = 0$. If the medium dyadic satisfies $\overline{\overline{\rm M}}\lfloor\lfloor\overline{\overline{\rm I}} = 0$, it has no axion or skewon parts. The principal part encompasses as special cases the free space relation as well as most standard linear media [14].

3.2. Skewonless P-medium

As an example let us consider the skewonless P-medium which corresponds to the eigendyadic $\overline{\overline{\rm M}} = \overline{\overline{\rm M}}_+$ of (31). In this case the $\overline{\overline{\rm P}}$ dyadic must satisfy

$$\overline{\overline{\rm I}}^{(4)}\lfloor\lfloor\overline{\overline{\rm P}}^{(2)} = \overline{\overline{\rm P}}^{(2)T}. \tag{39}$$

In the general case the skewonless medium is composed of an axion component and a principal component.

Applying the identity

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\overline{\mathsf{M}}}} = \left(\mathrm{tr}\overline{\overline{\overline{\mathsf{M}}}}\right)\overline{\overline{\mathsf{I}}}^{(2)} - \left(\overline{\overline{\overline{\mathsf{M}}}}^T\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} + \overline{\overline{\overline{\mathsf{M}}}}^T, \tag{40}$$

valid for any dyadic $\overline{\overline{\overline{\mathsf{M}}}} \in \mathbb{F}_2\mathbb{E}_2$, (39) is reduced to

$$\left(\mathrm{tr}\overline{\overline{\mathsf{P}}}^{(2)}\right)\overline{\overline{\mathsf{I}}}^{(2)} - \left(\overline{\overline{\mathsf{P}}}^{(2)T}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} = \left(\frac{1}{2}\left(\mathrm{tr}\overline{\overline{\mathsf{P}}}^{(2)}\right)\overline{\overline{\mathsf{I}}} - \overline{\overline{\mathsf{P}}}^{(2)T}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T\right){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} = 0. \tag{41}$$

This is possible only if the bracketed dyadic vanishes, which is equivalent to

$$\frac{1}{4}\left(\left(\mathrm{tr}\overline{\overline{\mathsf{P}}}\right)^2 - \mathrm{tr}\overline{\overline{\mathsf{P}}}^2\right)\overline{\overline{\mathsf{I}}} - \left(\mathrm{tr}\overline{\overline{\mathsf{P}}}\right)\overline{\overline{\mathsf{P}}}^T + \overline{\overline{\mathsf{P}}}^{2T} = 0, \tag{42}$$

or

$$\left(\overline{\overline{\mathsf{P}}} - \alpha\overline{\overline{\mathsf{I}}}^T\right)^2 = \beta^2\overline{\overline{\mathsf{I}}}^T, \quad \alpha = \frac{1}{2}\mathrm{tr}\overline{\overline{\mathsf{P}}}, \quad \beta = \frac{1}{2}\sqrt{\mathrm{tr}\overline{\overline{\mathsf{P}}}^2}. \tag{43}$$

The dyadic in brackets is a unipotent dyadic multiplied by β. The results (A16), (A17) and (A18) given in the Appendix give the following respective candidates for the dyadic $\overline{\overline{\mathsf{P}}}$:

$$\overline{\overline{\mathsf{P}}} = \gamma\overline{\overline{\mathsf{I}}}^T, \quad \overline{\overline{\mathsf{P}}} = \gamma\overline{\overline{\mathsf{I}}}^T + \delta\boldsymbol{\varepsilon}_1\mathbf{e}_1, \quad \overline{\overline{\mathsf{P}}} = \gamma\overline{\overline{\mathsf{I}}}^T + \delta\left(\boldsymbol{\varepsilon}_1\mathbf{e}_1 + \boldsymbol{\varepsilon}_2\mathbf{e}_2\right), \tag{44}$$

with $\gamma = \alpha - \beta$, $\delta = -2\beta$.

The expressions (44) are more general than the actual solutions of (39). Restricting conditions are obtained by enforcing (39) explicitly. The first possibility in (44) corresponds to a pure axion medium satisfying (39). The second possibility corresponds to to the medium dyadic

$$\overline{\overline{\overline{\mathsf{M}}}} = \overline{\overline{\mathsf{P}}}^{(2)} = \gamma^2\overline{\overline{\mathsf{I}}}^{(2)T} + \gamma\delta\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\mathbf{e}_1\boldsymbol{\varepsilon}_1\right)^T, \tag{45}$$

whose last term has a skewon component of the form (36). Requiring that it vanish yields $\delta = 0$, whence also this case leads to a pure axion medium.

The third case corresponds to

$$\overline{\overline{\overline{\mathsf{M}}}} = \gamma^2\overline{\overline{\mathsf{I}}}^{(2)T} + \gamma\delta\left(\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\left(\mathbf{e}_1\boldsymbol{\varepsilon}_1 + \mathbf{e}_2\boldsymbol{\varepsilon}_2\right)\right)^T + \delta^2\boldsymbol{\varepsilon}_{12}\mathbf{e}_{12}. \tag{46}$$

Let us consider the last term. Decomposing

$$\boldsymbol{\varepsilon}_{12}\mathbf{e}_{12} = \overline{\overline{\mathsf{B}}}_+ + \overline{\overline{\mathsf{B}}}_-, \tag{47}$$

with

$$\overline{\overline{\mathrm{B}}}_\pm = \frac{1}{2}(\boldsymbol{\varepsilon}_{12}\mathbf{e}_{12} \pm \boldsymbol{\varepsilon}_{34}\mathbf{e}_{34}) \mp \frac{1}{6}\overline{\overline{\mathrm{I}}}^{(2)T}, \tag{48}$$

we can easily check that the dyadic $\overline{\overline{\mathrm{B}}}_-$ can be written in the form

$$\overline{\overline{\mathrm{B}}}_- = \overline{\overline{\mathrm{I}}}^T {}^\wedge_\wedge \overline{\overline{\mathrm{C}}}, \qquad \overline{\overline{\mathrm{C}}} = \frac{1}{2}(\boldsymbol{\varepsilon}_1\mathbf{e}_1 + \boldsymbol{\varepsilon}_2\mathbf{e}_2) - \frac{1}{6}\overline{\overline{\mathrm{I}}}^T. \tag{49}$$

Since this is of the skewon-axion form, it can be added to the middle term of (46). Finally, the dyadic $\overline{\overline{\mathrm{B}}}_+$ can be shown to be a principal dyadic, because it satisfies (38):

$$\overline{\overline{\mathrm{B}}}_+ \lfloor\lfloor \overline{\overline{\mathrm{I}}} = \frac{1}{2}(\boldsymbol{\varepsilon}_3\mathbf{e}_3 + \boldsymbol{\varepsilon}_4\mathbf{e}_4 + \boldsymbol{\varepsilon}_1\mathbf{e}_1 + \boldsymbol{\varepsilon}_2\mathbf{e}_2) - \frac{1}{6}3\overline{\overline{\mathrm{I}}}^T = 0. \tag{50}$$

When requiring vanishing of the total skewon component of $\overline{\overline{\mathrm{M}}}$ we arrive at $\delta = 0$ or, again, the skewonless P-medium is a pure axion medium.

Thus, in conclusion, the principal part of any principal-axion P-medium vanishes. This also implies that a pure principal P-medium does not exist. In the general case, the principal part of a P-medium does not vanish, as will be seen from the example of the following Section.

4. UNIAXIAL P-MEDIUM

As a nontrivial example of a P-medium, let us assume that the spatial components are of the form

$$\overline{\overline{\mathrm{P}}}_s = P_t(\boldsymbol{\varepsilon}_1\mathbf{e}_1 + \boldsymbol{\varepsilon}_2\mathbf{e}_2) + P_3\boldsymbol{\varepsilon}_3\mathbf{e}_3, \tag{51}$$

$$\overline{\overline{\mathrm{P}}}{}_s^{(2)} = P_t^2\boldsymbol{\varepsilon}_{12}\mathbf{e}_{12} + P_tP_3(\boldsymbol{\varepsilon}_{23}\mathbf{e}_{23} + \boldsymbol{\varepsilon}_{31}\mathbf{e}_{31}), \tag{52}$$

$$\boldsymbol{\pi} = \pi_3\boldsymbol{\varepsilon}_3, \qquad \mathbf{p} = p_3\mathbf{e}_3. \tag{53}$$

Since the basis vector $\mathbf{e}_3$ and one-form $\boldsymbol{\varepsilon}_3$ take a special position in the medium dyadic, we may call such a medium uniaxial. The 3D medium dyadics can then be expressed as

$$\overline{\overline{\alpha}} = \alpha_t(\boldsymbol{\varepsilon}_{23}\mathbf{e}_{23} + \boldsymbol{\varepsilon}_{31}\mathbf{e}_{31}) + \alpha_3\boldsymbol{\varepsilon}_{12}\mathbf{e}_{12}, \tag{54}$$

$$\overline{\overline{\epsilon}}' = \epsilon_3\left(\boldsymbol{\varepsilon}_3 \wedge \overline{\overline{\mathrm{I}}}{}_s^T\right), \tag{55}$$

$$\overline{\overline{\mu}}^{-1} = \mu_3^{-1}\left(\overline{\overline{\mathrm{I}}}{}_s^T \wedge \mathbf{e}_3\right), \tag{56}$$

$$\overline{\overline{\beta}} = \beta_t\left(\boldsymbol{\varepsilon}_1\mathbf{e}_1 + \boldsymbol{\varepsilon}_2\mathbf{e}_2\right) + \beta_3\boldsymbol{\varepsilon}_3\mathbf{e}_3, \tag{57}$$

with

$$\alpha_t = P_t P_3, \quad \alpha_3 = P_t^2, \quad \epsilon_3 = -\pi_3 P_t, \tag{58}$$

$$\mu_3^{-1} = -p_3 P_t, \quad \beta_t = -p P_t, \quad \beta_3 = \pi_3 p_3 - p P_3. \tag{59}$$

Since the six parameters $\alpha_t \ldots \beta_3$ are defined in terms of five parameters $P_t \ldots p$, they are related by a condition which can be written as

$$\alpha_3\beta_3 - \alpha_t\beta_t = \epsilon_3\mu_3^{-1}. \tag{60}$$

Actually, this equals the condition (18).

The corresponding Gibbsian dyadics (elements of the space $\mathbb{E}_1\mathbb{E}_1$) justifying the uniaxial property of the medium are (see the Appendix)

$$\overline{\overline{\alpha}}_g = \left(\mathbf{e}_{123}\boldsymbol{\varepsilon}_{123}\lfloor\lfloor\overline{\overline{\alpha}}\right)|\overline{\overline{\mathrm{G}}}_s = \alpha_t(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2) + \alpha_3\mathbf{e}_3\mathbf{e}_3, \tag{61}$$

$$\overline{\overline{\epsilon}}'_g = \mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}' = \epsilon_3\mathbf{e}_3 \times \overline{\overline{\mathrm{G}}}_s, \tag{62}$$

$$\overline{\overline{\mu}}_g^{-1} = \overline{\overline{\mathrm{G}}}_s|\left(\overline{\overline{\mu}}^{-1}\rfloor\boldsymbol{\varepsilon}_{123}\right)|\overline{\overline{\mathrm{G}}}_s = \mu_3^{-1}\mathbf{e}_3 \times \overline{\overline{\mathrm{G}}}_s, \tag{63}$$

$$\overline{\overline{\beta}}_g = \overline{\overline{\mathrm{G}}}_s|\overline{\overline{\beta}} = \beta_t(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2) + \beta_3\mathbf{e}_3\mathbf{e}_3. \tag{64}$$

Here, the spatial metric dyadic $\overline{\overline{\mathrm{G}}}_s$ serves as the Gibbsian unit dyadic. It is noteworthy that the dyadics $\overline{\overline{\epsilon}}'_g$ and $\overline{\overline{\mu}}_g^{-1}$ are multiples of the same antisymmetric dyadic $\mathbf{e}_3 \times \overline{\overline{\mathrm{G}}}_s$, while $\overline{\overline{\alpha}}_g$ and $\overline{\overline{\beta}}_g$ are uniaxial Gibbsian dyadics.

The medium equations can be expressed in Gibbsian form as

$$\mathbf{D}_g = \epsilon_3\mathbf{e}_3 \times \mathbf{E}_g + \alpha_3\mathbf{e}_3 B_{g3} + \alpha_t\mathbf{B}_{gt}, \tag{65}$$

$$\mathbf{H}_g = \mu_3^{-1}\mathbf{e}_3 \times \mathbf{B}_{gt} + \beta_3\mathbf{e}_3 E_{g3} + \beta_t\mathbf{E}_{gt}, \tag{66}$$

where $\mathbf{a}_g = \mathbf{e}_3 a_{g3} + \mathbf{a}_{gt}$ denotes the decomposition of a Gibbsian vector $\mathbf{a}_g$ in its axial and transverse components.

It is interesting to note that the Gibbsian medium equations of the uniaxial P-medium, (65), (66), actually have one-to-one correspondence to those of a uniaxial skewon-axion medium (also termed as IB-medium) given in [15], Equation (67), with the obvious change in notation $\mathbf{e}_1 \to \mathbf{u}_x$, $\mathbf{e}_2 \to \mathbf{u}_y$, $\mathbf{e}_3 \to \mathbf{u}_z$. This raises a question on the relation between the two media, uniaxial P-medium on one hand and uniaxial skewon-axion medium on the other. After some algebraic juggling, the medium dyadic of the uniaxial P-medium can be written in the form

$$\overline{\overline{\mathrm{M}}} = \overline{\overline{\mathrm{I}}}^T {}^{\wedge}_{\wedge} \overline{\overline{\mathrm{A}}} + A\boldsymbol{\varepsilon}_{34}\mathbf{e}_{34}, \tag{67}$$

with

$$\overline{\overline{\mathrm{A}}} = \frac{1}{2}P_t^2\overline{\overline{\mathrm{I}}}^T + P_t(P_3 - P_t)\boldsymbol{\varepsilon}_3\mathbf{e}_3 + P_t\pi_3\boldsymbol{\varepsilon}_3\mathbf{e}_4 + P_t p_3\boldsymbol{\varepsilon}_4\mathbf{e}_3 + P_t(p - P_t)\boldsymbol{\varepsilon}_4\mathbf{e}_4, \tag{68}$$

$$A = (P_t - P_3)(P_t - p) - \pi_3 p_3. \tag{69}$$

The first term of (67) appears to be of the skewon-axion form. If the second term is decomposed as in (47),

$$A\boldsymbol{\varepsilon}_{34}\mathbf{e}_{34} = A\left(\overline{\overline{\mathrm{B}}}_{+} + \overline{\overline{\mathrm{B}}}_{-}\right), \tag{70}$$

with the definition

$$\overline{\overline{\mathrm{B}}}_{\pm} = \frac{1}{2}(\boldsymbol{\varepsilon}_{34}\mathbf{e}_{34} \pm \boldsymbol{\varepsilon}_{12}\mathbf{e}_{12}) \mp \frac{1}{6}\overline{\overline{\mathrm{I}}}^{(2)T}, \tag{71}$$

we attain the same outcome as in the previous section, with $\overline{\overline{\mathrm{B}}}_{-}$ corresponding the skewon part, and $\overline{\overline{\mathrm{B}}}_{+}$ the principal part, of $\boldsymbol{\varepsilon}_{34}\mathbf{e}_{34}$.

In conclusion, the principal part of the uniaxial P-medium dyadic $\overline{\overline{\mathrm{M}}}$, as represented by the term $A\overline{\overline{\mathrm{B}}}_{+}$, does not vanish, in general. Thus, the uniaxial P-medium equals a uniaxial skewon-axion medium only when the coefficient A vanishes, i.e., for

$$(P_t - P_3)(P_t - p) - \pi_3 p_3 = 0. \tag{72}$$

In terms of the 3D medium coefficients in (54)–(57) this reads

$$\alpha_t - \alpha_3 = \beta_t - \beta_3. \tag{73}$$

Actually, (73) equals the condition (18) in [15] which, despite the title of the paper, considers a more general six-parameter medium (65), (66) in which the skewon-axion condition (73) need not be valid. Thus, the problem of wave reflection from and transmission though a planar interface of a uniaxial half space applies to the six-parameter medium of which the uniaxial skewon-axion medium and P-medium are special cases. So, the main result of [15] is valid to both of these media.

The result can be stated as follows: independently of the values of the six medium parameters, the planar interface of the uniaxial medium can be interpreted as a boundary defined by the conditions [15–17],

$$\mathbf{n} \cdot \mathbf{D} = 0, \qquad \mathbf{n} \cdot \mathbf{B} = 0. \tag{74}$$

Here, the normal vector $\mathbf{n}$ coincides with the axial vector $\mathbf{e}_3 = \mathbf{u}_z$ of the medium. This result is not valid in the special case when the medium parameters satisfy a condition which in the present terminology takes the form

$$(\alpha_3 + \beta_t)(\alpha_3 - \alpha_t) = \epsilon_3 \mu_3^{-1}. \tag{75}$$

For example, this condition excludes the pure axion medium with $\alpha_3 = \alpha_t$, $\epsilon_3 = 0$, $\mu_3^{-1} = 0$. Thus, the uniaxial P-medium half space offers another possibility for the realization of the DB boundary. Interestingly, while the boundary defined by (74) was introduced already half a decade ago [18], it has only recently found applications in the study of electromagnetic cloaking [19–21].

5. PLANE WAVE IN P-MEDIUM

Let us consider the plane-wave solution

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{\Phi}\exp(\boldsymbol{\nu}|\mathbf{x}), \tag{76}$$

where $\boldsymbol{\nu}$ is the wave one-form. In a homogeneous linear medium we have

$$\mathbf{\Psi}(\mathbf{x}) = \overline{\overline{\mathrm{M}}}|\mathbf{\Phi}(\mathbf{x}) = \mathbf{\Psi}\exp(\boldsymbol{\nu}|\mathbf{x}), \tag{77}$$

and the Maxwell equations become

$$\boldsymbol{\nu}\wedge\mathbf{\Phi} = 0, \qquad \boldsymbol{\nu}\wedge\mathbf{\Psi} = 0, \tag{78}$$

whence the field two-forms can be expressed in terms of potential one-forms

$$\mathbf{\Phi} = \boldsymbol{\nu}\wedge\boldsymbol{\phi}, \qquad \mathbf{\Psi} = \boldsymbol{\nu}\wedge\boldsymbol{\psi}. \tag{79}$$

The potential $\boldsymbol{\phi}$ satisfies the equation

$$\boldsymbol{\nu}\wedge\mathbf{\Psi} = \boldsymbol{\nu}\wedge\overline{\overline{\mathrm{M}}}|(\boldsymbol{\nu}\wedge\boldsymbol{\phi}) = 0. \tag{80}$$

In the P-medium, this can be written as

$$\boldsymbol{\nu}\wedge\overline{\overline{\mathrm{P}}}^{(2)}|(\boldsymbol{\nu}\wedge\boldsymbol{\phi}) = \boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\phi}\right) = 0. \tag{81}$$

Assuming that $\boldsymbol{\nu}$ is not an eigen-one-form of the dyadic $\overline{\overline{\mathrm{P}}}$, we have $\boldsymbol{\nu}\wedge(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}) \neq 0$, whence $\overline{\overline{\mathrm{P}}}|\boldsymbol{\phi}$ must be a linear combination of the linearly independent one-forms $\boldsymbol{\nu}$ and $\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}$:

$$\overline{\overline{\mathrm{P}}}|\boldsymbol{\phi} = A\boldsymbol{\nu} + B\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}, \qquad \boldsymbol{\phi} = A\overline{\overline{\mathrm{P}}}^{-1}|\boldsymbol{\nu} + B\boldsymbol{\nu}. \tag{82}$$

The field two-forms then become

$$\mathbf{\Phi} = \boldsymbol{\nu}\wedge\boldsymbol{\phi} = A\boldsymbol{\nu}\wedge\overline{\overline{\mathrm{P}}}^{-1}|\boldsymbol{\nu}, \tag{83}$$

$$\mathbf{\Psi} = \overline{\overline{\mathrm{P}}}^{(2)}|\mathbf{\Phi} = A(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu})\wedge\boldsymbol{\nu}, \tag{84}$$

There is no restricting equation for the wave one-form $\boldsymbol{\nu}$. In fact, any one-form $\boldsymbol{\nu}$ can be used to define a plane wave in the P-medium and the field two-forms are obtained from (83), (84). Similar property was found to be connected to the skewon-axion medium (IB-medium) [13]. In the application of plane-wave reflection from a uniaxial skewon-axion medium half-space [15] it was shown that the plane-wave fields in the skewon-axion medium become uniquely determined through the conditions at the interface.

The strange free-choice property of the wave one-form $\boldsymbol{\nu}$ in the P-medium can be considered from a more general point of view as follows. Expressing (80) in the form

$$\overline{\overline{\mathrm{D}}}(\boldsymbol{\nu})|\boldsymbol{\phi} = 0, \qquad \overline{\overline{\mathrm{D}}}(\boldsymbol{\nu}) = -\mathbf{e}_N\lfloor\left(\boldsymbol{\nu}\wedge\overline{\overline{\mathrm{M}}}\lfloor\boldsymbol{\nu}\right) \in \mathbb{E}_1\mathbb{E}_1, \tag{85}$$

the dispersion dyadic $\overline{\overline{\mathrm{D}}}(\boldsymbol{\nu})$ satisfies for any medium [22]

$$\overline{\overline{\mathrm{D}}}^{(3)}(\boldsymbol{\nu}) = (\mathbf{e}_N\mathbf{e}_N\rfloor\rfloor\boldsymbol{\nu}\boldsymbol{\nu})\,D(\boldsymbol{\nu}), \tag{86}$$

where the scalar dispersion function $D(\boldsymbol{\nu})$ depends on the medium dyadic. Because of

$$\overline{\overline{\mathrm{D}}}^{(3)}\rfloor\boldsymbol{\phi} = \left(\overline{\overline{\mathrm{D}}}|\boldsymbol{\phi}\right)\wedge\overline{\overline{\mathrm{D}}}^{(2)} = 0, \tag{87}$$

the dispersion equation can be presented as

$$D(\boldsymbol{\nu}) = 0. \tag{88}$$

One can now easily show that for a P-medium the dispersion function is identically zero, so it does not limit the choice of the one-form $\boldsymbol{\nu}$. In fact, expanding

$$\overline{\overline{\mathrm{D}}}(\boldsymbol{\nu}) = -\mathbf{e}_N\rfloor\left(\boldsymbol{\nu}\wedge\overline{\overline{\mathrm{P}}}^{(2)}\rfloor\boldsymbol{\nu}\right) = -\mathbf{e}_N\rfloor\left(\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\wedge\overline{\overline{\mathrm{P}}}\right) = \mathbf{F}\rfloor\overline{\overline{\mathrm{P}}}, \tag{89}$$

with the bivector

$$\mathbf{F}(\boldsymbol{\nu}) = \mathbf{e}_N\rfloor\left(\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\right), \tag{90}$$

we have

$$\overline{\overline{\mathrm{D}}}^{(3)}(\boldsymbol{\nu}) = -\left(\mathbf{F}\rfloor\overline{\overline{\mathrm{I}}}^{T}\right)^{(3)}|\overline{\overline{\mathrm{P}}}^{(3)}. \tag{91}$$

Now from

$$\begin{aligned}\mathbf{F}\cdot\mathbf{F} &= \mathbf{F}|(\boldsymbol{\varepsilon}_N\rfloor\mathbf{F}) = \left(\mathbf{e}_N\rfloor\left(\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\right)\right)|\left(\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\right)\\ &= \mathbf{e}_N|\left(\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\wedge\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\right) = 0,\end{aligned} \tag{92}$$

$\mathbf{F}$ is a simple bivector, i.e., of the form $\mathbf{a}\wedge\mathbf{b}$, and it satisfies

$$\left(\mathbf{F}\rfloor\overline{\overline{\mathrm{I}}}^{T}\right)^{(2)} = \mathbf{F}\mathbf{F}, \qquad \left(\mathbf{F}\rfloor\overline{\overline{\mathrm{I}}}^{T}\right)^{(3)} = 0. \tag{93}$$

Thus, (91) does not give rise to a dispersion function $D(\boldsymbol{\nu})$ as in (86) to limit the choice of the wave one-form $\boldsymbol{\nu}$.

6. GENERALIZED P-MEDIUM

6.1. Medium Conditions

In analogy with the generalization of the Q-medium, the medium dyadic of the P-medium can be extended by adding a term of the form

$$\overline{\overline{\mathrm{M}}} = \overline{\overline{\mathrm{P}}}^{(2)} + \boldsymbol{\varepsilon}_N\rfloor\mathbf{D}\mathbf{C}, \tag{94}$$

or

$$\overline{\overline{\mathrm{M}}}_g = \mathbf{e}_N\lfloor\overline{\overline{\mathrm{P}}}^{(2)} + \mathbf{DC}, \tag{95}$$

where $\mathbf{D}, \mathbf{C} \in \mathbb{E}_2$ are two bivectors. A medium defined in this way can be called a generalized P-medium. Since the Hodge dual of a bivector again yields a bivector, generalized P-media are related to generalized Q-media of (7) by means of the Hodge duality operation (25). Let us now briefly consider some properties of the generalized P-media.

Expanding the bivectors as

$$\mathbf{D} = \mathbf{d}_1 \wedge \mathbf{d}_2 + \mathbf{d}_3 \wedge \mathbf{e}_4, \qquad \mathbf{C} = \mathbf{c}_1 \wedge \mathbf{c}_2 + \mathbf{c}_3 \wedge \mathbf{e}_4, \tag{96}$$

where the vectors $\mathbf{c}_i, \mathbf{d}_i$ are all spatial, we have

$$\boldsymbol{\varepsilon}_N\lfloor\mathbf{DC} = (-\boldsymbol{\varepsilon}_4 \wedge (\boldsymbol{\varepsilon}_{123}\lfloor(\mathbf{d}_1 \wedge \mathbf{d}_2)) + \boldsymbol{\varepsilon}_{123}\lfloor\mathbf{d}_3)(\mathbf{c}_1 \wedge \mathbf{c}_2 + \mathbf{c}_3 \wedge \mathbf{e}_4), \tag{97}$$

whence the 3D P-medium parameter dyadics can be generalized from those expressed by (14)–(17) to

$$\overline{\overline{\alpha}} = \overline{\overline{\mathrm{P}}}_s^{(2)} + (\boldsymbol{\varepsilon}_{123}\lfloor\mathbf{d}_3)(\mathbf{c}_1 \wedge \mathbf{c}_2), \tag{98}$$

$$\overline{\overline{\epsilon}}' = -\boldsymbol{\pi} \wedge \overline{\overline{\mathrm{P}}}_s + (\boldsymbol{\varepsilon}_{123}\lfloor\mathbf{d}_3)\mathbf{c}_3, \tag{99}$$

$$\overline{\overline{\mu}}^{-1} = -\overline{\overline{\mathrm{P}}}_s \wedge \mathbf{p} - \boldsymbol{\varepsilon}_{123}\lfloor(\mathbf{d}_1 \wedge \mathbf{d}_2)(\mathbf{c}_1 \wedge \mathbf{c}_2), \tag{100}$$

$$\overline{\overline{\beta}} = (\boldsymbol{\pi}\mathbf{p} - p\overline{\overline{\mathrm{P}}}_s) - \boldsymbol{\varepsilon}_{123}\lfloor(\mathbf{d}_1 \wedge \mathbf{d}_2))\mathbf{c}_3. \tag{101}$$

The generalization makes it possible for the dyadics $\overline{\overline{\epsilon}}'$ and $\overline{\overline{\mu}}^{-1}$ to have inverses.

6.2. Plane Wave

Considering a plane wave in a generalized P-medium, the dispersion dyadic (85) becomes

$$\overline{\overline{\mathrm{D}}}(\boldsymbol{\nu}) = -\mathbf{e}_N\lfloor\left(\boldsymbol{\nu} \wedge \left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right) \wedge \overline{\overline{\mathrm{P}}}\right) - (\mathbf{D}\lfloor\boldsymbol{\nu})(\mathbf{C}\lfloor\boldsymbol{\nu}). \tag{102}$$

Applying again the bivector $\mathbf{F}(\boldsymbol{\nu})$ defined by (90), the dispersion dyadic has the form

$$\overline{\overline{\mathrm{D}}}(\boldsymbol{\nu}) = \left(\mathbf{F}(\boldsymbol{\nu})\lfloor\overline{\overline{\mathrm{I}}}^T\right)|\overline{\overline{\mathrm{P}}} + (\mathbf{D}\lfloor\boldsymbol{\nu})\,(\mathbf{C}\lfloor\boldsymbol{\nu})\,. \tag{103}$$

With (93), we can write

$$\overline{\overline{\mathrm{D}}}^{(2)}(\boldsymbol{\nu}) = \mathbf{FF}|\overline{\overline{\mathrm{P}}}^{(2)} + \left(\left(\mathbf{F}\lfloor\overline{\overline{\mathrm{I}}}^T\right)|\overline{\overline{\mathrm{P}}}\right) {}^{\wedge}_{\wedge}(\mathbf{D}\lfloor\boldsymbol{\nu})(\mathbf{C}\lfloor\boldsymbol{\nu}), \tag{104}$$

$$\begin{aligned}\overline{\overline{\mathrm{D}}}^{(3)}(\boldsymbol{\nu}) &= \left(\mathbf{FF}|\overline{\overline{\mathrm{P}}}^{(2)}\right) {}^{\wedge}_{\wedge}(\mathbf{D}\lfloor\boldsymbol{\nu})\,(\mathbf{C}\lfloor\boldsymbol{\nu}) \\ &= (\mathbf{F} \wedge (\mathbf{D}\lfloor\boldsymbol{\nu}))\left(\mathbf{F}|\overline{\overline{\mathrm{P}}}^{(2)} \wedge (\mathbf{C}\lfloor\boldsymbol{\nu})\right).\end{aligned} \tag{105}$$

Invoking the identity [6]

$$\mathbf{F}\wedge(\mathbf{D}\lfloor\boldsymbol{\nu})+\mathbf{D}\wedge(\mathbf{F}\lfloor\boldsymbol{\nu})=(\mathbf{F}\wedge\mathbf{D})\lfloor\boldsymbol{\nu}=\boldsymbol{\varepsilon}_N|(\mathbf{F}\wedge\mathbf{D})(\mathbf{e}_N\lfloor\boldsymbol{\nu}), \quad (106)$$

from $\mathbf{F}\lfloor\boldsymbol{\nu}=0$, we can further write

$$\overline{\overline{\mathrm{D}}}^{(3)}(\boldsymbol{\nu})=\boldsymbol{\varepsilon}_N|(\mathbf{F}\wedge\mathbf{D})\,(\mathbf{e}_N\lfloor\boldsymbol{\nu})\left(\mathbf{F}|\overline{\overline{\mathrm{P}}}^{(2)}\wedge(\mathbf{C}\lfloor\boldsymbol{\nu})\right). \quad (107)$$

Substituted in (87) we finally obtain

$$\boldsymbol{\varepsilon}_N|(\mathbf{F}\wedge\mathbf{D})\left(\mathbf{F}|\overline{\overline{\mathrm{P}}}^{(2)}\wedge(\mathbf{C}\lfloor\boldsymbol{\nu})\right)\lfloor\boldsymbol{\phi}=0. \quad (108)$$

This condition, satisfied by the one-form $\boldsymbol{\phi}$, yields two possible dispersion equations. In the first case the scalar factor vanishes,

$$\boldsymbol{\varepsilon}_N|(\mathbf{F}\wedge\mathbf{D})=\left(\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\right)|\mathbf{D}=0, \quad \Rightarrow \quad \boldsymbol{\nu}|\left(\mathbf{D}\lfloor\overline{\overline{\mathrm{P}}}\right)|\boldsymbol{\nu}=0, \quad (109)$$

which is a second-order scalar equation for $\boldsymbol{\nu}$. For the second possibility we first expand

$$\begin{aligned}\mathbf{F}|\overline{\overline{\mathrm{P}}}^{(2)} &= \mathbf{e}_N|\left(\boldsymbol{\nu}\wedge\left(\overline{\overline{\mathrm{P}}}|\boldsymbol{\nu}\right)\wedge\overline{\overline{\mathrm{P}}}^{(2)}\right)=\mathbf{e}_N|\left(\boldsymbol{\nu}\wedge\overline{\overline{\mathrm{P}}}^{(3)}\lfloor\boldsymbol{\nu}\right)\\ &= \boldsymbol{\nu}|\left(\mathbf{e}_N\lfloor\overline{\overline{\mathrm{P}}}^{(3)})\lfloor\boldsymbol{\nu}\right)=\Delta_P\left(\left(\overline{\overline{\mathrm{P}}}^{-1}|\boldsymbol{\nu}\right)\rfloor\mathbf{e}_N\right)\lfloor\boldsymbol{\nu}, \quad (110)\end{aligned}$$

when applying the inverse rule [6]

$$\overline{\overline{\mathrm{P}}}^{-1}=\boldsymbol{\varepsilon}_N\mathbf{e}_N\lfloor\lfloor\overline{\overline{\mathrm{P}}}^{(3)T}/\Delta_P, \quad \Delta_P=\mathbf{e}_N\boldsymbol{\varepsilon}_N||\overline{\overline{\mathrm{P}}}^{(4)}. \quad (111)$$

The second condition arising from (108) becomes

$$\begin{aligned}&\left(\mathbf{F}|\overline{\overline{\mathrm{P}}}^{(2)}\wedge(\mathbf{C}\lfloor\boldsymbol{\nu})\right)\lfloor\boldsymbol{\phi}=\Delta_P\left(\left(\left(\overline{\overline{\mathrm{P}}}^{-1}|\boldsymbol{\nu}\right)\rfloor\mathbf{e}_N\right)\lfloor\boldsymbol{\nu}\right)\wedge(\mathbf{C}\lfloor\boldsymbol{\nu}))\lfloor\boldsymbol{\phi}\\ &=\Delta_P\left(\left(\overline{\overline{\mathrm{P}}}^{-1}|\boldsymbol{\nu}\right)\rfloor\mathbf{e}_N\right)\wedge(\mathbf{C}\lfloor\boldsymbol{\nu})\lfloor(\boldsymbol{\nu}\wedge\boldsymbol{\phi})\\ &=\Delta_P\left((\mathbf{C}\lfloor\boldsymbol{\nu})|\overline{\overline{\mathrm{P}}}^{-1}|\boldsymbol{\nu}\right)\mathbf{e}_N\lfloor\Phi=0, \quad \Rightarrow \quad \boldsymbol{\nu}|\left(\mathbf{C}\lfloor\overline{\overline{\mathrm{P}}}^{-1}\right)|\boldsymbol{\nu}=0, (112)\end{aligned}$$

assuming $\Delta_P=\mathbf{e}_N\boldsymbol{\varepsilon}_N||\overline{\overline{\mathrm{P}}}^{(4)}\neq 0$.

As a conclusion, for the generalized P-medium the wave one-form $\boldsymbol{\nu}$ must satisfy either of the two dispersion equations

$$\left(\mathbf{D}\lfloor\overline{\overline{\mathrm{P}}}\right)||\boldsymbol{\nu}\boldsymbol{\nu} = 0, \quad (113)$$

$$\left(\mathbf{C}\lfloor\overline{\overline{\mathrm{P}}}^{-1}\right)||\boldsymbol{\nu}\boldsymbol{\nu} = 0. \quad (114)$$

For $\overline{\overline{\mathrm{P}}}^{(4)}=0$, we must replace $\overline{\overline{\mathrm{P}}}^{-1}$ in (114) by $\boldsymbol{\varepsilon}_N\mathbf{e}_N\lfloor\lfloor\overline{\overline{\mathrm{P}}}^{(3)T}$. For $\overline{\overline{\mathrm{P}}}^{(3)}=0$ (114) does not exist. In the case of basic P-medium, $\mathbf{C}=\mathbf{D}=0$, neither of the dispersion Equations (113), (114) limits the choice of the one-form $\boldsymbol{\nu}$.

6.3. Field Conditions

Defining the dot product for two two-forms $\mathbf{\Phi}, \mathbf{\Psi}$ as

$$\mathbf{\Phi} \cdot \mathbf{\Psi} = \mathbf{\Phi}|(\mathbf{e}_N \lfloor \mathbf{\Psi}) = \mathbf{e}_N|(\mathbf{\Phi} \wedge \mathbf{\Psi}) = \mathbf{\Psi} \cdot \mathbf{\Phi}, \tag{115}$$

the fields of any plane wave in any linear medium satisfy the orthogonality conditions $\mathbf{\Psi} \cdot \mathbf{\Psi} = 0$ and $\mathbf{\Phi} \cdot \mathbf{\Phi} = 0$. For the field two-form $\mathbf{\Phi}$ of a plane wave in a generalized P-medium we obtain the condition

$$\begin{aligned} &\mathbf{\Psi} \cdot \mathbf{\Psi} - \Delta_P \mathbf{\Phi} \cdot \mathbf{\Phi} = \mathbf{\Phi}| \left(\overline{\overline{\mathrm{M}}}^T | \mathbf{e}_N \lfloor \overline{\overline{\mathrm{M}}} - \Delta_P \mathbf{e}_N \lfloor \overline{\overline{\mathrm{I}}}^{(2)T} \right) |\mathbf{\Phi} \\ &= \mathbf{\Phi}| \left(\overline{\overline{\mathrm{P}}}^{(2)T} | \mathbf{e}_N \lfloor \overline{\overline{\mathrm{P}}}^{(2)} - \Delta_P \mathbf{e}_N \lfloor \overline{\overline{\mathrm{I}}}^{(2)T} + 2\mathbf{C}\mathbf{D} | \overline{\overline{\mathrm{P}}}^{(2)} + (\mathbf{D} \cdot \mathbf{D}) \mathbf{C}\mathbf{C} \right) |\mathbf{\Phi} \\ &= (\mathbf{\Phi}|\mathbf{C}) \left(\left(2\mathbf{D} | \overline{\overline{\mathrm{P}}}^{(2)} + (\mathbf{D} \cdot \mathbf{D})\, \mathbf{C} \right) |\mathbf{\Phi} \right) = 0, \end{aligned} \tag{116}$$

since $\overline{\overline{\mathrm{P}}}^{(2)T} | \mathbf{e}_N \lfloor \overline{\overline{\mathrm{P}}}^{(2)} - \Delta_P \mathbf{e}_N \lfloor \overline{\overline{\mathrm{I}}}^{(2)T} = 0$. From this we conclude that the field of a plane wave in a generalized P-medium must satisfy either of the two conditions:

$$\mathbf{C}|\mathbf{\Phi} = 0, \tag{117}$$

$$\left(2\mathbf{D} | \overline{\overline{\mathrm{P}}}^{(2)} + (\mathbf{D} \cdot \mathbf{D}) \mathbf{C} \right) |\mathbf{\Phi} = 0. \tag{118}$$

In 3D expansions these conditions have the respective form

$$(\mathbf{c}_1 \wedge \mathbf{c}_2)\, |\mathbf{B} + \mathbf{c}_3 | \mathbf{E} = 0, \tag{119}$$

$$\begin{aligned} &\left((\mathbf{d}_1 \wedge \mathbf{d}_2)\, | \overline{\overline{\mathrm{P}}}_s^{(2)} + \mathbf{d}_3 | \overline{\overline{\mathrm{P}}}_s \wedge \mathbf{p} + \Delta_D \mathbf{c}_1 \wedge \mathbf{c}_2 \right) |\mathbf{B} \\ &- \left(((\mathbf{d}_1 \wedge \mathbf{d}_2)\, \lfloor \boldsymbol{\pi} - p \mathbf{d}_3)\, | \overline{\overline{\mathrm{P}}}_s + (\mathbf{d}_3 | \boldsymbol{\pi})\, \mathbf{p} - \Delta_D \mathbf{c}_3 \right) |\mathbf{E} = 0, \end{aligned} \tag{120}$$

$$\Delta_D = \frac{1}{2} \mathbf{D} \cdot \mathbf{D} = \frac{1}{2} \boldsymbol{\varepsilon}_{123} |\, (\mathbf{d}_1 \wedge \mathbf{d}_2 \wedge \mathbf{d}_3)\, . \tag{121}$$

Obviously, (117) and (118) are associated to the two dispersion Equations (113), (114). The question remains which one corresponds to which one. Let us consider the wave associated to (117). Because in this case, the generalized P-medium can be replaced by the ungeneralized P-medium, the field two-form must be of the form (83). Requiring (117) to be satisfied,

$$\mathbf{C}|\mathbf{\Phi} = A\mathbf{C}| \left(\boldsymbol{\nu} \wedge \overline{\overline{\mathrm{P}}}^{-1} | \boldsymbol{\nu} \right) = -A\boldsymbol{\nu}| \left(\mathbf{C} \lfloor \overline{\overline{\mathrm{P}}}^{-1} \right) |\boldsymbol{\nu} = 0, \tag{122}$$

we arrive at (114). Since (114) corresponds to (117), (113) must correspond to (118).

7. CONCLUSION

In this paper, a class of media is considered whose constitutive relation resembles that of the class of Q-media studied a few years ago. Because of its definition in four-dimensional formalism is so simple, the medium class deserves attention. Applying three-dimensional expansions, it is demonstrated that the medium equations cannot be expressed with Gibbsian vectors and dyadics as **D** and **B** in terms of **E** and **H**, because the dyadic $\overline{\overline{\mu}}_g^{-1}$ has no inverse. As a special case, P-media with no skewon component are considered. It is shown that, in such a case, there cannot be any principal component. Equivalently, a pure principal P-medium does not exist, contrary to what is known for Q-media. As another example, the special case of uniaxial P-media is considered in relation to uniaxial skewon-axion (or IB)-media. The half space of uniaxial P-media can be used to realize the recently studied DB boundary condition requiring vanishing normal components of **D** and **B**. As another property for the P-medium it is shown that there is no restricting condition (dispersion equation) for the wave one-form of a plane wave, which property is shared with the skewon-axion medium. Finally, a generalization to P-media, similar to that of Q-media, is introduced and the corresponding plane-wave properties are briefly studied. It is shown that, for the generalized P-medium, the dispersion equation is factorized in two simpler ones the solutions of which correspond to certain polarization properties of the plane wave.

APPENDIX A. PROPERTIES OF SOME DYADICS

A.1. Gibbsian Dyadics

Because spatial vectors, bivectors, one-forms and two-forms have three components, they can be represented by Gibbsian vectors, elements of $\mathbb{E}_1$. The same applies to various 3D dyadics. Let us consider the different cases.

The Gibbsian representation depends on a chosen spatial metric dyadic $\overline{\overline{\mathrm{G}}}_s \in \mathbb{E}_1\mathbb{E}_1$

$$\overline{\overline{\mathrm{G}}}_s = \sum_1^3 \mathbf{e}_i\mathbf{e}_i = \overline{\overline{\Gamma}}_s^{\,-1}, \qquad \overline{\overline{\Gamma}}_s = \sum_1^3 \boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i, \tag{A1}$$

the one-form fields $\mathbf{E}, \mathbf{H}$ can be transformed to Gibbsian vectors as

$$\mathbf{E}_g = \overline{\overline{\mathrm{G}}}_s|\mathbf{E}, \quad \mathbf{H}_g = \overline{\overline{\mathrm{G}}}_s|\mathbf{H}. \tag{A2}$$

The two-form fields $\mathbf{D}, \mathbf{B}$ are transformed by

$$\mathbf{D}_g = \mathbf{e}_{123}\lfloor\mathbf{D}, \quad \mathbf{B}_g = \mathbf{e}_{123}\lfloor\mathbf{B}. \tag{A3}$$

A 3D bivector $\mathbf{A}$ is first transformed to a one-form $\boldsymbol{\varepsilon}_{123}\lfloor\mathbf{A}$:

$$\mathbf{A}_g = \overline{\overline{\mathsf{G}}}_s|(\boldsymbol{\varepsilon}_{123}\lfloor\mathbf{A}). \tag{A4}$$

The Gibbsian counterparts of the spatial medium dyadics are defined by

$$\overline{\overline{\alpha}}_g = \mathbf{e}_{123}\lfloor\overline{\overline{\alpha}}\rfloor\boldsymbol{\varepsilon}_{123}|\overline{\overline{\mathsf{G}}}_s, \quad \overline{\overline{\epsilon}}'_g = \mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}', \tag{A5}$$

$$\overline{\overline{\mu}}_g^{-1} = \overline{\overline{\mathsf{G}}}_s|\overline{\overline{\mu}}^{-1}\rfloor\boldsymbol{\varepsilon}_{123}|\overline{\overline{\mathsf{G}}}_s, \quad \overline{\overline{\beta}}_g = \overline{\overline{\mathsf{G}}}_s|\overline{\overline{\beta}}, \quad \in \mathbb{E}_1\mathbb{E}_1. \tag{A6}$$

The dot and cross products between two Gibbsian quantities are defined by

$$\overline{\overline{\beta}}_g \cdot \mathbf{E}_g = \overline{\overline{\beta}}_g|\overline{\overline{\Gamma}}_s|\mathbf{E}_g = \overline{\overline{\mathsf{G}}}_s|\overline{\overline{\beta}}|\overline{\overline{\Gamma}}_s|\left(\overline{\overline{\mathsf{G}}}_s|\mathbf{E}\right) = \overline{\overline{\mathsf{G}}}_s|\overline{\overline{\beta}}|\mathbf{E}, \tag{A7}$$

$$\mathbf{a}_g \times \mathbf{b}_g = \mathbf{e}_{123}\lfloor\left(\left(\overline{\overline{\Gamma}}_s|\mathbf{a}_g\right) \wedge \left(\overline{\overline{\Gamma}}_s|\mathbf{b}_g\right)\right). \tag{A8}$$

A.2. Unipotent Dyadics

Dyadics $\overline{\overline{\mathsf{D}}} \in \mathbb{E}_1\mathbb{F}_1$ satisfying

$$\overline{\overline{\mathsf{D}}}^2 = \overline{\overline{\mathsf{I}}}, \tag{A9}$$

are called unipotent. A unipotent dyadic has an inverse which equals the dyadic itself. Expressing the condition (A9) as

$$\left(\overline{\overline{\mathsf{D}}} - \overline{\overline{\mathsf{I}}}\right) | \left(\overline{\overline{\mathsf{D}}} + \overline{\overline{\mathsf{I}}}\right) = 0, \tag{A10}$$

and considering solutions in the form

$$\overline{\overline{\mathsf{D}}} = \overline{\overline{\mathsf{I}}} - 2\overline{\overline{\Pi}}, \tag{A11}$$

we see that the dyadic $\overline{\overline{\Pi}}$ satisfies

$$\overline{\overline{\Pi}}^2 = \overline{\overline{\Pi}}. \tag{A12}$$

This means that $\overline{\overline{\Pi}}$ must be a projection dyadic. The complementary projection dyadic

$$\overline{\overline{\Pi}}' = \overline{\overline{\mathsf{I}}} - \overline{\overline{\Pi}}, \tag{A13}$$

can be shown to satisfy the same Equation (A12) and the conditions

$$\overline{\overline{\Pi}}|\overline{\overline{\Pi}}' = 0, \quad \overline{\overline{\Pi}} + \overline{\overline{\Pi}}' = \overline{\overline{\mathsf{I}}}. \tag{A14}$$

The unipotent dyadic can be expressed in various forms as

$$\overline{\overline{\mathsf{D}}} = \overline{\overline{\Pi}}' - \overline{\overline{\Pi}} = \overline{\overline{\mathsf{I}}} - 2\overline{\overline{\Pi}} = -\left(\overline{\overline{\mathsf{I}}} - 2\overline{\overline{\Pi}}'\right). \tag{A15}$$

Any projection dyadic $\overline{\overline{\Pi}}$ maps vectors of the four-dimensional vector space into a subspace of dimension p and acts as a unit dyadic for vectors in that subspace. Correspondingly, $\overline{\overline{\Pi}}'$ maps any vector in the complementary subspace of dimension $4-p$ and acts as a unit dyadic in that subspace. We can choose a vector basis $\{\mathbf{e}_i\}$ and the reciprocal basis of one-forms $\{\boldsymbol{\varepsilon}_i\}$ in a suitable manner so that vectors $\mathbf{e}_1 \dots \mathbf{e}_p$ and $\mathbf{e}_{p+1} \dots \mathbf{e}_4$ belong to the corresponding subspaces. In this, we can separate five different cases,

$$\begin{aligned}
&\text{(i)}\ \ p=0, \ \Rightarrow\ \overline{\overline{\Pi}}=0 \quad \overline{\overline{\Pi}}' = \mathbf{e}_1\boldsymbol{\varepsilon}_1+\mathbf{e}_2\boldsymbol{\varepsilon}_2+\mathbf{e}_3\boldsymbol{\varepsilon}_3+\mathbf{e}_4\boldsymbol{\varepsilon}_4=\overline{\overline{\mathrm{I}}},\\
&\text{(ii)}\ \ p=1, \ \Rightarrow\ \overline{\overline{\Pi}}=\mathbf{e}_1\boldsymbol{\varepsilon}_1, \quad \overline{\overline{\Pi}}' = \mathbf{e}_2\boldsymbol{\varepsilon}_2+\mathbf{e}_3\boldsymbol{\varepsilon}_3+\mathbf{e}_4\boldsymbol{\varepsilon}_4,\\
&\text{(iii)}\ \ p=2, \ \Rightarrow\ \overline{\overline{\Pi}}=\mathbf{e}_1\boldsymbol{\varepsilon}_1+\mathbf{e}_2\boldsymbol{\varepsilon}_2, \quad \overline{\overline{\Pi}}' = \mathbf{e}_3\boldsymbol{\varepsilon}_3+\mathbf{e}_4\boldsymbol{\varepsilon}_4,\\
&\text{(iv)}\ \ p=3, \ \Rightarrow\ \overline{\overline{\Pi}}=\mathbf{e}_1\boldsymbol{\varepsilon}_1+\mathbf{e}_2\boldsymbol{\varepsilon}_2+\mathbf{e}_3\boldsymbol{\varepsilon}_3, \quad \overline{\overline{\Pi}}' = \mathbf{e}_4\boldsymbol{\varepsilon}_4,\\
&\text{(v)}\ \ p=4, \ \Rightarrow\ \overline{\overline{\Pi}}=\mathbf{e}_1\boldsymbol{\varepsilon}_1+\mathbf{e}_2\boldsymbol{\varepsilon}_2+\mathbf{e}_3\boldsymbol{\varepsilon}_3+\mathbf{e}_4\boldsymbol{\varepsilon}_4=\overline{\overline{\mathrm{I}}}, \quad \overline{\overline{\Pi}}' = 0
\end{aligned}$$

From the previous it follows that there are three different types of solutions to (A9). The cases 1 and 5 correspond to the solutions

$$\overline{\overline{\mathrm{D}}} = \pm\overline{\overline{\mathrm{I}}}, \tag{A16}$$

while the cases 2 and 4 lead to

$$\overline{\overline{\mathrm{D}}} = \pm(\overline{\overline{\mathrm{I}}} - 2\mathbf{e}_1\boldsymbol{\varepsilon}_1), \tag{A17}$$

and the case 3 yields

$$\overline{\overline{\mathrm{D}}} = \pm\left(\overline{\overline{\mathrm{I}}} - 2(\mathbf{e}_1\boldsymbol{\varepsilon}_1 + \mathbf{e}_2\boldsymbol{\varepsilon}_2)\right). \tag{A18}$$

It is easy to check that all these expressions satisfy (A9).

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