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APPROXIMATE BOUNDARY RELATIONS ON ANISOTROPIC SHEETS

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Abstract—Approximate boundary relations on general anisotropic sheets of arbitrary shape as well as the special case when they are backed by a perfect electrical conductor are investigated based on a generalization of the procedure introduced by İdemen in 1993 for uniaxially anisotropic planar sheets to general anisotropic and arbitrarily shaped surfaces. The ranges of validity of the approximations in the methodology are also tested numerically for the impedance boundary condition obtained in the case of a PEC backed uniaxially anisotropic sheet.

1. INTRODUCTION

Simulation of natural or man-made thin layers by approximate boundary conditions is one of the most important directions of research in scattering theory connected with complementing the associated boundary value problems in antenna engineering, radio wave propagation and planar microwave technologies. The historical development and descriptions of a large variety of such conditions including resistive, conductive, standard impedance, generalized impedance, and absorbing boundary conditions as well as a systematic treatment of scattering by canonical bodies simulated by such conditions has been introduced in 1995 in the monograph [1].

In the present work we study approximate boundary relations on a closed, arbitrarily shaped thin material surface simulated by general anisotropic constitutive parameters using an original analytical approach by combining the distributional tools on a regular surface of arbitrary shape with the methodology devised by İdemen in [2] for

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uniaxially anisotropic planar sheets. The problem of radiation and scattering by anisotropic thin sheets has earned much importance in parallel to their use in microwave technology as substrate layers. One may refer to [3] and [4] for a historical review and early developments in that field. In that regard the present work is purely analytical with no specific area of application addressed and its contribution to literature can be outlined as follows:

i) The results obtained in [2] for uniaxially anisotropic planar sheets are generalized for arbitrarily shaped sheets by covering the influence of geometry (surface curvature) along with boundary relations tabulated for 8 different sets of tensor constitutive parameters of practical interest.
ii) The procedure is extended rigorously to cover the most general results when the sheet has arbitrary constitutive parameters as well as the case when it is backed by a perfect electrical conductor (PEC). The availability of impedance boundary relations for PEC backed sheets are observed for the same 8 sets of tensor constitutive parameters along with a numerical test of its range of validity when the sheet is assumed nonmagnetic.

The methodology introduced in [2] and extended in the present work in simulating thin material surfaces has the following features:

i) The analytical tools are adopted from the powerful theory of Schwartz-Sobolev distributions and therefore the methodology is capable of taking into account polarization mechanisms of all orders inside the material sheet and yields elegant results in a straightforward manner through general theorems on the orders of field singularities.
ii) The derived boundary relations apply regardless of the structures, frequency, polarizations and locations of the sources outside the material sheet as well as the constitutive parameters of the medium surrounding the sheet. Without losing generality, in the present work we assume the surrounding half spaces as simple media as in [2] while expressing the end results.
iii) The fields inside the material sheet are approximated by a zeroth order averaging procedure so that one obtains first order boundary relations including the practically interesting special cases of impedance, resistive and conductive sheets.

The investigation starts with the description of boundary and compatibility relations and general theorems on the orders of field singularities on an arbitrarily shaped material sheet which may support singular field quantities of arbitrary order. In our simulation we are interested with the results for the special case when the field quantities display singularities of first order as introduced in Sec. 3. In Sec. 4 we introduce the averaging procedure which relates the field quantities at an arbitrary point inside the material sheet to their limit values as one approaches to the surface boundaries. The constitutive relations inside the anisotorpic sheet help express the free and polarizational sources inside the material sheet in terms of electrical and magnetic fields. Through the boundary relations on the upper and lower interfaces between the material sheet and the adjoining media, we reach at the boundary relations between the limiting values of the fields as one approaches on the material sheet from outside. The results for a uniaxially anisotropic sheet as well as for the case it is backed by a PEC constitute the simplest special cases and therefore are derived in the first place in sufficient detail in Secs. 5 and 6, respectively. In Secs. 7 and 8 we present the end results when the procedure is extended rigorously for biaxially and general anisotropic sheets, respectively. Finally, in Sec. 9 we test numerically the ranges of validity of approximations in the methodology over the scalar impedance condition obtained in the case of a PEC backed uniaxially anisotropic sheet.

The analytical tools from the Schwartz-Sobolev theory of distributions as utilized in the investigation are summarized in Appendix, which require to be checked in the first place. We also conform utmost to the notation introduced in [2] where a time convention $\exp(-i\omega t)$ is assumed and suppressed.

2. GENERAL RESULTS FOR BOUNDARY AND COMPATIBILITY RELATIONS ON AN ARBITRARY MATERIAL SHEET

Let us introduce the phasor field equations of arbitrary fixed continuous material media as

$$curl\vec{E}(\vec{r}) - i\omega\vec{B}(\vec{r}) = \vec{0} \tag{1a}$$

$$curl\,\vec{H}(\vec{r}) + i\omega\vec{D}(\vec{r}) = \vec{J}_C(\vec{r}) \tag{1b}$$

$$div\vec{D}(\vec{r}) = \rho_f(\vec{r}) \tag{1c}$$

$$div\vec{B}(\vec{r}) = 0 \tag{1d}$$

and the principle of continuity

$$div\vec{J}_C(\vec{r}) - i\omega\rho_f(\vec{r}) = 0, \tag{2}$$

where the free currents are assumed to comprise conduction currents. The general constitutive relations read

$$\vec{D}(\vec{r}) = \varepsilon_0\vec{E}(\vec{r}) + \vec{P}^e(\vec{r}), \quad \vec{B}(\vec{r}) = \mu_0\vec{H}(\vec{r}) + \vec{P}^m(\vec{r}) \tag{3}$$

When (3) are substituted into (1) we get the alternative set of field equations

$$curl\vec{E}(\vec{r}) - i\omega\mu_0\vec{H}(\vec{r}) = -\vec{J}_P^m(\vec{r}) \tag{4a}$$

$$curl\vec{H}(\vec{r}) + i\omega\varepsilon_0\vec{E}(\vec{r}) = \vec{J}_C(\vec{r}) + \vec{J}_P^e(\vec{r}) \tag{4b}$$

$$\varepsilon_0 div\vec{E}(\vec{r}) = \rho_f(\vec{r}) + \rho_P^e(\vec{r}) \tag{4c}$$

$$\mu_0 div\vec{H}(\vec{r}) = \rho_P^m(\vec{r}) \tag{4d}$$

where four more phasor source quantities

$$\rho_P^{e,m}(\vec{r}) = -div\vec{P}^{e,m}(\vec{r}), \quad \vec{J}_P^{e,m}(\vec{r}) = -i\omega\vec{P}^{e,m}(\vec{r}) \tag{5}$$

derived from the electrical/magnetic polarization density vectors $\vec{P}^{e,m}(\vec{r})$ are introduced. Namely,

- $\rho_P^e(\vec{r}) \equiv -div\vec{P}^e(\vec{r})$: volume density of fixed electrically polarized sources in material media.
- $\rho_P^m(\vec{r}) \equiv -div\vec{P}^m(\vec{r})$: volume density of fixed magnetically polarized sources in material media.
- $\vec{J}_P^e(\vec{r}) \equiv -i\omega\vec{P}^e(\vec{r})$:volume density of electrical polarization current flowing in material media.
- $\vec{J}_P^m(\vec{r}) \equiv -i\omega\vec{P}^m(\vec{r})$:volume density of magnetic polarization current flowing in material media.

The potential fields are beyond our concern. Now let us consider a regular closed surface S that supports singularities of arbitrary order such that in virtue of (A8) all field quantities may be expressed in the general form

$$\vec{E}(\vec{r}) = \left\{\vec{E}(\vec{r})\right\} + \sum_{k=0}^{\infty} \vec{E}_k(\vec{r}_S)\delta^{(k)}(S) \tag{6a}$$

$$\vec{H}(\vec{r}) = \left\{\vec{H}(\vec{r})\right\} + \sum_{k=0}^{\infty} \vec{H}_k(\vec{r}_S)\delta^{(k)}(S) \tag{6b}$$

$$\vec{D}(\vec{r}) = \left\{\vec{D}(\vec{r})\right\} + \sum_{k=0}^{\infty} \vec{D}_k(\vec{r}_S)\delta^{(k)}(S) \tag{6c}$$

$$\vec{B}(\vec{r}) = \left\{\vec{B}(\vec{r})\right\} + \sum_{k=0}^{\infty} \vec{B}_k(\vec{r}_S)\,\delta^{(k)}(S) \tag{6d}$$

$$\vec{P}^e(\vec{r}) = \left\{\vec{P}^e(\vec{r})\right\} + \sum_{k=0}^{\infty} \vec{P}_k^e(\vec{r}_S)\delta^{(k)}(S) \tag{6e}$$

$$\vec{P}^m(\vec{r}) = \left\{\vec{P}^m(\vec{r})\right\} + \sum_{k=0}^{\infty} \vec{P}_k^m(\vec{r}_S)\delta^{(k)}(S) \tag{6f}$$

$$\rho_f(\vec{r}) = \{\rho_f(\vec{r})\} + \sum_{k=0}^{\infty} \rho_k(\vec{r}_S)\delta^{(k)}(S) \tag{6g}$$

$$\vec{J}_C(\vec{r}) = \left\{\vec{J}_C(\vec{r})\right\} + \sum_{k=0}^{\infty} \vec{J}_k(\vec{r}_S)\delta^{(k)}(S) \tag{6h}$$

The surface distribution $\delta^{(k)}(S)$ is described in (A7) and has MKSA unit $[m^{-k-1}]$. Therefore the density fields which are regular point functions on S have the following units

$$\vec{E}_k \left[V \cdot m^k\right]; \quad \vec{H}_k \left[A \cdot m^k\right]; \quad \vec{D}_k, \vec{P}_k^e \quad \left[C \cdot m^{k-1}\right];$$
$$\vec{B}_k, \vec{P}_k^m \left[Wb \cdot m^{k-2}\right]; \quad \rho_k \left[C \cdot m^{k-2}\right]; \quad \vec{J}_k \left[A \cdot m^{k-1}\right].$$

In virtue of the fundamental postulation first introduced in [5] that the Maxwell equations hold in the sense of Schwartz-Sobolev distributions and also Theorem A5, one may substitute (6) into (1)–(3), which yield

$$\vec{D}_k(\vec{r}_S) = \varepsilon_0\vec{E}_k(\vec{r}_S) + \vec{P}_k^e(\vec{r}_S), \quad \vec{B}_k(\vec{r}_S) = \mu_0\vec{H}_k(\vec{r}_S) + \vec{P}_k^m(\vec{r}_S), \ \forall k \tag{7}$$

and an infinite system of equations for $\forall k$. For $k = 0$, the corresponding equations

$$\hat{n} \times [\vec{E}_t^{II} - \vec{E}_t^I] = i\omega\vec{P}_0^m - curl_S\vec{E}_0 + i\omega\mu_0\vec{H}_0 \tag{8a}$$
$$\hat{n} \times [\vec{H}_t^{II} - \vec{H}_t^I] = \vec{J}_0 - i\omega\vec{P}_0^e - curl_S\vec{H}_0 - i\omega\varepsilon_0\vec{E}_0 \tag{8b}$$
$$D_n^{II} - D_n^I = \rho_0 - div_S\vec{P}_0^e - \varepsilon_0 div_S\vec{E}_0 \tag{8c}$$
$$B_n^{II} - B_n^I = -div_S\vec{P}_0^m - \mu_0 div_S\vec{H}_0 \tag{8d}$$
$$J_{Cn}^{II} - J_{Cn}^I = -div_S\vec{J}_0 + i\omega\rho_0 \tag{8e}$$

describe the boundary relations on the material sheet, whereas the recursive sets for $k \geq 1$

$$\hat{n} \times \vec{E}_{k-1} + curl_S\vec{E}_k - i\omega\mu_0\vec{H}_k = i\omega\vec{P}_k^m, \quad k \geq 1 \tag{9a}$$
$$\hat{n} \times \vec{H}_{k-1} + curl_S\vec{H}_k + i\omega\varepsilon_0\vec{E}_k = \vec{J}_k - i\omega\vec{P}_k^e, \quad k \geq 1 \tag{9b}$$
$$\hat{n} \cdot \vec{E}_{k-1} + div_S\vec{E}_k = (1/\varepsilon_0)\left(\rho_k - \hat{n} \cdot \vec{P}_{k-1}^e - div_S\vec{P}_k^e\right), \quad k \geq 1 \tag{9c}$$
$$\hat{n} \cdot \vec{H}_{k-1} + div_S\vec{H}_k = (1/\mu_0)\left(-\hat{n} \cdot \vec{P}_{k-1}^m - div_S\vec{P}_k^m\right), \quad k \geq 1 \tag{9d}$$
$$\hat{n} \cdot \vec{J}_{k-1} + div_S\vec{J}_k - i\omega\rho_k = 0, \quad k \geq 1 \tag{9e}$$

are called the compatibility relations. Here we assume the surface S divides the entire space into two regions I and II with its unit normal $\hat{n}$ directed into region II. Regarding the notation, the field quantities in the form $\vec{A}^I$, $\vec{A}^{II}$ in (8) and the rest of the paper denote the limiting values of a vector $\vec{A}$ as one approaches on a point on S from the related regions, while the subscripts t and n indicate the components of any field quantity tangential and normal to S which satisfy $\vec{A} = \vec{A}_t + \hat{n}A_n = (\hat{n} \times \vec{A}) \times \hat{n} + \hat{n}(\hat{n} \cdot \vec{A})$. The relations (8), (9) were first introduced by İdemen in [6, 7] for the special case of a planar interface. In virtue of Theorem A3, the fields (6) are supposed to possess singularities of finite order which also renders the dimension of the sets of the compatibility equations finite and consequently yield the following theorems regarding the orders of singularity on an arbitrary material sheet.

Theorem 1: If there exists a natural number N such that $\rho_k = 0$, $\vec{P}_k^e = \vec{0}$, $\vec{P}_k^m = \vec{0}$, $\vec{J}_k = \vec{0}$, $k \geq N+1$ and $P_{Nn}^e = 0$, $P_{Nn}^m = 0$, then one has $\vec{E}_k = \vec{0}$, $\vec{H}_k = \vec{0}$, $\vec{D}_k = \vec{0}$, $\vec{B}_k = \vec{0}$, $k \geq N$ and $J_{Nn} = 0$.

Theorem 2: On a simple interface between two arbitrary media described electrically by $\vec{P}_k^e = \vec{0}$, $\vec{P}_k^m = \vec{0}$, $\forall k$ and $\rho_k = 0$, $\vec{J}_k = \vec{0}$, $k \geq 1$, one has $\vec{E}_k = \vec{0}$, $\vec{H}_k = \vec{0}$, $\vec{D}_k = \vec{0}$, $\vec{B}_k = \vec{0}$, $\forall k$ and $J_{0n} = 0$. In this case the boundary conditions (8) reduce into

$$\hat{n} \times [\vec{E}_t^{II} - \vec{E}_t^{I}] = \vec{0} \tag{10a}$$

$$\hat{n} \times [\vec{H}_t^{II} - \vec{H}_t^{I}] = \vec{J}_{0t} \tag{10b}$$

$$D_n^{II} - D_n^{I} = \rho_0 \tag{10c}$$

$$B_n^{II} - B_n^{I} = 0 \tag{10d}$$

$$J_{Cn}^{II} - J_{Cn}^{I} = -div_S \vec{J}_{0t} + i\omega\rho_0 \tag{10e}$$

3. RESULTS FOR THE SPECIAL CASE OF AN ARBITRARY MATERIAL SHEET SUPPORTING FIRST ORDER SINGULARITIES

The results in this section are a generalization of the similar results first introduced by İdemen in [8] for the special case of a planar material sheet. Disregarding the constitutive parameters and the thickness of the material sheet, the assumption of its electrical thickness being small enough helps us approximate the actual sources inside by equivalent first order free and polarizational currents with densities $(\vec{J}_0, \vec{P}_0^e, \vec{P}_0^m)$ which flow in both normal and tangential directions on S. Due to the principle of continuity in (2), the presence of free current flow in the normal direction inside the sheet also requires an accumulation of fixed point charges at the two faces of the sheet. This addresses a double layer charge configuration with dipole moment density denoted as $-\rho_1$ so that we can express all equivalent sources *inside* the material sheet as follows:

$$\rho_f(\vec{r}) = \rho_0\delta(S) + \rho_1\delta^{(1)}(S) \tag{11a}$$

$$\vec{J}_C(\vec{r}) = \vec{J}_0\delta(S) \tag{11b}$$

$$\vec{P}^{e,m}(\vec{r}) = \vec{P}_0^{e,m}\delta(S) \tag{11c}$$

The corresponding fields *inside* the material sheet can be determined from Theorem 1 and the compatibility relations (9) as

$$\vec{E}(\vec{r}) = \vec{E}_0\delta(S) \tag{12a}$$

$$\vec{H}(\vec{r}) = \vec{H}_0\delta(S) \tag{12b}$$

$$\vec{D}(\vec{r}) = \vec{D}_0\delta(S) \tag{12c}$$

$$\vec{B}(\vec{r}) = \vec{B}_0\delta(S) \tag{12d}$$

with

$$\vec{D}_0 = \varepsilon_0\vec{E}_0 + \vec{P}_0^e \tag{12e}$$

$$\vec{B}_0 = \mu_0\vec{H}_0 + \vec{P}_0^m \tag{12f}$$

$$\hat{n}\times\vec{E}_0 = \vec{0} \tag{12g}$$

$$\varepsilon_0 E_{0n} = \rho_1 - P_{0n}^e \tag{12h}$$

$$\hat{n}\times\vec{H}_0 = \vec{0} \tag{12i}$$

$$\mu_0 H_{0n} = -P_{0n}^m \tag{12j}$$

$$J_{0n} - i\omega\rho_1 = 0 \tag{12k}$$

from which the density fields can be calculated as

$$\vec{E}_0 = \hat{n}(\rho_1 - P_{0n}^e)/\varepsilon_0 \tag{13a}$$

$$\vec{D}_0 = \hat{n}\rho_1 \tag{13b}$$

$$\vec{H}_0 = -\hat{n}P_{0n}^m/\mu_0 \tag{13c}$$

$$\vec{B}_0 = \vec{0} \tag{13d}$$

Next we invoke (12k) and (13) to shape (8) into

$$\hat{n}\times[\vec{E}_t^{II} - \vec{E}_t^I] = (1/\varepsilon_0)\hat{n}\times grad_S(\rho_1 - P_{0n}^e) + i\omega\vec{P}_{0t}^m \tag{14a}$$

$$\hat{n}\times[\vec{H}_t^{II} - \vec{H}_t^I] = -(1/\mu_0)\hat{n}\times grad_S P_{0n}^m + \vec{J}_{0t} - i\omega\vec{P}_{0t}^e \tag{14b}$$

$$D_n^{II} - D_n^I = \rho_0 + 2\Omega\rho_1 - div_S\vec{P}_0^e - 2\Omega P_{0n}^e = \rho_0 + 2\Omega\rho_1 - div_S\vec{P}_{0t}^e \tag{14c}$$

$$B_n^{II} - B_n^I = -div_S\vec{P}_0^m - 2\Omega P_{0n}^m = -div_S\vec{P}_{0t}^m \tag{14d}$$

$$J_{Cn}^{II} - J_{Cn}^I = -div_S\vec{J}_0 + i\omega\rho_0 = 2\Omega J_{0n} - div_S\vec{J}_{0t} + i\omega\rho_0 \tag{14e}$$

We observe fixed sources ρ_0, ρ_1 in (14a), (14c). If one rather wishes to replace these sources with currents by invoking the continuity relations (12k) and (14e) while preserving the dual nature of the boundary relations, then the corresponding representation of (14a)–(d) reads

$$-i\omega\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^{I}]=-(1/\varepsilon_0)\hat{n}\times grad_S(J_{0n}-i\omega P_{0n}^e)-(-i\omega)^2\,\vec{P}_{0t}^m \quad (15a)$$
$$-i\omega\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^{I}]=i\omega(1/\mu_0)\hat{n}\times grad_S P_{0n}^m-i\omega\vec{J}_{0t}+(-i\omega)^2\,\vec{P}_{0t}^e \quad (15b)$$
$$-i\omega[D_n^{II}-D_n^{I}]=-[J_{Cn}^{II}-J_{Cn}^{I}]-div_S\vec{J}_{0t}+i\omega div_S\vec{P}_{0t}^e \quad (15c)$$
$$-i\omega[B_n^{II}-B_n^{I}]=i\omega div_S\vec{P}_{0t}^m \quad (15d)$$

4. THE AVERAGING PROCEDURE

The next step in the simulation of the material sheet is to relate its geometrical and constitutive parameters to the equivalent zeroth order singular sources with densities ($\vec{J}_{0t}$, J_{0n}) & ($\vec{P}_{0t}^{e,m}$, $P_{0n}^{e,m}$). For this purpose we apply an averaging procedure: Let $\vec{A}_{Slab}(\vec{r})$ be a regular phasor field with its support lying inside the material sheet. Then an equivalence between this physical field and a corresponding first order distribution $\vec{A}(\vec{r}_S)\delta(S)$ can be established in the sense of distributions by the inner product

$$\langle\delta(S),\phi(\vec{r})\rangle=\oint\phi(\vec{r})\,\delta(S)\,dn=\phi(\vec{r}_S), \quad (16a)$$

where dn is the differential line element of the coordinate curve n which is normal to S. Accordingly one can write

$$\left\langle\vec{A}_{Slab}(\vec{r}),\phi(\vec{r})\right\rangle=\left\langle\vec{A}(\vec{r}_S)\delta(S),\phi(\vec{r})\right\rangle. \quad (16b)$$

Let us assume the thickness of the material sheet as $2d$. Since the material sheet is assumed thin enough electrically, variations in the normal direction are assumed linear inside the sheet. Such a zeroth order approximation helps one to express the l.h.s. of (16b) as

$$\begin{aligned}\left\langle\vec{A}_{Slab}(\vec{r}),\phi(\vec{r})\right\rangle &= \oint\vec{A}_{Slab}(\vec{r})\phi(\vec{r})dn=\int_{\substack{\text{“across}\\ \text{the slab''}}}\vec{A}_{Slab}(\vec{r})\phi(\vec{r})dn\\ &\simeq\int_{\substack{\text{“across}\\ \text{the slab''}}}\vec{A}_{Slab}(\vec{r}_S)\phi(\vec{r}_S)dn=2d\vec{A}_{Slab}(\vec{r}_S)\phi(\vec{r}_S)\\ &= \left\langle 2d\vec{A}_{Slab}(\vec{r})\delta(S),\phi(\vec{r})\right\rangle\end{aligned}$$

which yields the distributional equivalence

$$\vec{A}(\vec{r}_S) \simeq 2d\,\vec{A}_{Slab}(\vec{r}). \tag{17a}$$

The same result (17a) is derived in [2] by referring to Riesz representation theorem, while we remain inside the distribution theory. Next, let us define S_1 and S_2 as the two simple interfaces of the sheet; and (S_1^+, S_1^-) and (S_2^+, S_2^-) as surfaces infinitely close to S_1 and S_2 as depicted in Fig. 1.

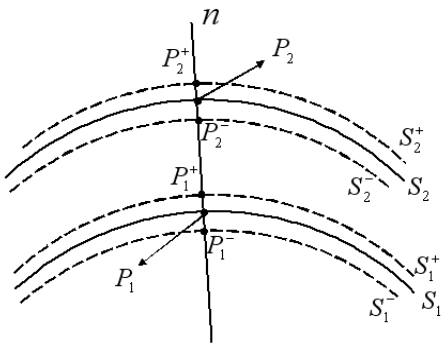

Figure 1. A cross section of the two interfaces of an arbitrary material sheet.

Let the four position vectors $\vec{r}_{S1}^{\pm}$ and $\vec{r}_{S2}^{\pm}$ indicate the points of intersection of $S_1^{\pm}$ and $S_2^{\pm}$ with the normal coordinate curve n. In that case the field inside the sheet can be averaged as

$$\vec{A}_{Slab}(\vec{r}) = \left[\vec{A}_{Slab}(\vec{r}_{S1}^{+}) + \vec{A}_{Slab}(\vec{r}_{S2}^{-})\right]\Big/2 \tag{17b}$$

and we reach to the desired relation

$$\vec{A}(\vec{r}_S) = d\left[\vec{A}_{Slab}(\vec{r}_{S1}^{+}) + \vec{A}_{Slab}(\vec{r}_{S2}^{-})\right]. \tag{17c}$$

(17c) can be adapted to the six equivalent sources of the material sheet as

$$\vec{J}_{0t}(\vec{r}_S) = d\left[\vec{J}_{Ct}(\vec{r}_{S1}^{+}) + \vec{J}_{Ct}(\vec{r}_{S2}^{-})\right] \tag{18a}$$

$$J_{0n}(\vec{r}_S) = d\left[J_{Cn}(\vec{r}_{S1}^{+}) + J_{Cn}(\vec{r}_{S2}^{-})\right] \tag{18b}$$

$$\vec{P}_{0t}^{e,m}(\vec{r}_S) = d\left[\vec{P}_{t}^{e,m}(\vec{r}_{S1}^{+}) + \vec{P}_{t}^{e,m}(\vec{r}_{S2}^{-})\right] \tag{18c}$$

$$P_{0n}^{e,m}(\vec{r}_S) = d\left[P_{n}^{e,m}(\vec{r}_{S1}^{+}) + P_{n}^{e,m}(\vec{r}_{S2}^{-})\right] \tag{18d}$$

5. SIMULATION OF A UNIAXIALLY ANISOTROPIC SHEET

For simplicity we shall assume the entire space (the material sheet and the ambient region) has constant constitutive parameters. Then, in virtue of (10), the fields at the two sides of the interfaces S_1 and S_2 are linked through the boundary relations

$$\vec{E}_t(\vec{r}_{S1}^{\,+}) = \vec{E}_t(\vec{r}_{S1}^{\,-}) \tag{19a}$$
$$D_n(\vec{r}_{S1}^{\,+}) = D_n(\vec{r}_{S1}^{\,-}) \tag{19b}$$
$$\vec{H}_t(\vec{r}_{S1}^{\,+}) = \vec{H}_t(\vec{r}_{S1}^{\,-}) \tag{19c}$$
$$B_n(\vec{r}_{S1}^{\,+}) = B_n(\vec{r}_{S1}^{\,-}) \tag{19d}$$
$$J_{Cn}(\vec{r}_{S1}^{\,+}) = J_{Cn}(\vec{r}_{S1}^{\,-}) \tag{19e}$$

and

$$\vec{E}_t(\vec{r}_{S2}^{\,+}) = \vec{E}_t(\vec{r}_{S2}^{\,-}) \tag{20a}$$
$$D_n(\vec{r}_{S2}^{\,+}) = D_n(\vec{r}_{S2}^{\,-}) \tag{20b}$$
$$\vec{H}_t(\vec{r}_{S2}^{\,+}) = \vec{H}_t(\vec{r}_{S2}^{\,-}) \tag{20c}$$
$$B_n(\vec{r}_{S2}^{\,+}) = B_n(\vec{r}_{S2}^{\,-}) \tag{20d}$$
$$J_{Cn}(\vec{r}_{S2}^{\,+}) = J_{Cn}(\vec{r}_{S2}^{\,-}) \tag{20e}$$

respectively. Let the uniaxially anisotropic constitutive tensors of the sheet be given as

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_n \end{bmatrix}, \quad \bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_n \end{bmatrix}, \quad \bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_n \end{bmatrix} \tag{21}$$

while the ambient simple half spaces have electrical parameters (ε_1, μ_1, σ_1) and (ε_2, μ_2, σ_2) as illustrated in Fig. 2.

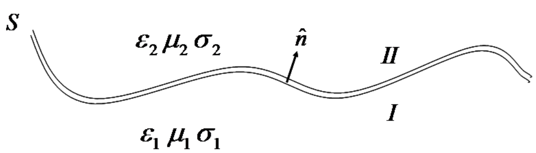

Figure 2. A cross section of an arbitrarily shaped material sheet in a simple medium.

The corresponding complex permittivities shall be given as

$$\varepsilon_{1,2}^c = \varepsilon_{1,2} + i\sigma_{1,2}/\omega \quad \varepsilon^c = \varepsilon + i\sigma/\omega \quad \varepsilon_n^c = \varepsilon_n + i\sigma_n/\omega \tag{22}$$

Next, these medium parameters are placed into (19b), (d), (e) as

$$\varepsilon_n E_n(\vec{r}_{S1}^{+}) = \varepsilon_1 E_n^I(\vec{r}_{S1}^{-}) \tag{23a}$$
$$\mu_n H_n(\vec{r}_{S1}^{+}) = \mu_1 H_n^I(\vec{r}_{S1}^{-}) \tag{23b}$$
$$\sigma_n E_n(\vec{r}_{S1}^{+}) = \sigma_1 E_n^I(\vec{r}_{S1}^{-}) \tag{23c}$$

One can combine (23a) and (23c) as

$$\varepsilon_n^c E_n(\vec{r}_{S1}^{+}) = \varepsilon_1^c E_n^I(\vec{r}_{S1}^{-}). \tag{23d}$$

Similarly, on S_2 we have

$$\varepsilon_n E_n(\vec{r}_{S2}^{-}) = \varepsilon_2 E_n^{II}(\vec{r}_{S2}^{+}) \tag{24a}$$
$$\mu_n H_n(\vec{r}_{S2}^{-}) = \mu_2 H_n^{II}(\vec{r}_{S2}^{+}) \tag{24b}$$
$$\sigma_n E_n(\vec{r}_{S2}^{-}) = \sigma_2 E_n^{II}(\vec{r}_{S2}^{+}) \tag{24c}$$
$$\varepsilon_n^c E_n(\vec{r}_{S2}^{-}) = \varepsilon_2^c E_n^{II}(\vec{r}_{S2}^{+}) \tag{24d}$$

Accordingly, the twelve sources at the r.h.s. of (18) can be expressed in terms of electrical and magnetic fields as

$$\vec{J}_{Ct}(\vec{r}_{S1}^{+}) = \sigma \vec{E}_t(\vec{r}_{S1}^{+}) = \sigma \vec{E}_t^I \tag{25a}$$
$$J_{Cn}(\vec{r}_{S1}^{+}) = \sigma_n E_n(\vec{r}_{S1}^{+}) = \sigma_n \frac{\varepsilon_1^c}{\varepsilon_n^c} E_n^I \tag{25b}$$
$$\vec{P}_t^e(\vec{r}_{S1}^{+}) = (\varepsilon - \varepsilon_0)\vec{E}_t(\vec{r}_{S1}^{+}) = (\varepsilon - \varepsilon_0)\vec{E}_t^I \tag{25c}$$
$$P_n^e(\vec{r}_{S1}^{+}) = (\varepsilon_n - \varepsilon_0)E_n(\vec{r}_{S1}^{+}) = (\varepsilon_n - \varepsilon_0)\frac{\varepsilon_1^c}{\varepsilon_n^c} E_n^I \tag{25d}$$
$$\vec{P}_t^m(\vec{r}_{S1}^{+}) = (\mu - \mu_0)\vec{H}_t(\vec{r}_{S1}^{+}) = (\mu - \mu_0)\vec{H}_t^I \tag{25e}$$
$$P_n^m(\vec{r}_{S1}^{+}) = (\mu_n - \mu_0)H_n(\vec{r}_{S1}^{+}) = \mu_1 \frac{(\mu_n - \mu_0)}{\mu_n} H_n^I \tag{25f}$$
$$\vec{J}_{Ct}(\vec{r}_{S2}^{-}) = \sigma \vec{E}_t(\vec{r}_{S2}^{-}) = \sigma \vec{E}_t^{II} \tag{25g}$$
$$J_{Cn}(\vec{r}_{S2}^{-}) = \sigma_n E_n(\vec{r}_{S2}^{-}) = \sigma_n \frac{\varepsilon_2^c}{\varepsilon_n^c} E_n^{II} \tag{25h}$$
$$\vec{P}_t^e(\vec{r}_{S2}^{-}) = (\varepsilon - \varepsilon_0)\vec{E}_t(\vec{r}_{S2}^{-}) = (\varepsilon - \varepsilon_0)\vec{E}_t^{II} \tag{25i}$$
$$P_n^e(\vec{r}_{S2}^{-}) = (\varepsilon_n - \varepsilon_0)E_n(\vec{r}_{S2}^{-}) = (\varepsilon_n - \varepsilon_0)\frac{\varepsilon_2^c}{\varepsilon_n^c} E_n^{II} \tag{25j}$$
$$\vec{P}_t^m(\vec{r}_{S2}^{-}) = (\mu - \mu_0)\vec{H}_t(\vec{r}_{S2}^{-}) = (\mu - \mu_0)\vec{H}_t^{II} \tag{25k}$$
$$P_n^m(\vec{r}_{S2}^{-}) = (\mu_n - \mu_0)H_n(\vec{r}_{S2}^{-}) = \mu_2 \frac{(\mu_n - \mu_0)}{\mu_n} H_n^{II} \tag{25l}$$

where we have dropped the arguments in the resultant expressions for brevity. Next, (25) are placed into (18) to yield

$$\vec{J}_{0t} \simeq d\sigma \left[\vec{E}_t^I + \vec{E}_t^{II}\right] \tag{26a}$$

$$J_{0n} \simeq d\frac{\sigma_n}{\varepsilon_n^c}\left[\varepsilon_1^c E_n^I + \varepsilon_2^c E_n^{II}\right] \tag{26b}$$

$$\vec{P}_{0t}^e \simeq d(\varepsilon - \varepsilon_0)\left[\vec{E}_t^I + \vec{E}_t^{II}\right] \tag{26c}$$

$$P_{0n}^e \simeq d\frac{(\varepsilon_n - \varepsilon_0)}{\varepsilon_n^c}\left[\varepsilon_1^c E_n^I + \varepsilon_2^c E_n^{II}\right] \tag{26d}$$

$$\vec{P}_{0t}^m \simeq d(\mu - \mu_0)\left[\vec{H}_t^I + \vec{H}_t^{II}\right] \tag{26e}$$

$$P_{0n}^m \simeq d\frac{(\mu_n - \mu_0)}{\mu_n}\left[\mu_1 H_n^I + \mu_2 H_n^{II}\right] \tag{26f}$$

Finally, one places (26) into (15) to obtain

$$\begin{aligned} \hat{n} \times [\vec{E}_t^{II} - \vec{E}_t^I] &= \hat{n} \times \left[e_{1n}\, grad_S E_n^I + e_{2n}\, grad_S E_n^{II}\right] \\ &\quad -h\left(\vec{H}_t^I + \vec{H}_t^{II}\right) \end{aligned} \tag{27a}$$

$$\begin{aligned} \hat{n} \times [\vec{H}_t^{II} - \vec{H}_t^I] &= \hat{n} \times \left[h_{1n}\, grad_S H_n^I + h_{2n}\, grad_S H_n^{II}\right] \\ &\quad +e\left(\vec{E}_t^I + \vec{E}_t^{II}\right) \end{aligned} \tag{27b}$$

$$\begin{aligned} i\omega[D_n^{II} - D_n^I] &= [J_{Cn}^{II} - J_{Cn}^I] + e\, div_S\left(\vec{E}_t^I + \vec{E}_t^{II}\right) \\ \text{or} \quad i\omega\left[\varepsilon_2^c E_n^{II} - \varepsilon_1^c E_n^I\right] &= e div_S\left(\vec{E}_t^I + \vec{E}_t^{II}\right) \end{aligned} \tag{27c}$$

$$\begin{aligned} i\omega[B_n^{II} - B_n^I] &= h div_S\left(\vec{H}_t^I + \vec{H}_t^{II}\right) \\ \text{or} \quad i\omega[\mu_2 H_n^{II} - \mu_1 H_n^I] &= h div_S\left(\vec{H}_t^I + \vec{H}_t^{II}\right) \end{aligned} \tag{27d}$$

where we define

$$e = -i\omega d(\varepsilon^c - \varepsilon_0) \tag{28a}$$

$$h = -i\omega d(\mu - \mu_0) \tag{28b}$$

$$e_{1n} = -d\frac{\varepsilon_1^c}{\varepsilon_0}\frac{(\varepsilon_n^c - \varepsilon_0)}{\varepsilon_n^c} \tag{28c}$$

$$h_{1n} = -d\frac{\mu_1}{\mu_0}\frac{(\mu_n - \mu_0)}{\mu_n} \tag{28d}$$

$$e_{2n} = -d\frac{\varepsilon_2^c}{\varepsilon_0}\frac{(\varepsilon_n^c - \varepsilon_0)}{\varepsilon_n^c} \tag{28e}$$

$$h_{2n} = -d\frac{\mu_2}{\mu_0}\frac{(\mu_n - \mu_0)}{\mu_n} \tag{28f}$$

We observe that the analytical results obtained so far for an arbitrarily shaped material sheet have a very similar structure to those obtained for a planar one in [2] in the sense that the Gaussian curvature parameter is observable only in the distributional interpretation of the Gauss Laws (1c), (1d) and the principle of continuity (2). For this reason the methodology followed and the evidences obtained regarding the derivation of impedance type boundary conditions for a planar sheet in [2, Secs. 4.4 and 5] also apply for an arbitrarily shaped one as in Fig. 2.

We consider 7 special cases of practical interest for the tensor constitutive parameters of a uniaxially anisotropic sheet as outlined in Table 1. The corresponding boundary relations for each case are provided separately in Table 2.

6. THE SPECIAL CASE OF A PEC BACKED UNIAXIALLY ANISOTROPIC SHEET

In this example we consider a PEC lying on the lower face S_1 of the material sheet as depicted in Fig. 3, which supports free surface charge and current densities ρ_{S1}, $\vec{J}_{S1}$, while the fields in region I are set to zero.

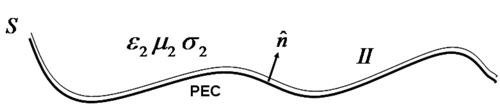

Figure 3. A cross section of an arbitrarily shaped PEC backed material sheet in a simple medium.

Then the boundary relations on S_1 are given as

$$\vec{E}_t(\vec{r}_{S1}^{\,+}) = \vec{0} \tag{29a}$$

$$\varepsilon_n E_n(\vec{r}_{S1}^{\,+}) = \rho_{S1} \tag{29b}$$

$$\vec{H}_t(\vec{r}_{S1}^{\,+}) = \vec{J}_{S1} \times \hat{n} \tag{29c}$$

$$\mu_n H_n(\vec{r}_{S1}^{\,+}) = 0 \tag{29d}$$

$$\sigma_n E_n(\vec{r}_{S1}^{\,+}) + div_S \vec{J}_{S1} - i\omega\rho_{S1} = 0 \tag{29e}$$

$$\hat{n} \cdot \vec{J}_{S1} = 0 \tag{29f}$$

while the relations (24) on S_2 still apply. One can combine (29b), (e) as

$$div_S \vec{J}_{S1} = i\omega \left(\varepsilon_n^c/\varepsilon_n\right) \rho_{S1}. \tag{29g}$$

Accordingly, the six sources on S_1 in (25a)–(f) can be expressed in terms of electrical and magnetic fields as

$$\vec{J}_{Ct}(\vec{r}_{S1}^{+}) = \sigma \vec{E}_t(\vec{r}_{S1}^{+}) = \vec{0} \tag{30a}$$

Table 1. Physical parameters of certain types of uniaxially anisotropic materials.

	TYPE	SPECIFIC PARAMETERS	CONSTITUTIVE PARAMETERS
1	GENERAL UNIAXIALLY ANISOTROPIC SHEET	$e, e_{in}, e_{2n} \neq 0$ $h, h_{1n}, h_{2n} \neq 0$	$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_n \end{bmatrix}$ $\bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_n \end{bmatrix}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$
2	UNIAXIALLY ANISOTROPIC DIELECTRIC SHEET	$e, e_{1n}, e_{2n} \neq 0$ $h = h_{1n} = h_{2n} = 0$	$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_n \end{bmatrix}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$
3	UNIAXIALLY ANISOTROPIC CONDUCTOR SHEET	$e, e_{1n}, e_{2n} \neq 0$ $h = h_{1n} = h_{2n} = 0$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$
4	UNIAXIALLY ANISOTROPIC MAGNETIC SHEET	$e = e_{1n} = e_{2n} = 0$ $h, h_{1n}, h_{2n} \neq 0$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_n \end{bmatrix}$ $\bar{\bar{\sigma}} = 0$

5	RESISTIVE SHEET	$e_{1n} = e_{2n} = 0$ $h = h_{1n} = h_{2n} = 0$ $R = \frac{1}{2e} = \frac{i}{2\omega d[\varepsilon - \varepsilon_0 + i\sigma/\omega]}$	$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_n \end{bmatrix}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$
6	CONDUCTIVE SHEET	$e = e_{1n} = e_{2n} = 0$ $h_{1n} = h_{2n} = 0$ $G = -\frac{1}{2h} = \frac{-i}{2\omega d(\mu - \mu_0)}$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}$ $\bar{\bar{\sigma}} = 0$
7	ISOTROPIC LOSSY DIELECTRIC SHEET	$e, e_{1n}, e_{2n} \neq 0$ $h = h_{1n} = h_{2n} = 0$	$\bar{\bar{\varepsilon}} = \varepsilon \bar{\bar{I}}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \sigma \bar{\bar{I}}$
8	ISOTROPIC MAGNETIC SHEET	$e = e_{1n} = e_{2n} = 0$ $h, h_{1n}, h_{2n} \neq 0$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \mu \bar{\bar{I}}$ $\bar{\bar{\sigma}} = 0$

$$J_{Cn}(\vec{r}_{S1}^{\,+}) = \sigma_n E_n(\vec{r}_{S1}^{\,+}) = (\sigma_n/\varepsilon_n)\,\rho_{S1} = (\sigma_n/i\omega\varepsilon_n^c)\,div_S \vec{J}_{S1} \tag{30b}$$
$$\vec{P}_t^e(\vec{r}_{S1}^{\,+}) = (\varepsilon - \varepsilon_0)\vec{E}_t(\vec{r}_{S1}^{\,+}) = \vec{0} \tag{30c}$$
$$\begin{aligned} P_n^e(\vec{r}_{S1}^{\,+}) &= (\varepsilon_n - \varepsilon_0)E_n(\vec{r}_{S1}^{\,+}) = ((\varepsilon_n - \varepsilon_0)/\varepsilon_n)\,\rho_{S1} \\ &= ((\varepsilon_n - \varepsilon_0)/i\omega\varepsilon_n^c)\,div_S \vec{J}_{S1} \end{aligned} \tag{30d}$$
$$\vec{P}_t^m(\vec{r}_{S1}^{\,+}) = (\mu - \mu_0)\vec{H}_t(\vec{r}_{S1}^{\,+}) = (\mu - \mu_0)\vec{J}_{S1} \times \hat{n} \tag{30e}$$
$$P_n^m(\vec{r}_{S1}^{\,+}) = (\mu_n - \mu_0)H_n(\vec{r}_{S1}^{\,+}) = 0 \tag{30f}$$

while (25g)–(l) on S_2 still apply. Then (30a)–(f) and (25g)–(l) are placed into (18) to yield

$$\vec{J}_{0t} \simeq \vec{J}_{S1} + d\sigma \vec{E}_t^{II} \tag{31a}$$
$$J_{0n} \simeq d\frac{\sigma_n}{\varepsilon_n^c}\left[(1/i\omega)\,div_S \vec{J}_{S1} + \varepsilon_2^c E_n^{II}\right] \tag{31b}$$
$$\vec{P}_{0t}^e \simeq d(\varepsilon - \varepsilon_0)\vec{E}_t^{II} \tag{31c}$$
$$P_{0n}^e \simeq d\frac{(\varepsilon_n - \varepsilon_0)}{\varepsilon_n^c}\left[(1/i\omega)\,div_S \vec{J}_{S1} + \varepsilon_2^c E_n^{II}\right] \tag{31d}$$
$$\vec{P}_{0t}^m \simeq d(\mu - \mu_0)\left[\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}\right] \tag{31e}$$

Table 2. Boundary relations on certain types of uniaxially anisotropic material sheets of arbitrary shape.

TYPE	BOUNDARY RELATIONS
1	$\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^{I}]=\hat{n}\times[e_{1n}grad_S E_n^I+e_{2n}grad_S E_n^{II}]-h[\vec{H}_t^I+\vec{H}_t^{II}]$ $\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^{I}]=\hat{n}\times[h_{1n}grad_S H_n^I+h_{2n}grad_S H_n^{II}]+e[\vec{E}_t^I+\vec{E}_t^{II}]$ $i\omega[\varepsilon_2^c E_n^{II}-\varepsilon_1^c E_n^I]=ediv_S[\vec{E}_t^I+\vec{E}_t^{II}]$ $i\omega[B_n^{II}-B_n^I]=hdiv_S[\vec{H}_t^I+\vec{H}_t^{II}]$
2,3,7	$\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^{I}]=\hat{n}\times[e_{1n}grad_S E_n^I+e_{2n}grad_S E_n^{II}]$ $\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^{I}]=e[\vec{E}_t^I+\vec{E}_t^{II}]$ $i\omega[\varepsilon_2^c E_t^{II}-\varepsilon_1^c E_t^I]=ediv_S[\vec{E}_t^I+\vec{E}_t^{II}]$ $i\omega[B_n^{II}-B_t^I]=0$, i.e., $B_n^{II}=B_n^I$
4,8	$\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^{I}]=-h[\vec{H}_t^I-\vec{H}_t^{II}]$ $\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^{I}]=\hat{n}\times[h_{1n}grad_S H_t^I+h_{2n}grad_S H_n^{II}]$ $i\omega[\varepsilon_2^c E_t^{II}-\varepsilon_1^c E_t^I]=0$, i.e., $\varepsilon_2^c E_n^{II}=\varepsilon_1^c E_n^I$ $i\omega[B_n^{II}-B_t^I]=hdiv_S[\vec{H}_t^I+\vec{H}_t^{II}]$
5	$\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^{I}]=0$, i.e., $\vec{E}_t^{II}=\vec{E}_t^I$ $\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^{I}]=e[\vec{E}_t^I+\vec{E}_t^{II}]=2e\vec{E}_t^{I,II}=\frac{1}{R}\vec{E}_t^{I,II}$ or $\vec{E}_t^{I,II}=R\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^I]$ $i\omega[\varepsilon_2^c E_t^{II}-\varepsilon_1^c E_t^I]=2ediv_S\vec{E}_t^{I,II}=\frac{1}{R}div_S\vec{E}_t^{I,II}$ $i\omega[B_n^{II}-B_t^I]=0$, i.e., $B_n^{II}=B_n^I$
6	$\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^{I}]=-h[\vec{H}_t^I+\vec{H}_t^I]=-2h\vec{H}_t^{I,II}=\frac{1}{G}\vec{H}_t^{I,II}$ or $\vec{H}_t^{I,II}=G\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^I]$ $\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^{I}]=\vec{0}$, i.e., $\vec{H}_t^{II}=\vec{H}_t^I$ $i\omega[\varepsilon_2^c E_t^{II}-\varepsilon_1^c E_t^I]=0$, i.e., $\varepsilon_2^c E_n^{II}=\varepsilon_1^c E_n^I$ $i\omega[B_n^{II}-B_t^I]=hdiv_S[\vec{H}_t^I+\vec{H}_t^I]=2hdiv_S\vec{H}_t^{I,II}$

$$P_{0n}^m \simeq d\frac{(\mu_n-\mu_0)}{\mu_n}\mu_2 H_n^{II} \tag{31f}$$

Finally, one places (31) into (15) to obtain

$$\hat{n}\times\vec{E}_t^{II}=\hat{n}\times\left[(e_n/i\omega)\,grad_S div_S\vec{J}_{S1}+e_{2n}grad_S E_n^{II}\right]-h\left(\vec{J}_{S1}\times\hat{n}+\vec{H}_t^{II}\right) \tag{32a}$$

$$\hat{n}\times\vec{H}_t^{II} = h_{2n}\hat{n}\times grad_S H_n^{II}+e\vec{E}_t^{II} \tag{32b}$$

$$i\omega D_n^{II} = J_{Cn}^{II}+ediv_S\vec{E}_t^{II} \quad \text{or} \quad i\omega\varepsilon_2^c E_n^{II}=ediv_S\vec{E}_t^{II} \tag{32c}$$

$$i\omega B_n^{II} = hdiv_S\left(\vec{J}_{S1}\times\hat{n}+\vec{H}_t^{II}\right) \tag{32d}$$

Table 3. Boundary relations on certain types of PEC backed uniaxially anisotropic material sheets of arbitrary shape.

TYPE	BOUNDARY RELATIONS
1	$\hat{n} \times \vec{E}^{II} = \hat{n} \times [(e_n/i\omega) grad_S div_S \vec{J}_{S1} + e_{2n} grad_S E_n^{II}] - h[\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}]$ $\hat{n} \times \vec{H}_t^{II} = h_{2n} \hat{n} \times grad_S H_n^{II} + e\vec{E}_t^{II}$ $i\omega D_n^{II} = J_{Cn}^{II} + e div_S \vec{E}_t^{II}$ or $i\omega \varepsilon_2^c E_n^{II} = e div_S \vec{E}_t^{II}$ $i\omega B_n^{II} = h div_S [\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}]$ *Comment*: The impedance boundary condition $\vec{E}_t^{II} = Z\hat{n} \times \vec{H}_t^{II}$ appears in the second equation for $h_{2n} = 0\,(\mu_n = \mu_0)$ with $Z = 1/e$ $= i/[\omega d(\varepsilon^c - \varepsilon_0)]$ regardless of the rest of the constitutive parameters.
2, 3, 7	$\hat{n} \times \vec{E}_t^{II} = \hat{n} \times [(e_n/i\omega) grad_S div_S \vec{J}_{S1} + e_{2n} grad_S E_n^{II}]$ $\hat{n} \times \vec{H}_t^{II} = e\vec{E}_t^{II}$ $i\omega D_n^{II} = J_{Cn}^{II} + e div_S \vec{E}_t^{II}$ or $i\omega \varepsilon_2^c E_n^{II} = e div_S \vec{E}_t^{II}$ $i\omega B_n^{II} = 0$, i.e., $B_n^{II} = 0$, $H_n^{II} = 0$ *Comment*: The impedance boundary condition appears in the second equation naturally with $Z = 1/e = i/[\omega d(\varepsilon^c - \varepsilon_0)]$. For a Type 3 material the impedance becomes real valued as $Z = 1\sigma d$.
4, 8	$\hat{n} \times \vec{E}_t^{II} = -h[\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}]$ or $\vec{E}_t^{II} = h[\vec{J}_{S1} + \hat{n} \times \vec{H}_t^{II}]$ $\hat{n} \times \vec{H}_t^{II} = h_{2n} \hat{n} \times grad_S H_n^{II}$ $i\omega \varepsilon_2^c E_n^{II} = 0$, i.e., $E_n^{II} = 0$ $i\omega B_n^{II} = h div_S [\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}] = h div_S (\hat{n} \times \vec{E}_t^{II})$ *Comment*: We cannot obtain an impedance boundary condition.
5	$\hat{n} \times \vec{E}_t^{II} = \vec{0}$, i.e., $\vec{E}_t^{II} = \vec{0}$ $\hat{n} \times \vec{H}_t^{II} = e\vec{E}_t^{II}$ so that $\hat{n} \times \vec{H}_t^{II} = \vec{0}$, i.e., $\vec{H}_t^{II} = \vec{0}$ $i\omega D_n^{II} = J_{Cn}^{II} + e div_S \vec{E}_t^{II}$ or $i\omega \varepsilon_2^c E_n^{II} = e div_S \vec{E}_t^{II} = 0$, i.e., $\vec{E}_n^{II} = 0$ $i\omega B_n^{II} = h div_S [\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}] = 0$, i.e., $B_n^{II} = 0$, $H_n^{II} = 0$ *Comment*: We obtain the end results $\vec{E}^{II} = \vec{0}$, $\vec{H}^{II} = \vec{0}$, which reveal that the thin sheet simulation breaks down for this parametrization.
6	$\hat{n} \times \vec{E}_t^{II} = -h[\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}] = -h\vec{J}_{S1} \times \hat{n}$, i.e., $\vec{E}_t^{II} = h\vec{J}_{S1}$ $\hat{n} \times \vec{H}_t^{II} = \vec{0}$, i.e., $\vec{H}_t^{II} = \vec{0}$ $i\omega D_n^{II} = J_{Cn}^{II}$ or $i\omega \varepsilon_2^c E_n^{II} = 0$, i.e., $E_n^{II} = 0$ $i\omega B_n^{II} = h div_S [\vec{J}_{S1} \times \hat{n} + \vec{H}_t^{II}] = h div_S (\vec{J}_{S1} \times \hat{n}) = div_S (\vec{E}_t^{II} \times \hat{n})$ *Comment*: We cannot obtain an impedance boundary condition.

where in (32a) we additionally introduce

$$e_n = -d\frac{(\varepsilon_n^c - \varepsilon_0)}{\varepsilon_0 \varepsilon_n^c}. \tag{33}$$

It can be seen that the end results (32) appear as the special case (27) upon setting $\vec{H}_t^I \rightarrow \vec{J}_{S1} \times \hat{n}$, $H_n^I \rightarrow 0$, $\vec{E}_t^I \rightarrow \vec{0}$, $e_{1n}E_n^I \rightarrow (e_n/i\omega)div_S\vec{J}_{S1}$. The corresponding jump relations for the 8 types of materials in Table 1 are provided in Table 3 with comments on the availability and form of the impedance boundary condition. As a general comment covering all 8 types of materials one can say that an impedance boundary condition can be obtained (regardless of the operating frequency) as long as the following three conditions are satisfied:

i) $\mu_n = \mu_0$

ii) $\varepsilon \neq \varepsilon_0$ or $\sigma \neq 0$

iii) $\varepsilon_n \neq \varepsilon_0$ or $\sigma_n \neq 0$ or $\mu \neq \mu_0$.

7. BOUNDARY RELATIONS ON A BIAXIALLY ANISOTROPIC SHEET

In this section we generalize the results obtained in Secs. 5 and 6 for the biaxial case when the constitutive parameters are given by

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_u & 0 & 0 \\ 0 & \varepsilon_v & 0 \\ 0 & 0 & \varepsilon_n \end{bmatrix}, \quad \bar{\bar{\mu}} = \begin{bmatrix} \mu_u & 0 & 0 \\ 0 & \mu_v & 0 \\ 0 & 0 & \mu_n \end{bmatrix}, \quad \bar{\bar{\sigma}} = \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & \sigma_v & 0 \\ 0 & 0 & \sigma_n \end{bmatrix} \tag{34}$$

with

$$\varepsilon_{u,v}^c = \varepsilon_{u,v} + i\sigma_{u,v}/\omega. \tag{35}$$

Here we consider (u, v) as the curvilinear coordinate curves of S such that any tangential field component in (27) and (32) is expressed through its components along these curves in the form $\vec{A}_t = \vec{A}_u + \vec{A}_v$. Then the modified expressions for the tangential source quantities in (25a), (c), (e), (g), (i), (k) read

$$\vec{J}_{Ct}(\vec{r}_{S1}^{\,+}) = \sigma_u\vec{E}_u(\vec{r}_{S1}^{\,+}) + \sigma_v\vec{E}_v(\vec{r}_{S1}^{\,+}) = \sigma_u\vec{E}_u^I + \sigma_v\vec{E}_v^I \tag{36a}$$

$$\begin{aligned}\vec{P}_t^e(\vec{r}_{S1}^{\,+}) &= (\varepsilon_u - \varepsilon_0)\vec{E}_u(\vec{r}_{S1}^{\,+}) + (\varepsilon_v - \varepsilon_0)\vec{E}_v(\vec{r}_{S1}^{\,+}) \\ &= (\varepsilon_u - \varepsilon_0)\vec{E}_u^I + (\varepsilon_v - \varepsilon_0)\vec{E}_v^I \end{aligned} \tag{36b}$$

$$\begin{aligned}\vec{P}_t^m(\vec{r}_{S1}^{\,+}) &= (\mu_u - \mu_0)\vec{H}_u(\vec{r}_{S1}^{\,+}) + (\mu_v - \mu_0)\vec{H}_v(\vec{r}_{S1}^{\,+}) \\ &= (\mu_u - \mu_0)\vec{H}_u^I + (\mu_v - \mu_0)\vec{H}_v^I \end{aligned} \tag{36c}$$

$$\vec{J}_{Ct}(\vec{r}_{S2}^{\,-}) = \sigma_u\vec{E}_u(\vec{r}_{S2}^{\,-}) + \sigma_v\vec{E}_v(\vec{r}_{S2}^{\,-}) = \sigma_u\vec{E}_u^{II} + \sigma_v\vec{E}_v^{II} \tag{36d}$$

$$\begin{aligned}\vec{P}_t^e(\vec{r}_{S2}^{\,-}) &= (\varepsilon_u - \varepsilon_0)\vec{E}_u(\vec{r}_{S2}^{\,-}) + (\varepsilon_v - \varepsilon_0)\vec{E}_v(\vec{r}_{S2}^{\,-}) \\ &= (\varepsilon_u - \varepsilon_0)\vec{E}_u^{II} + (\varepsilon_v - \varepsilon_0)\vec{E}_v^{II} \end{aligned} \tag{36e}$$

$$\begin{aligned}\vec{P}_t^m(\overleftarrow{r_{S2}}) &= (\mu_u-\mu_0)\vec{H}_u(\overleftarrow{r_{S2}})+(\mu_v-\mu_0)\vec{H}_v(\overleftarrow{r_{S2}}) \\ &= (\mu_u-\mu_0)\vec{H}_u^{II}+(\mu_v-\mu_0)\vec{H}_v^{II}\end{aligned} \tag{36f}$$

while the expressions of the normally oriented sources (25b), (d), (f), (h), (j), (l) remain unchanged. Accordingly, the tangential equivalent sources are written as

$$(1/d)\,\vec{J}_{0t} \simeq \sigma_u\left(\vec{E}_u^I+\vec{E}_u^{II}\right)+\sigma_v\left(\vec{E}_v^I+\vec{E}_v^{II}\right) \tag{37a}$$

$$(1/d)\,\vec{P}_{0t}^e \simeq (\varepsilon_u-\varepsilon_0)\left(\vec{E}_u^I+\vec{E}_u^{II}\right)+(\varepsilon_v-\varepsilon_0)\left(\vec{E}_v^I+\vec{E}_v^{II}\right) \tag{37b}$$

$$(1/d)\,\vec{P}_{0t}^m \simeq (\mu_u-\mu_0)\left(\vec{H}_u^I+\vec{H}_u^{II}\right)+(\mu_v-\mu_0)\left(\vec{H}_v^I+\vec{H}_v^{II}\right) \tag{37c}$$

while (26b), (d), (f) still applies. As a result, the modified boundary relations can be obtained rigorously as

$$\begin{aligned}\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^I] &= \hat{n}\times\left[e_{1n}\,grad_S E_n^I+e_{2n}\,grad_S E_n^{II}\right] \\ &\quad -h_u\left(\vec{H}_u^I+\vec{H}_u^{II}\right)-h_v\left(\vec{H}_v^I+\vec{H}_v^{II}\right)\end{aligned} \tag{38a}$$

$$\begin{aligned}\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^I] &= \hat{n}\times\left[h_{1n}\,grad_S H_n^I+h_{2n}\,grad_S H_n^{II}\right] \\ &\quad +e_u\left(\vec{E}_u^I+\vec{E}_u^{II}\right)+e_v\left(\vec{E}_v^I+\vec{E}_v^{II}\right)\end{aligned} \tag{38b}$$

$$i\omega[D_n^{II}-D_n^I]=[J_{Cn}^{II}-J_{Cn}^I]+e_u div_S\left(\vec{E}_u^I+\vec{E}_u^{II}\right)+e_v div_S\left(\vec{E}_v^I+\vec{E}_v^{II}\right)$$

$$\text{or}\quad i\omega\left[\varepsilon_2^c E_n^{II}-\varepsilon_1^c E_n^I\right]=e_u div_S\left(\vec{E}_u^I+\vec{E}_u^{II}\right)+e_v div_S\left(\vec{E}_v^I+\vec{E}_v^{II}\right) \tag{38c}$$

$$i\omega[B_n^{II}-B_n^I] = h_u div_S\left(\vec{H}_u^I+\vec{H}_u^{II}\right)+h_v div_S\left(\vec{H}_v^I+\vec{H}_v^{II}\right) \tag{38d}$$

where we additionally define

$$e_u = -i\omega d(\varepsilon_u^c-\varepsilon_0) \tag{39a}$$

$$e_v = -i\omega d(\varepsilon_v^c-\varepsilon_0) \tag{39b}$$

$$h_u = -i\omega d(\mu_u-\mu_0) \tag{39c}$$

$$h_v = -i\omega d(\mu_v-\mu_0) \tag{39d}$$

The resistive and conductive boundary relations for a biaxially anisotropic sheet read

$$\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^I]=\frac{1}{R_u}\vec{E}_u^{I,II}+\frac{1}{R_v}\vec{E}_v^{I,II}\quad\text{with}\quad R_u=\frac{1}{2e_u},\quad R_v=\frac{1}{2e_v} \tag{40}$$

and

$$\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^I]=\frac{1}{G_u}\vec{H}_u^{I,II}+\frac{1}{G_v}\vec{H}_v^{I,II}\quad\text{with}\quad G_u=-\frac{1}{2h_u},\quad G_v=-\frac{1}{2h_v} \tag{41}$$

Similarly, the boundary relations for a PEC backed biaxially anisotropic sheet as in Fig. 3 read

$$\begin{aligned}\hat{n}\times\vec{E}_t^{II} &= \hat{n}\times\left[(e_n/i\omega)\; grad_S div_S \vec{J}_{S1} + e_{2n}\, grad_S E_n^{II}\right]\\ &\quad -(\hat{u}h_u J_{S1v} - \hat{v}h_v J_{S1u}) - \left(h_u\vec{H}_u^{II} + h_v\vec{H}_v^{II}\right) \qquad (42a)\end{aligned}$$

$$\hat{n}\times\vec{H}_t^{II} = h_{2n}\hat{n}\times grad_S H_n^{II} + e_u\vec{E}_u^{II} + e_v\vec{E}_v^{II} \qquad (42b)$$

$$i\omega\varepsilon_2^c E_n^{II} = div_S\left(e_u\vec{E}_u^{II} + e_v\vec{E}_v^{II}\right) \qquad (42c)$$

$$i\omega B_n^{II} = div_S(\hat{u}h_u J_{S1v} - \hat{v}h_v J_{S1u}) + div_S\left(h_u\vec{H}_u^{II} + h_v\vec{H}_v^{II}\right) \qquad (42d)$$

where the surface current density is expressed by its tangential components as $\vec{J}_{S1} = \hat{u}J_{S1u} + \hat{v}J_{S1v}$. In this case similar arguments as in Table 3 hold for the availability of the impedance condition which is modified as

$$\hat{n}\times\vec{H}_t^{II} = \frac{1}{Z_u}\vec{E}_u^{II} + \frac{1}{Z_v}\vec{E}_v^{II} \quad \text{with} \quad Z_u = \frac{1}{e_u}, \quad Z_v = \frac{1}{e_v} \qquad (43)$$

(43a) can be written in matrix form as

$$\begin{bmatrix} E_u^{II} \\ E_v^{II} \end{bmatrix} = \begin{bmatrix} 0 & -Z_u \\ Z_v & 0 \end{bmatrix}\begin{bmatrix} H_u^{II} \\ H_v^{II} \end{bmatrix}. \qquad (44)$$

8. BOUNDARY RELATIONS ON A GENERAL ANISOTROPIC SHEET

We consider the constitutive parameters of a general anisotropic sheet in the form

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{uu} & \varepsilon_{uv} & \varepsilon_{un} \\ \varepsilon_{vu} & \varepsilon_{vv} & \varepsilon_{vn} \\ \varepsilon_{nu} & \varepsilon_{nv} & \varepsilon_{nn} \end{bmatrix} \qquad (45a)$$

$$\bar{\bar{\mu}} = \begin{bmatrix} \mu_{uu} & \mu_{uv} & \mu_{un} \\ \mu_{vu} & \mu_{vv} & \mu_{vn} \\ \mu_{nu} & \mu_{nv} & \mu_{nn} \end{bmatrix} \qquad (45b)$$

$$\bar{\bar{\sigma}} = \begin{bmatrix} \sigma_{uu} & \sigma_{uv} & \sigma_{un} \\ \sigma_{vu} & \sigma_{vv} & \sigma_{vn} \\ \sigma_{nu} & \sigma_{nv} & \sigma_{nn} \end{bmatrix} \qquad (45c)$$

with

$$\varepsilon_{ab}^c = \varepsilon_{ab} + i\sigma_{ab}/\omega, \quad a, b = n, u, v. \qquad (46)$$

The boundary relations on the interfaces $S_{1,2}$ read

$$\vec{E}_u(\vec{r}_{S1}^{+}) = \vec{E}_u^I \quad (47a)$$

$$\vec{E}_v(\vec{r}_{S1}^{+}) = \vec{E}_v^I \quad (47b)$$

$$\varepsilon_{nu}^c E_u(\vec{r}_{S1}^{+}) + \varepsilon_{nv}^c E_v(\vec{r}_{S1}^{+}) + \varepsilon_{nn}^c E_n(\vec{r}_{S1}^{+}) = \varepsilon_1^c E_n^I \quad (47c)$$

$$\vec{H}_u(\vec{r}_{S1}^{+}) = \vec{H}_u^I \quad (47d)$$

$$\vec{H}_v(\vec{r}_{S1}^{+}) = \vec{H}_v^I \quad (47e)$$

$$\mu_{nu} H_u(\vec{r}_{S1}^{+}) + \mu_{nv} H_v(\vec{r}_{S1}^{+}) + \mu_{nn} H_n(\vec{r}_{S1}^{+}) = \mu_1 H_n^I \quad (47f)$$

and

$$\vec{E}_u(\vec{r}_{S2}^{-}) = \vec{E}_u^{II} \quad (48a)$$

$$\vec{E}_v(\vec{r}_{S2}^{-}) = \vec{E}_v^{II} \quad (48b)$$

$$\varepsilon_{nu}^c E_u(\vec{r}_{S2}^{-}) + \varepsilon_{nv}^c E_v(\vec{r}_{S2}^{-}) + \varepsilon_{nn}^c E_n(\vec{r}_{S2}^{-}) = \varepsilon_2^c E_n^{II} \quad (48c)$$

$$\vec{H}_u(\vec{r}_{S2}^{-}) = \vec{H}_u^{II} \quad (48d)$$

$$\vec{H}_v(\vec{r}_{S2}^{-}) = \vec{H}_v^{II} \quad (48e)$$

$$\mu_{nu} H_u(\vec{r}_{S2}^{-}) + \mu_{nv} H_v(\vec{r}_{S2}^{-}) + \mu_{nn} H_n(\vec{r}_{S2}^{-}) = \mu_2 H_n^{II} \quad (48f)$$

One can express the tangential electrical and magnetic field components via the unit vectors of the coordinates curves of S as $\vec{E}_u^{I,II} = \hat{u} E_u^{I,II}$, $\vec{E}_v^{I,II} = \hat{v} E_v^{I,II}$, and similar for the magnetic field. Accordingly, the twelve sources at the r.h.s. of (18) can be expressed in terms of electrical and magnetic fields as

$$\vec{J}_{Ct}(\vec{r}_{S1}^{+}) = \hat{u}\left(\sigma_{uu} E_u^I + \sigma_{uv} E_v^I\right) + \hat{v}\left(\sigma_{vu} E_u^I + \sigma_{vv} E_v^I\right) \quad (49a)$$

$$J_{Cn}(\vec{r}_{S1}^{+}) = \sigma_{nn}\frac{\varepsilon_1^c}{\varepsilon_{nn}^c} E_n^I + \left(\sigma_{nu} - \sigma_{nn}\frac{\varepsilon_{nu}^c}{\varepsilon_{nn}^c}\right) E_u^I + \left(\sigma_{nv} - \sigma_{nn}\frac{\varepsilon_{nv}^c}{\varepsilon_{nn}^c}\right) E_v^I \quad (49b)$$

$$\vec{P}_t^e(\vec{r}_{S1}^{+}) = \hat{u}\left((\varepsilon_{uu} - \varepsilon_0) E_u^I + \varepsilon_{uv} E_v^I\right) + \hat{v}\left(\varepsilon_{vu} E_u^I + (\varepsilon_{vv} - \varepsilon_0) E_v^I\right) \quad (49c)$$

$$P_n^e(\vec{r}_{S1}^{+}) = (\varepsilon_{nn} - \varepsilon_0)\frac{\varepsilon_1^c}{\varepsilon_{nn}^c} E_n^I + \left(\varepsilon_{nu} - (\varepsilon_{nn} - \varepsilon_0)\frac{\varepsilon_{nu}^c}{\varepsilon_{nn}^c}\right) E_u^I + \left(\varepsilon_{nv} - (\varepsilon_{nn} - \varepsilon_0)\frac{\varepsilon_{nv}^c}{\varepsilon_{nn}^c}\right) E_v^I \quad (49d)$$

$$\vec{P}_t^m(\vec{r}_{S1}^{+}) = \hat{u}\left((\mu_{uu} - \mu_0) H_u^I + \mu_{uv} H_v^I\right) + \hat{v}\left(\mu_{vu} H_u^I + (\mu_{vv} - \mu_0) H_v^I\right) \quad (49e)$$

$$P_n^m(\vec{r}_{S1}^{+}) = (\mu_{nn} - \mu_0)\frac{\mu_1}{\mu_{nn}} H_n^I + \frac{\mu_0 \mu_{nu}}{\mu_{nn}} H_u^I + \frac{\mu_0 \mu_{nv}}{\mu_{nn}} H_v^I \quad (49f)$$

$$\vec{J}_{Ct}(\vec{r}_{S2}^{-}) = \hat{u}\left(\sigma_{uu} E_u^{II} + \sigma_{uv} E_v^{II}\right) + \hat{v}\left(\sigma_{vu} E_u^{II} + \sigma_{vv} E_v^{II}\right) \quad (49g)$$

$$J_{Cn}(\vec{r}_{S2}^{-}) = \sigma_{nn}\frac{\varepsilon_2^c}{\varepsilon_{nn}^c} E_n^{II} + \left(\sigma_{nu} - \sigma_{nn}\frac{\varepsilon_{nu}^c}{\varepsilon_{nn}^c}\right) E_u^{II} + \left(\sigma_{nv} - \sigma_{nn}\frac{\varepsilon_{nv}^c}{\varepsilon_{nn}^c}\right) E_v^{II} \quad (49h)$$

$$\vec{P}_t^e(\vec{r_{S2}}) = \hat{u}\left((\varepsilon_{uu}-\varepsilon_0)E_u^{II}+\varepsilon_{uv}E_v^{II}\right) + \hat{v}\left(\varepsilon_{vu}E_u^{II}+(\varepsilon_{vv}-\varepsilon_0)E_v^{II}\right) \quad (49i)$$

$$P_n^e(\vec{r_{S2}}) = (\varepsilon_{nn}-\varepsilon_0)\frac{\varepsilon_2^c}{\varepsilon_{nn}^c}E_n^{II} + \left(\varepsilon_{nu} - (\varepsilon_{nn}-\varepsilon_0)\frac{\varepsilon_{nu}^c}{\varepsilon_{nn}^c}\right)E_u^{II} + \left(\varepsilon_{nv} - (\varepsilon_{nn}-\varepsilon_0)\frac{\varepsilon_{nv}^c}{\varepsilon_{nn}^c}\right)E_v^{II} \quad (49j)$$

$$\vec{P}_t^m(\vec{r_{S2}}) = \hat{u}\left((\mu_{uu}-\mu_0)H_u^{II}+\mu_{uv}H_v^{II}\right)+\hat{v}\left(\mu_{vu}H_u^{II}+(\mu_{vv}-\mu_0)H_v^{II}\right) \quad (49k)$$

$$P_n^m(\vec{r_{S2}}) = (\mu_{nn}-\mu_0)\frac{\mu_2}{\mu_{nn}}H_n^{II} + \frac{\mu_0\mu_{nu}}{\mu_{nn}}H_u^{II} + \frac{\mu_0\mu_{nv}}{\mu_{nn}}H_v^{II} \quad (49l)$$

Next, (49) are placed into (18) to yield

$$(1/d)\,\vec{J}_{0t} \simeq \hat{u}\left(\sigma_{uu}\left(E_u^I+E_u^{II}\right)+\sigma_{uv}\left(E_v^I+E_v^{II}\right)\right) + \hat{v}\left(\sigma_{vu}\left(E_u^I+E_u^{II}\right)+\sigma_{vv}\left(E_v^I+E_v^{II}\right)\right) \quad (50a)$$

$$(1/d)\,J_{0n} \simeq \frac{\sigma_{nn}}{\varepsilon_{nn}^c}\left(\varepsilon_1^c E_n^I+\varepsilon_2^c E_n^{II}\right) + \left(\sigma_{nu}-\sigma_{nn}\frac{\varepsilon_{nu}^c}{\varepsilon_{nn}^c}\right)\left(E_u^I+E_u^{II}\right) + \left(\sigma_{nv}-\sigma_{nn}\frac{\varepsilon_{nv}^c}{\varepsilon_{nn}^c}\right)\left(E_v^I+E_v^{II}\right) \quad (50b)$$

$$(1/d)\,\vec{P}_{0t}^e \simeq \hat{u}\left((\varepsilon_{uu}-\varepsilon_0)\left(E_u^I+E_u^{II}\right)+\varepsilon_{uv}\left(E_v^I+E_v^{II}\right)\right) + \hat{v}\left(\varepsilon_{vu}\left(E_u^I+E_u^{II}\right)+(\varepsilon_{vv}-\varepsilon_0)\left(E_v^I+E_v^{II}\right)\right) \quad (50c)$$

$$(1/d)\,P_{0n}^e \simeq \frac{(\varepsilon_{nn}-\varepsilon_0)}{\varepsilon_{nn}^c}\left(\varepsilon_1^c E_n^I+\varepsilon_2^c E_n^{II}\right) + \left(\varepsilon_{nu}-(\varepsilon_{nn}-\varepsilon_0)\frac{\varepsilon_{nu}^c}{\varepsilon_{nn}^c}\right)\left(E_u^I+E_u^{II}\right)+\left(\varepsilon_{nv}-(\varepsilon_{nn}-\varepsilon_0)\frac{\varepsilon_{nv}^c}{\varepsilon_{nn}^c}\right)\left(E_v^I+E_v^{II}\right) \quad (50d)$$

$$(1/d)\,\vec{P}_{0t}^m \simeq \hat{u}\left((\mu_{uu}-\mu_0)\left(H_u^I+H_u^{II}\right)+\mu_{uv}\left(H_v^I+H_v^{II}\right)\right) + \hat{v}\left(\mu_{vu}\left(H_u^I+H_u^{II}\right)+(\mu_{vv}-\mu_0)\left(H_v^I+H_v^{II}\right)\right) \quad (50e)$$

$$(1/d)\,P_{0n}^m \simeq \frac{(\mu_{nn}-\mu_0)}{\mu_{nn}}\left(\mu_1 H_n^I+\mu_2 H_n^{II}\right) + \frac{\mu_0\mu_{nu}}{\mu_{nn}}\left(H_u^I+H_u^{II}\right) + \frac{\mu_0\mu_{nv}}{\mu_{nn}}\left(H_v^I+H_v^{II}\right) \quad (50f)$$

Finally, one places (50) into (15) to obtain

$$\begin{aligned}
&\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^I] \\
&= \hat{n}\times grad_S\left[e_{1nn}\,E_n^I+e_{2nn}\,E_n^{II}+e_{nu}\left(E_u^I+E_u^{II}\right)\right. \\
&\left.+e_{nv}\left(E_v^I+E_v^{II}\right)\right]-h_{uu}\left(\vec{H}_u^I+\vec{H}_u^{II}\right)-h_{vv}\left(\vec{H}_v^I+\vec{H}_v^{II}\right) \\
&+h_{uv}\hat{n}\times\left(\vec{H}_v^I+\vec{H}_v^{II}\right)-h_{vu}\hat{n}\times\left(\vec{H}_u^I+\vec{H}_u^{II}\right) \qquad (51a)
\end{aligned}$$

$$\begin{aligned}
&\hat{n} \times [\vec{H}_t^{II} - \vec{H}_t^I] \\
= \; &\hat{n} \times grad_S \left[h_{1nn} H_n^I + h_{2nn} H_n^{II} + h_{nu} \left(H_u^I + H_u^{II} \right) \right. \\
&\left. + h_{nv} \left(H_v^I + H_v^{II} \right) \right] + \mathrm{e}_{uu} \left(\vec{E}_u^I + \vec{E}_u^{II} \right) + e_{vv} \left(\vec{E}_v^I + \vec{E}_v^{II} \right) \\
&- e_{uv} \hat{n} \times \left(\vec{E}_v^I + \vec{E}_v^{II} \right) + e_{vu} \hat{n} \times \left(\vec{E}_u^I + \vec{E}_u^{II} \right) \qquad (51b)
\end{aligned}$$

$$\begin{aligned}
&i\omega \left[\varepsilon_2^c E_n^{II} - \varepsilon_1^c E_n^I \right] \\
= \; &div_S \left[e_{uu} \left(\vec{E}_u^I + \vec{E}_u^{II} \right) + e_{vv} \left(\vec{E}_v^I + \vec{E}_v^{II} \right) \right. \\
&\left. - e_{uv} \hat{n} \times \left(\vec{E}_v^I + \vec{E}_v^{II} \right) + e_{vu} \hat{n} \times \left(\vec{E}_u^I + \vec{E}_u^{II} \right) \right] \qquad (51c)
\end{aligned}$$

$$\begin{aligned}
&i\omega [B_n^{II} - B_n^I] \\
= \; &div_S \left[h_{uu} \left(\vec{H}_u^I + \vec{H}_u^{II} \right) + h_{vv} \left(\vec{H}_v^I + \vec{H}_v^{II} \right) \right. \\
&\left. - h_{uv} \hat{n} \times \left(\vec{H}_v^I + \vec{H}_v^{II} \right) + h_{vu} \hat{n} \times \left(\vec{H}_u^I + \vec{H}_u^{II} \right) \right] \qquad (51d)
\end{aligned}$$

where we define

$$e_{uu} = -i\omega d(\varepsilon_{uu}^c - \varepsilon_0) \qquad (52a)$$
$$e_{vv} = -i\omega d(\varepsilon_{vv}^c - \varepsilon_0) \qquad (52b)$$
$$e_{uv} = -i\omega d\varepsilon_{uv}^c \qquad (52c)$$
$$e_{vu} = -i\omega d\varepsilon_{vu}^c \qquad (52d)$$
$$e_{1nn} = -d\frac{\varepsilon_1^c}{\varepsilon_0} \frac{(\varepsilon_{nn}^c - \varepsilon_0)}{\varepsilon_{nn}^c} \qquad (52e)$$
$$e_{2nn} = -d\frac{\varepsilon_2^c}{\varepsilon_0} \frac{(\varepsilon_{nn}^c - \varepsilon_0)}{\varepsilon_{nn}^c} \qquad (52f)$$
$$e_{nu} = -d\frac{\varepsilon_{nu}^c}{\varepsilon_{nn}^c} \qquad (52g)$$
$$e_{nv} = -d\frac{\varepsilon_{nv}^c}{\varepsilon_{nn}^c} \qquad (52h)$$

and

$$h_{uu} = -i\omega d(\mu_{uu} - \mu_0) \qquad (53a)$$
$$h_{vv} = -i\omega d(\mu_{vv} - \mu_0) \qquad (53b)$$
$$h_{uv} = -i\omega d\mu_{uv} \qquad (53c)$$
$$h_{vu} = -i\omega d\mu_{vu} \qquad (53d)$$
$$h_{1nn} = -d\frac{\mu_1}{\mu_0} \frac{(\mu_{nn} - \mu_0)}{\mu_{nn}} \qquad (53e)$$

$$h_{2nn} = -d\frac{\mu_2}{\mu_0}\frac{(\mu_{nn}-\mu_0)}{\mu_{nn}} \tag{53f}$$

$$h_{nu} = -d\frac{\mu_{nu}}{\mu_{nn}} \tag{53g}$$

$$h_{nv} = -d\frac{\mu_{nv}}{\mu_{nn}} \tag{53h}$$

The resistive and conductive boundary relations for a general anisotropic sheet read

$$\begin{aligned}\hat{n}\times[\vec{H}_t^{II}-\vec{H}_t^{I}] &= \frac{1}{R_{uu}}\vec{E}_u^{I,II}+\frac{1}{R_{vv}}\vec{E}_v^{I,II}\\ &\quad -\frac{1}{R_{uv}}\hat{n}\times\vec{E}_v^{I,II}+\frac{1}{R_{vu}}\hat{n}\times\vec{E}_u^{I,II}\end{aligned} \tag{54a}$$

with

$$R_{uu}=\frac{1}{2e_{uu}},\quad R_{vv}=\frac{1}{2e_{vv}},\quad R_{uv}=\frac{1}{2e_{uv}},\quad R_{vu}=\frac{1}{2e_{vu}} \tag{54b}$$

and

$$\begin{aligned}\hat{n}\times[\vec{E}_t^{II}-\vec{E}_t^{I}] &= \frac{1}{G_{uu}}\vec{H}_u^{I,II}+\frac{1}{G_{vv}}\vec{H}_v^{I,II}-\frac{1}{G_{uv}}\hat{n}\times\vec{H}_v^{I,II}\\ &\quad +\frac{1}{G_{vu}}\hat{n}\times\vec{H}_u^{I,II}\end{aligned} \tag{55a}$$

with

$$G_{uu}=-\frac{1}{2h_{uu}},\quad G_{vv}=-\frac{1}{2h_{vv}},\quad G_{uv}=-\frac{1}{2h_{uv}},\quad G_{vu}=-\frac{1}{2h_{vu}} \tag{55b}$$

These equations can also be given in matrix form as

$$\begin{aligned}\begin{bmatrix} H_u^{II}-H_u^{I}\\ H_v^{II}-H_v^{I}\end{bmatrix} &= \begin{bmatrix} 1/R_{vu} & 1/R_{vv}\\ -1/R_{uu} & -1/R_{uv}\end{bmatrix}\begin{bmatrix} E_u^{I,II}\\ E_v^{I,II}\end{bmatrix}\\ &=2\begin{bmatrix} e_{vu} & e_{vv}\\ -e_{uu} & -e_{uv}\end{bmatrix}\begin{bmatrix} E_u^{I,II}\\ E_v^{I,II}\end{bmatrix}\end{aligned} \tag{56}$$

and

$$\begin{aligned}\begin{bmatrix} E_u^{II}-E_u^{I}\\ E_v^{II}-E_v^{I}\end{bmatrix} &= \begin{bmatrix} 1/G_{vu} & 1/G_{vv}\\ -1/G_{uu} & -1/G_{uv}\end{bmatrix}\begin{bmatrix} H_u^{I,II}\\ H_v^{I,II}\end{bmatrix}\\ &=2\begin{bmatrix} -h_{vu} & -h_{vv}\\ h_{uu} & h_{uv}\end{bmatrix}\begin{bmatrix} H_u^{I,II}\\ H_v^{I,II}\end{bmatrix}\end{aligned} \tag{57}$$

When the general anisotropic sheet is backed by a PEC as in Fig. 3, the boundary relations on S_1 are given as

$$E_u(\vec{r}^+_{S1}) = 0 \tag{58a}$$
$$E_v(\vec{r}^+_{S1}) = 0 \tag{58b}$$
$$\varepsilon_{nn} E_n(\vec{r}^+_{S1}) = \rho_{S1} \tag{58c}$$
$$\vec{H}_t(\vec{r}^+_{S1}) = \vec{J}_{S1} \times \hat{n} \tag{58d}$$
$$\mu_{nu} H_u(\vec{r}^+_{S1}) + \mu_{nv} H_v(\vec{r}^+_{S1}) + \mu_{nn} H_n(\vec{r}^+_{S1}) = 0 \tag{58e}$$
$$\sigma_{nn} E_n(\vec{r}^+_{S1}) + div_S \vec{J}_{S1} - i\omega\rho_{S1} = 0 \tag{58f}$$
$$\hat{n} \cdot \vec{J}_{S1} = 0 \tag{58g}$$

while the relations (48) on S_2 still apply. One can combine (58c), (f) as

$$div_S \vec{J}_{S1} = i\omega \left(\varepsilon^c_{nn}/\varepsilon_{nn}\right) \rho_{S1}. \tag{58h}$$

Accordingly, the six sources on S_1 in (49a)–(f) can be expressed in terms of electrical and magnetic fields as

$$\vec{J}_{Ct}(\vec{r}^+_{S1}) = \vec{0} \tag{59a}$$
$$J_{Cn}(\vec{r}^+_{S1}) = \sigma_{nn} E_n(\vec{r}^+_{S1}) = (\sigma_{nn}/\varepsilon_{nn})\,\rho_{S1} = (\sigma_{nn}/i\omega\varepsilon^c_{nn})\, div_S \vec{J}_{S1} \tag{59b}$$
$$\vec{P}^e_t(\vec{r}^+_{S1}) = \vec{0} \tag{59c}$$

$$\begin{aligned} P^e_n(\vec{r}^+_{S1}) &= (\varepsilon_{nn} - \varepsilon_0) E_n(\vec{r}^+_{S1}) = \left((\varepsilon_{nn} - \varepsilon_0)/\varepsilon_{nn}\right) \rho_{S1} \\ &= \left((\varepsilon_{nn} - \varepsilon_0)/i\omega\varepsilon^c_{nn}\right) div_S \vec{J}_{S1} \end{aligned} \tag{59d}$$

$$\begin{aligned} \vec{P}^m_t(\vec{r}^+_{S1}) &= \hat{u}\left((\mu_{uu} - \mu_0) H_u(\vec{r}^+_{S1}) + \mu_{uv} H_v(\vec{r}^+_{S1})\right) \\ &\quad + \hat{v}\left(\mu_{vu} H_u(\vec{r}^+_{S1}) + (\mu_{vv} - \mu_0) H_v(\vec{r}^+_{S1})\right) \\ &= \hat{u}\left((\mu_{uu} - \mu_0) J_{S1v} - \mu_{uv} J_{S1u}\right) \\ &\quad + \hat{v}\left(\mu_{vu} J_{S1v} - (\mu_{vv} - \mu_0) J_{S1u}\right) \end{aligned} \tag{59e}$$

$$\begin{aligned} P^m_n(\vec{r}^+_{S1}) &= \mu_{nu} H_u(\vec{r}^+_{S1}) + \mu_{nv} H_v(\vec{r}^+_{S1}) + (\mu_{nn} - \mu_0) H_n(\vec{r}^+_{S1}) \\ &= -\mu_0 H_n(\vec{r}^+_{S1}) = \frac{\mu_0}{\mu_{nn}}\left(\mu_{nu} H_u(\vec{r}^+_{S1}) + \mu_{nv} H_v(\vec{r}^+_{S1})\right) \\ &= \frac{\mu_0}{\mu_{nn}}\left(\mu_{nu} J_{S1v} - \mu_{nv} J_{S1u}\right) \end{aligned} \tag{59f}$$

while (49g)–(l) on S_2 still apply. Next, (59a)–(f) and (49g)–(l) are placed into (18) to yield

$$(1/d)\, \vec{J}_{0t} \simeq \hat{u}\left(\sigma_{uu} E^{II}_u + \sigma_{uv} E^{II}_v\right) + \hat{v}\left(\sigma_{vu} E^{II}_v + \sigma_{vv} E^{II}_v\right) \tag{60a}$$

$$(1/d)\, J_{0n} \simeq (\sigma_{nn}/i\omega\varepsilon^c_{nn})\, div_S \vec{J}_{S1} + \frac{\sigma_{nn}}{\varepsilon^c_{nn}} \varepsilon^c_2 E^{II}_n + \left(\sigma_{nu} - \sigma_{nn}\frac{\varepsilon^c_{nu}}{\varepsilon^c_{nn}}\right) E^{II}_u + \left(\sigma_{nv} - \sigma_{nn}\frac{\varepsilon^c_{nv}}{\varepsilon^c_{nn}}\right) E^{II}_v \tag{60b}$$

$$(1/d)\, \vec{P}^e_{0t} \simeq \hat{u}\big((\varepsilon_{uu}-\varepsilon_0)E^{II}_u + \varepsilon_{uv}E^{II}_v\big) + \hat{v}\big(\varepsilon_{vu}E^{II}_u + (\varepsilon_{vv}-\varepsilon_0)E^{II}_v\big) \tag{60c}$$

$$(1/d)\, P^e_{0n} \simeq ((\varepsilon_{nn} - \varepsilon_0)/i\omega\varepsilon^c_{nn})\, div_S \vec{J}_{S1} + \frac{(\varepsilon_{nn} - \varepsilon_0)}{\varepsilon^c_{nn}} \varepsilon^c_2 E^{II}_n + \left(\varepsilon_{nu} - (\varepsilon_{nn}-\varepsilon_0)\frac{\varepsilon^c_{nu}}{\varepsilon^c_{nn}}\right) E^{II}_u + \left(\varepsilon_{nv} - (\varepsilon_{nn}-\varepsilon_0)\frac{\varepsilon^c_{nv}}{\varepsilon^c_{nn}}\right) E^{II}_v \tag{60d}$$

$$\begin{aligned}(1/d)\vec{P}^m_{0t} \simeq\ & \hat{u}((\mu_{uu}-\mu_0)J_{S1v} - \mu_{uv}J_{S1u}) + \hat{v}(\mu_{vu}J_{S1v} - (\mu_{vv} - \mu_0)J_{S1u}) \\ & + \hat{u}\big((\mu_{uu}-\mu_0)H^{II}_u + \mu_{uv}H^{II}_v\big) \\ & + \hat{v}\big(\mu_{vu}H^{II}_u + (\mu_{vv}-\mu_0)H^{II}_v\big)\end{aligned} \tag{60e}$$

$$(1/d)\, P^m_{0n} \simeq \frac{\mu_0}{\mu_{nn}} (\mu_{nu}J_{S1v} - \mu_{nv}J_{S1u}) + \frac{(\mu_{nn} - \mu_0)}{\mu_{nn}} \mu_2 H^{II}_n + \frac{\mu_0\mu_{nu}}{\mu_{nn}} H^{II}_u + \frac{\mu_0\mu_{nv}}{\mu_{nn}} H^{II}_v \tag{60f}$$

Finally, one places (60) into (15) to obtain

$$\begin{aligned}&\hat{n} \times [\vec{E}^{II}_t - \vec{E}^I_t] \\ =\ & \hat{n} \times grad_S \left[(e_{nn}/i\omega)\, div_S \vec{J}_{S1} + e_{2nn} E^{II}_n + e_{nu}E^{II}_u + e_{nv}E^{II}_v\right] \\ & - \hat{u}\left[h_{uu}\left(J_{S1v} + H^{II}_u\right) + h_{uv}\left(-J_{S1u} + H^{II}_v\right)\right] \\ & - \hat{v}\left[h_{vu}\left(J_{S1v} + H^{II}_u\right) + h_{vv}\left(-J_{S1u} + H^{II}_v\right)\right]\end{aligned} \tag{61a}$$

$$\begin{aligned}&\hat{n} \times [\vec{H}^{II}_t - \vec{H}^I_t] \\ =\ & \hat{n} \times grad_S \left[h_{nu}\left(J_{S1v} + H^{II}_u\right) + h_{nv}\left(J_{S1u} + H^{II}_v\right) + h_{nn}H^{II}_n\right] \\ & + e_{uu}\vec{E}^{II}_u + e_{vv}\vec{E}^{II}_v - e_{uv}\hat{n} \times \vec{E}^{II}_v + e_{vu}\hat{n} \times \vec{E}^{II}_u\end{aligned} \tag{61b}$$

$$i\omega\varepsilon^c_2 E^{II}_n = div_S \left[e_{uu}\vec{E}^{II}_u + e_{vv}\vec{E}^{II}_v - e_{uv}\hat{n}\times\vec{E}^{II}_v + e_{vu}\hat{n}\times\vec{E}^{II}_u\right] \tag{61c}$$

$$\begin{aligned}i\omega\, B^{II}_n = div_S\ & [\hat{u}\,(h_{uu}J_{S1v} - h_{uv}J_{S1u}) + \hat{v}\,(-h_{vv}J_{S1u} + h_{vu}J_{S1v}) \\ & + h_{uu}\vec{H}^{II}_u + h_{vv}\vec{H}^{II}_v - h_{uv}\hat{n}\times\vec{H}^{II}_v + h_{vu}\hat{n}\times\vec{H}^{II}_u\Big]\end{aligned} \tag{61d}$$

where in (61) we additionally define

$$e_{nn} = -d\frac{(\varepsilon^c_{nn} - \varepsilon_0)}{\varepsilon_0\varepsilon^c_{nn}} \tag{62}$$

Again, similar arguments as in Table 3 hold for the availability of the impedance condition which is modified as

$$\hat{n} \times \vec{H}_t^{II} = \frac{1}{Z_{uu}} \vec{E}_u^{II} + \frac{1}{Z_{vv}} \vec{E}_v^{II} - \frac{1}{Z_{uv}} \hat{n} \times \vec{E}_v^{II} + \frac{1}{Z_{vu}} \hat{n} \times \vec{E}_u^{II} \quad (63a)$$

with

$$Z_{uu} = \frac{1}{e_{uu}}, \quad Z_{vv} = \frac{1}{e_{vv}}, \quad Z_{uv} = \frac{1}{e_{uv}}, \quad Z_{vu} = \frac{1}{e_{vu}} \quad (63b)$$

The impedance condition can also be given in matrix form as

$$\begin{aligned} \begin{bmatrix} H_u^{II} - H_u^I \\ H_v^{II} - H_v^I \end{bmatrix} &= \begin{bmatrix} 1/Z_{vu} & 1/Z_{vv} \\ -1/Z_{uu} & -1/Z_{uv} \end{bmatrix} \begin{bmatrix} E_u^{I,II} \\ E_v^{I,II} \end{bmatrix} \\ &= \begin{bmatrix} e_{vu} & e_{vv} \\ -e_{uu} & -e_{uv} \end{bmatrix} \begin{bmatrix} E_u^{I,II} \\ E_v^{I,II} \end{bmatrix}. \end{aligned} \quad (64)$$

9. A NUMERICAL TEST OF SCALAR IMPEDANCE BOUNDARY CONDITION

In this section we investigate the frequency range of validity of the impedance boundary for a PEC backed planar uniaxially anisotropic sheet in free space when illuminated by a normally incident plane wave as depicted in Fig. 4.

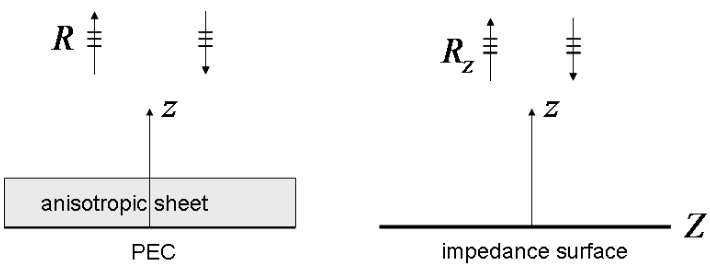

Figure 4. Plane wave reflection from a uniaxially anisotropic and nonmagnetic sheet in free space.

The sheet is assumed of Type 2 material described in Table 2 and the fields of the incoming wave shall be given as

$$\vec{E}^{inc} = \hat{x}\, e^{-ik_0 z}, \quad (65a)$$

$$\vec{H}^{inc} = -\hat{y} \frac{e^{-ik_0 z}}{Z_0} \quad (65b)$$

$$\text{with} \quad k_0 = \omega\sqrt{\mu_0 \varepsilon_0}, \quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \quad (65c)$$

Then one obtains the exact reflection coefficients for the sheet and the impedance boundary respectively as

$$R = -\frac{1 + i\eta^c \tan(2kd)}{1 - i\eta^c \tan(2kd)} e^{-i4k_0 d} \tag{66a}$$

with

$$k = \omega\sqrt{\mu_0\left(\varepsilon + i\sigma/\omega\right)}, \tag{66b}$$

$$\eta^c = k_0/k = 1\Big/\sqrt{\varepsilon_r + i\sigma/(\omega\varepsilon_0)}, \tag{66c}$$

$$\varepsilon_r = \varepsilon/\varepsilon_0 \tag{66d}$$

and

$$R_Z = \frac{\eta - 1}{\eta + 1} \tag{67a}$$

$$\text{with} \quad \eta = Z/Z_0, \quad Z = i/\left[d\left(\omega\left(\varepsilon - \varepsilon_0\right) + i\sigma\right)\right] \tag{67b}$$

The wavenumber inside the sheet can be written as

$$k = \alpha + i\beta \tag{68a}$$

where

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\sqrt{1+p^2}+1}{2}}, \quad \beta = \frac{\omega}{c}\sqrt{\frac{\sqrt{1+p^2}-1}{2}} \tag{68b}$$

with

$$p = \frac{\sigma}{\omega\varepsilon}, \quad c = \frac{1}{\sqrt{\mu_0\varepsilon}} \tag{68c}$$

It allows us to express the wavelength inside the sheet directly as

$$\lambda = \frac{2\pi}{\alpha} \tag{69}$$

The unique assumption in the derivation of all approximate boundary conditions investigated in the manuscript is the averaging procedure which assumes a linear variation of amplitude and phase for all field quantities inside the sheet. Such an approximation obviously requires the electrical thickness of the sheet be "small enough"; namely,

$$\frac{2d}{\lambda} \ll 1. \tag{70}$$

In what follows we provide 8 numerical illustrations for the reflection coefficients R and R_Z in the frequency range 10 MHz–10 GHz, where we assume reasonable fixed values $\varepsilon_r = 5$, $2d = 1\,\text{cm}$ regarding microwave applications. Our observations are summarized under the following 3 items.

i) In all observations the validity of the impedance boundary condition seems to corrupt gradually for increasing values of $\frac{2d}{\lambda}$, as expected.

ii) When $\sigma = 0$ we have $p = 0$, $\beta = 0$, $k = \frac{\omega}{c}$, $\eta^c = 1/\sqrt{\varepsilon_r}$, $Z = i/[d\omega(\varepsilon - \varepsilon_0)]$. In this case the reflection coefficients are purely imaginary (with $|R| = |R_Z| = 1$) indicating perfect reflection, while a considerable theoretical discrepancy is observed in their phases calculated as

$$phase(R) = \pi + 2\tan^{-1}\left[\frac{1}{\sqrt{\varepsilon_r}}\tan\left(2d\frac{\omega}{c}\right)\right] - 4k_0 d \quad \text{(71a)}$$

$$phase(R_Z) = \pi - 2\tan^{-1}\left[\frac{1}{Z_0 d\omega(\varepsilon - \varepsilon_0)}\right] \quad \text{(71b)}$$

The reflection coefficients (in degrees) given in Fig. 5 are opposite in phase in a stable manner in the frequency range satisfying (70),

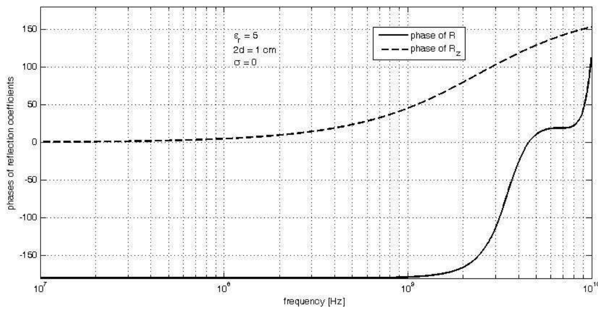

Figure 5. Variation of the phases of reflection coefficients versus frequency for $\sigma = 0$.

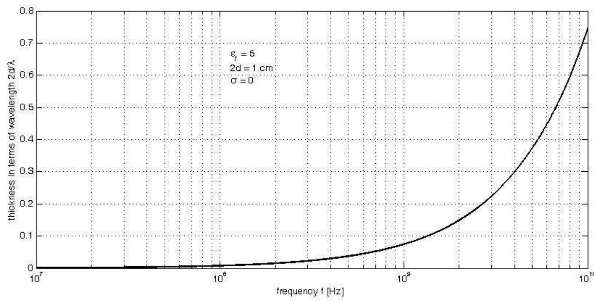

Figure 6. Variation of the electrical thickness of the sheet versus frequency for $\sigma = 0$.

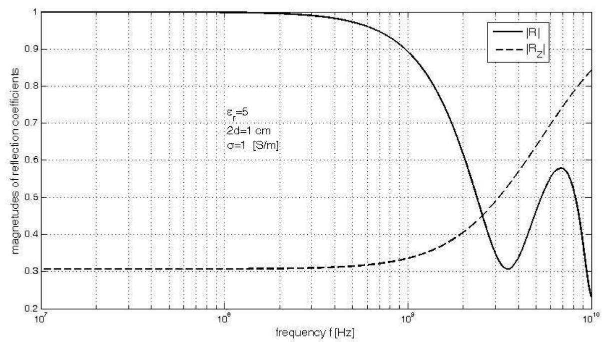

Figure 7. Variation of the magnitudes of reflection coefficients versus frequency for $\sigma = 1$.

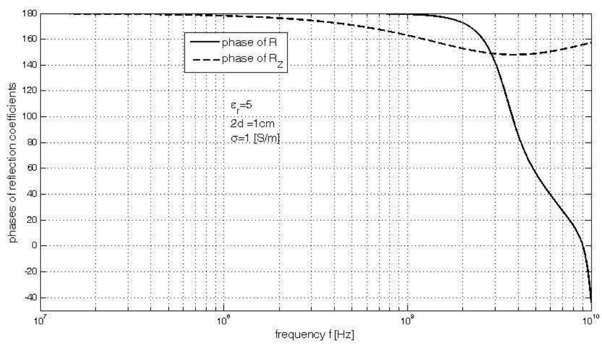

Figure 8. Variation of the phases of reflection coefficients versus frequency for $\sigma = 1$.

which can be followed from Fig. 6.

iii) We provide two sets of graphs for the illustration of the influence of conductivity in improving the impedance boundary approximation. The only difference in between is that the conductivity is chosen as $\sigma = 1\,[\text{S/m}]$ in the first set of Figs. 7–9, while it is increased to $\sigma = 5\,[\text{S/m}]$ in the second set of Figs. 10–12. In the first set we observe a considerable mismatch in the magnitudes of the reflection coefficients. The field reflected from the impedance boundary has a very low amplitude, while the error in the phase patterns surprisingly disappears in the frequency range satisfying (70), which, in this case, is up to 1 GHz as seen in Fig. 8.

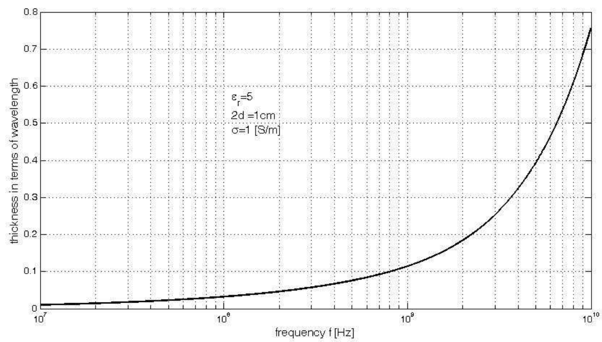

Figure 9. Variation of the electrical thickness of the sheet versus frequency for $\sigma = 1$.

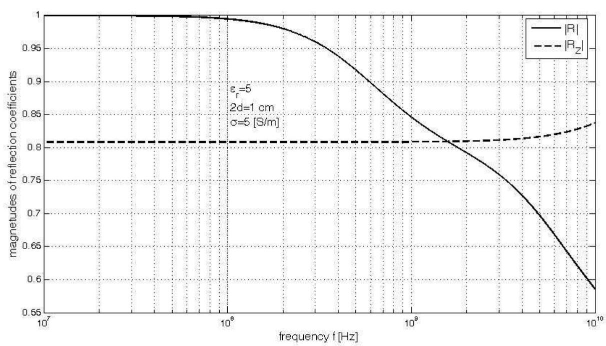

Figure 10. Variation of the magnitudes of reflection coefficients versus frequency for $\sigma = 5$.

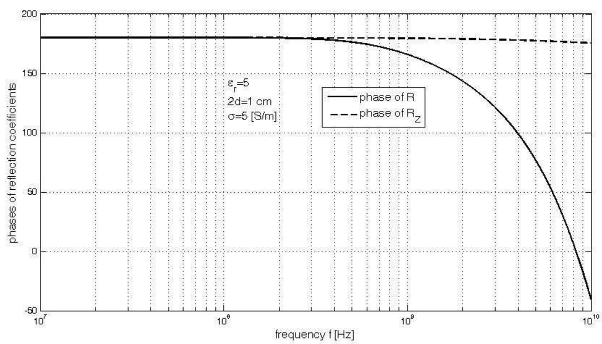

Figure 11. Variation of the phases of reflection coefficients versus frequency for $\sigma = 5$.

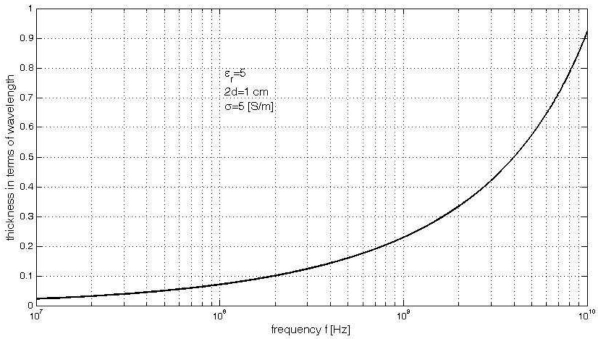

Figure 12. Variation of the electrical thickness of the sheet versus frequency for $\sigma = 5$.

In Figs. 10–12, the influence of the increase in conductivity in (67) in the convergence of the impedance boundary approximation can be seen very clearly by a much better fit in amplitude and phase patterns over a relatively larger frequency range. It is observed that the convergence is proportional with the increase in conductivity, while the phase of R_Z is not much sensitive to frequency in general regardless of the conductivity. A comparison of Figs. 9 and 12 reveal numerically the increase in the electrical thickness of the sheet with increasing conductivity, and therefore the tangent loss p, and real part of the wavenumber α, which indicate the decrease in wavelength λ in the sheet.

10. CONCLUDING REMARKS

Similar convergence properties are expected for biaxially and general anisotropic sheets since all derivations rely on the same averaging procedure. An attempt to improve further the frequency range of convergence of the boundary relations derived in the manuscript requires a better (first or high order) averaging procedure. Since that involves the spatial derivatives of the fields inside the sheet, such an investigation can also be carried out in the context of "generalized impedance boundary conditions", a research field developed and reviewed comprehensively in [1] and [2, Sec. 6]. The presented methodology can also be extended rigorously to derive approximate boundary relations on multilayered sheets with the motivation of investigating scattering by integrated planar microwave circuit structures.

APPENDIX A. CERTAIN DISTRIBUTIONAL TOOLS ON AN ARBITRARY REGULAR SURFACE

The theory of generalized functions were first introduced by Sobolev (cf. [9]) in early 20th century and later developed extensively by mid 20th century as documented in the reference works by many great mathematicians led by the pioneers Schwartz [10], Gelfand and Shilov [11]. The requirement for generalized functions in a field theory can be realized immediately in an attempt to express analytically the volume density function of a source quantity concentrated in a non-volumetric domain. Since the point form field equations in mathematical physics are always given through density functions in space and time, such equations would otherwise not permit algebraic operations in non-volumetric domains.

Regardless of the structure of the singularity domain being a surface, a space curve or a point, in a Schwartz-Sobolev setting we may assume any arbitrary scalar and vector field quantity to be expressed in the conventional form

$$V(\vec{r};t)=\{V(\vec{r};t)\}+[V(\vec{r};t)]_S\,,\quad \vec{A}(\vec{r};t)=\left\{\vec{A}(\vec{r};t)\right\}+\left[\vec{A}(\vec{r};t)\right]_S \quad \text{(A1)}$$

The quantities in curly brackets signify the "regular component" of a distribution. They are defined outside the singularity domain and are assumed to be of L^1_{loc}, the class of locally integrable functions in the Lebesgue sense, in any compact subspace of 3-D Euclidean space R_3. The second terms at the r.h.s. of (A1) with representation $[\,]_S$ stand for the "singular component" of a distribution. They are assumed to be of D', the class of Schwartz-Sobolev distributions, in the present case concentrated on a regular, two-sided, fixed, isolated and closed surface S. The spatial derivatives of locally integrable functions in L^1_{loc} may generate distributions in D' as addressed in Theorem A1. To ensure the smooth behavior of the regular components in (A1) in presence of a surface of singularity S, we require them to be *regular singular* functions ([12], Sec. 5.5) w.r.t. S in a subspace $E_S \subset L^1_{loc}$ defined with the properties

i) $\{V(\vec{r};t)\}$ and $\{\vec{A}(\vec{r};t)\}$ have spatial derivatives of all orders outside S, and

ii) $\{V(\vec{r};t)\}$, $\{\vec{A}(\vec{r};t)\}$ and all their spatial derivatives have boundary values (in other words, possess bounded (or jump) discontinuities) from both sides of S.

The surface distribution $\delta(S)$ is described by the inner product

$$\langle \delta(S), \phi(\vec{r};t)\rangle = \int_{-\infty}^{\infty}\int_{S} \phi(\vec{r}_S;t)dS\,dt \tag{A2}$$

where $\phi(\vec{r};t) \in D$ is a test function infinitely differentiable (in C^{∞}) with a compact support and dS is the surface measure on S.

Theorem A1: The gradient, divergence and curl of the regular components of distributions in presence of an isolated closed surface of singularity S have the general form

$$grad\{V(\vec{r};t)\} = \{gradV(\vec{r};t)\} + \hat{n}\,\Delta[V]\,\delta(S) \tag{A3a}$$

$$div\left\{\vec{A}(\vec{r};t)\right\} = \left\{div\vec{A}(\vec{r};t)\right\} + \hat{n}\cdot\Delta[\vec{A}]\,\delta(S) \tag{A3b}$$

$$curl\left\{\vec{A}(\vec{r};t)\right\} = \left\{curl\vec{A}(\vec{r};t)\right\} + \hat{n}\times\Delta[\vec{A}]\,\delta(S) \tag{A3c}$$

where

$$\Delta[V] \stackrel{\Delta}{=} V(\vec{r}_S^{\,+};t) - V(\vec{r}_S^{\,-};t), \quad \Delta[\vec{A}] \stackrel{\Delta}{=} \vec{A}(\vec{r}_S^{\,+};t) - \vec{A}(\vec{r}_S^{\,-};t) \tag{A4}$$

represents the spatial jump on S when one approaches to the same point from the two sides. A proof can be found in [12, Sec. 5.5].

Let (u_1, u_2) be the real valued orthogonal parametric curves of S described by the position vector $\vec{r}_S = \vec{r}(u_1, u_2)$. A quantity that assumes one or more definite values at each point of a surface is called a function of position, or a 'point function' for the surface. Let us consider a scalar point function $\psi(u_1, u_2)$ and a vector point function

$$\vec{A}(u_1,u_2) = A_1(u_1,u_2)\hat{u}_1 + A_2(u_1,u_2)\hat{u}_2 + A_n(u_1,u_2)\hat{n},$$

where $\hat{u}_1$, $\hat{u}_2$ are unit tangent vectors along the curves u_1 = const. and u_2 = const. and $\hat{n}$ is the unit normal of S, which constitute a right handed system. Then it can be shown (cf. [13, Ch. 12]) that the gradient, divergence and curl operators acting on the point functions on a surface are as follows:

Theorem A2: (Surficial Vector Differential Operators)

$$grad_S\psi = \frac{\hat{u}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial\psi}{\partial u_2} \tag{A5a}$$

$$div_S\vec{A} = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_1) + \frac{\partial}{\partial u_2}(h_1A_2)\right] - 2\Omega A_n \tag{A5b}$$

$$curl_S\vec{A} = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\hat{n} + \frac{A_2}{\alpha_2}\hat{u}_1 - \frac{A_1}{\alpha_1}\hat{u}_2 + grad_SA_n\times\hat{n} \tag{A5c}$$

Here h_1, h_2 are the metric coefficients of the parametric curves; the principle radii of curvature $\alpha_{1,2}$ are related to the metric coefficients through

$$\frac{1}{\alpha_1} = -\frac{1}{h_1}\frac{dh_1}{dn}, \quad \frac{1}{\alpha_2} = -\frac{1}{h_2}\frac{dh_2}{dn} \tag{A5d}$$

and

$$2\Omega = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} = -div_S(\hat{n}) \tag{A5e}$$

is called the first curvature of S.

Let us write the tangential component of $\vec{A}$ as $\vec{A}_t$. Then for this component one also has

$$div_S\vec{A}_t = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_1) + \frac{\partial}{\partial u_2}(h_1A_2)\right] \tag{A6a}$$

$$\begin{aligned} curl_S\vec{A}_t &= \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\hat{n} + \frac{A_2}{\alpha_2}\hat{u}_1 - \frac{A_1}{\alpha_1}\hat{u}_2 \\ &= \hat{n}\, div_S(\vec{A}_t \times \hat{n}) + \frac{A_2}{\alpha_2}\hat{u}_1 - \frac{A_1}{\alpha_1}\hat{u}_2 \end{aligned} \tag{A6b}$$

$$div_S(\vec{A}_t \times \hat{n}) = \hat{n} \cdot curl_S\vec{A}_t \tag{A6c}$$

$$curl_S\left(\hat{n} \times \vec{A}_t\right) = \hat{n}\, div_S\vec{A}_t + \frac{A_1}{\alpha_2}\hat{u}_1 + \frac{A_2}{\alpha_1}\hat{u}_2 \tag{A6d}$$

$$div_S\vec{A}_t = \hat{n} \cdot curl_S\left(\hat{n} \times \vec{A}_t\right) \tag{A6e}$$

Definition A1: The normal derivatives of the surface distribution (A2) of arbitrary order $k \geq 0$ as $\delta^{(k+1)}(\bar{S}) \equiv \sum\limits_{i=1}^{3} n_i \frac{\partial}{\partial x_i}\delta^{(k)}(\bar{S}) = \hat{n} \cdot grad\delta^{(k)}(\bar{S}) = \frac{d^{k+1}}{dn^{k+1}}\delta(\bar{S})$ are described by

$$\left\langle \delta^{(k)}(\bar{S}), \phi(\vec{r}; t)\right\rangle = (-1)^k \left\langle \delta(\bar{S}), \left(\frac{d}{dn} - 2\Omega\right)^k \phi(\vec{r}; t)\right\rangle \tag{A7}$$

The distributional postulation of Maxwell equations [2, 5–8] require that scalar and vector fields constructed in the general form

$$\begin{aligned} V(\vec{r}; t) &= \{V(\vec{r}; t)\} + \sum_{k=0}^{\infty} V_k(\vec{r}_S; t)\delta^{(k)}(S), \\ \vec{A}(\vec{r}; t) &= \left\{\vec{A}(\vec{r}; t)\right\} + \sum_{k=0}^{\infty} \vec{A}_k(\vec{r}_S; t)\delta^{(k)}(S) \end{aligned} \tag{A8}$$

can represent any source and field quantity successfully for all types of polarization (molecular displacement) mechanisms in the normal direction on the singularity surface S.

Theorem A3: Every distribution that has compact support is of finite order.

This general theorem is quite well known and many alternative approaches to its proof are available in literature (cf. [14, Ch. 3], [15, Ch. 3, Sec. 6]).

The reflection of this theorem for a surface type distribution whose support lies on the regular surface S is that the singular components of scalar/vector field quantities have the unique representation

$$[V(\vec{r};t)]_S = \sum_{k=0}^{N} V_k(\vec{r}_S;t)\delta^{(k)}(S), \quad \left[\vec{A}(\vec{r};t)\right]_S = \sum_{k=0}^{N} \vec{A}_k(\vec{r}_S;t)\delta^{(k)}(S) \tag{A9}$$

where N is a finite number.

Theorem A4: The Cartesian derivatives of surface distributions of arbitrary order are given by

$$\frac{\partial}{\partial x_i}\delta^{(k)}(S) = n_i\delta^{(k+1)}(S), \quad k \geq 0, \quad i = 1,2,3. \tag{A10}$$

A comprehensive investigation of the properties of surface distributions of arbitrary order including the proof of Theorem A4 can be found in [12, Ch. 5] and the references cited therein.

Corollary A1: For $k \geq 0$,

$$grad\delta^{(k)}(S) = \hat{n}\delta^{(k+1)}(S) \tag{A11a}$$

$$grad\left(V_k(\vec{r}_S;t)\delta^{(k)}(S)\right) = (grad_S V_k)\,\delta^{(k)}(S) + \hat{n}V_k\delta^{(k+1)}(S) \tag{A11b}$$

$$div\left(\vec{A}_k(\vec{r}_S;t)\delta^{(k)}(S)\right) = \left(div_S\vec{A}_k\right)\delta^{(k)}(S) + \hat{n}\cdot\vec{A}_k\delta^{(k+1)}(S) \tag{A11c}$$

$$curl\left(\vec{A}_k(\vec{r}_S;t)\delta^{(k)}(S)\right) = \left(curl_S\vec{A}_k\right)\delta^{(k)}(S) + \hat{n}\times\vec{A}_k\delta^{(k+1)}(S) \tag{A11d}$$

Corollary A2:

$$\begin{aligned} & grad\left(\{V(\vec{r};t)\} + \sum_{k=0}^{N} V_k(\vec{r}_S;t)\delta^{(k)}(S)\right) \\ = \; & \{gradV(\vec{r};t)\} + \left(\hat{n}\Delta[V] + grad_S V_0\right)\delta(S) \\ & + \sum_{k=1}^{N}\left(grad_S V_k + \hat{n}V_{k-1}\right)\delta^{(k)}(S) + \hat{n}V_N\delta^{(N+1)}(S) \end{aligned} \tag{A12a}$$

$$\begin{aligned} & div\left(\left\{\vec{A}(\vec{r};t)\right\}+\sum_{k=0}^{N}\vec{A}_k(\vec{r}_S;t)\delta^{(k)}(S)\right) \\ = & \left\{div\vec{A}(\vec{r};t)\right\}+\left(\hat{n}\cdot\Delta[\vec{A}]+div_S\vec{A}_0\right)\delta(S) \\ & +\sum_{k=1}^{N}\left(div_S\vec{A}_k+\hat{n}\cdot\vec{A}_{k-1}\right)\delta^{(k)}(S)+\hat{n}\cdot\vec{A}_N\delta^{(N+1)}(S) \end{aligned} \tag{A12b}$$

$$\begin{aligned} & curl\left(\left\{\vec{A}(\vec{r};t)\right\}+\sum_{k=0}^{N}\vec{A}_k(\vec{r}_S;t)\delta^{(k)}(S)\right) \\ = & \left\{curl\vec{A}(\vec{r};t)\right\}+\left(\hat{n}\times\Delta[\vec{A}]+curl_S\vec{A}_0\right)\delta(S) \\ & +\sum_{k=1}^{N}\left(curl_S\vec{A}_k+\hat{n}\times\vec{A}_{k-1}\right)\delta^{(k)}(S)+\hat{n}\times\vec{A}_N\delta^{(N+1)}(S) \end{aligned} \tag{A12c}$$

Finally, regarding the unique solution of distributional Maxwell equations we meet in the text it requires to introduce the following theorem.

Theorem A5: The unique solution of the relation

$$\left\{\vec{A}(\vec{r};t)\right\}+\sum_{k=0}^{N}\vec{A}_k(\vec{r}_S;t)\delta^{(k)}(S)=0 \tag{A13a}$$

with $\vec{A}_k(\vec{r}_S;t)$, $k=0,1,\ldots,N$ being smooth point functions of the coordinates of surface S is

$$\left\{\vec{A}(\vec{r};t)\right\}=\vec{0} \quad \text{and} \quad \vec{A}_k(\vec{r}_S;t)=\vec{0}, \quad \forall k. \tag{A13b}$$

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