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TWO EFFICIENT UNCONDITIONALLY-STABLE FOUR-STAGES SPLIT-STEP FDTD METHODS WITH LOW NUMERICAL DISPERSION

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Abstract—Two efficient unconditionally-stable four-stages split-step (SS) finite-difference time-domain (FDTD) methods based on controlling parameters are presented, which provide low numerical dispersion. Firstly, in the first proposed method, the Maxwell's matrix is split into four sub-matrices. Simultaneously, two controlling parameters are introduced to decrease the numerical dispersion error. Accordingly, the time step is divided into four sub-steps. The second proposed method is obtained by adjusting the sequence of the sub-matrices deduced in the first method. Secondly, the theoretical proofs of the unconditional stability and dispersion relations of the proposed methods are given. Furthermore, the processes of obtaining the controlling parameters for the proposed methods are shown. Thirdly, the dispersion characteristics of the proposed methods are also investigated, and numerical dispersion errors of the proposed methods can be decreased significantly. Finally, to substantiate the efficiency of the proposed methods, numerical experiments are presented.

1. INTRODUCTION

Recently, to remove the Courant-Friedrichs-Lewy (CFL) [1] limitation on the time step size of the finite-difference time-domain (FDTD) method [2], an unconditionally-stable FDTD method based on the alternating direction implicit (ADI) technique has been developed [3, 4]. The ADI-FDTD method has second-order accuracy both in time and

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space. Moreover, the numerical dispersion of the ADI-FDTD method was analyzed in [5]. Nevertheless, it presents large numerical dispersion error with large time steps. To improve the dispersion performance, several methods were proposed, such as error-reduced [6], iterative [7, 8], parameter-optimized [9–12], and artificial-anisotropy methods [13–18].

Along the same line, other unconditionally stable methods such as split-step [19–27], locally-one-dimensional (LOD) [28] and leapfrog ADI [29, 30] FDTD methods were developed. The high-order split-step FDTD method in [21] has six stages and is represented as 6-stages SS-FDTD herein. Moreover, the method in [22] has four stages and is denoted as 4-stages SS-FDTD_1 herein, and the four-stages split-step FDTD method for low anisotropy in [23] is denoted as 4-stages SS-FDTD_2. An improved six-stages split-step FDTD method was presented in [27]. The LOD-FDTD method can be considered as the split-step approach (SS1) with first-order accuracy in time, which consumes less CPU time than that of the ADI-FDTD method. Moreover, 3-D LOD-FDTD methods were shown in [31, 32]. The fourth-order LOD-FDTD was presented in [33]. An arbitrary-order 3-D LOD-FDTD method was proposed in [34]. However, similar to the ADI-FDTD method, the LOD-FDTD method also has a larger numerical dispersion error at larger time steps. Subsequently, modified LOD-FDTD methods with low dispersion proposed in [35–38]. Moreover, an efficient method to reduce the numerical dispersion in the LOD-FDTD method based on the (2, 4) stencil was proposed in [39].

To reduce the numerical dispersion error further, two efficient four-stages split-step FDTD methods in two dimensions are proposed in this paper. Firstly, the Maxwell's matrix is split into four sub-matrices. Simultaneously, the controlling parameters are added to reduce the numerical dispersion error. Accordingly, the time step is divided into four sub-steps. Then, the first proposed method is generated, which is denoted by efficient 4-stages SS-FDTD_1. The second proposed method is deduced by adjusting the sequence of the sub-matrices, denoted by efficient 4-stages SS-FDTD_2. Secondly, the proposed methods are proven to be unconditionally stable by using the Fourier method. Furthermore, the dispersion analyses are given. Moreover, the processes of obtaining the controlling parameters are shown. Thirdly, the numerical dispersion characteristics of the proposed methods are analyzed and compared with the ADI-FDTD, 3-stages SS-FDTD, 6-stages SS-FDTD, and the initial 4-stages SS-FDTD methods, which can be improved significantly. Finally, numerical experiments are presented to verify the properties of the proposed methods.

2. NEW NUMERICAL FORMULATIONS

2.1. The Efficient 4-stages SS-FDTD_1 Method

For simplicity, the 2-D TM_z wave propagation in a linear, isotropic, non-dispersive and lossless medium is considered here. ε and μ are the electric permittivity and magnetic permeability, respectively. Then, the Maxwell's equations can be written in a matrix form as

$$\partial\vec{u}/\partial t = [M]\,\vec{u}. \tag{1}$$

where $\vec{u} = [E_z, H_x, H_y]^T$, and $[M]$ is the Maxwell's matrix as

$$[M] = \begin{bmatrix} 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial y} & \frac{1}{\varepsilon}\frac{\partial}{\partial x} \\ -\frac{1}{\mu}\frac{\partial}{\partial y} & 0 & 0 \\ \frac{1}{\mu}\frac{\partial}{\partial x} & 0 & 0 \end{bmatrix} [A] = \begin{bmatrix} 0 & 0 & \frac{1}{\varepsilon}\frac{\partial}{\partial x} \\ 0 & 0 & 0 \\ \frac{1}{\mu}\frac{\partial}{\partial x} & 0 & 0 \end{bmatrix} [B]$$
$$= \begin{bmatrix} 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial y} & 0 \\ -\frac{1}{\mu}\frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

According to x and y directions, the matrix $[M]$ is decomposed into four parts $[A]/2$, $[B]/2$, $[A]/2$, $[B]/2$. Simultaneously, controlling parameters of C_x and C_y are introduced to decrease the numerical dispersion error. Then, (1) can be written as

$$\partial\vec{u}/\partial t = C_x\cdot[A]/2\cdot\vec{u}+C_y\cdot[B]/2\cdot\vec{u}+C_x\cdot[A]/2\cdot\vec{u}+C_y\cdot[B]/2\cdot\vec{u}. \tag{2}$$

By using the split-step scheme [19], (2) is divided into four sub-equations. From n to $n+1$, one time step is divided into four sub-steps accordingly, $n \to n+1/4$, $n+1/4 \to n+2/4$, $n+2/4 \to n+3/4$, and $n+3/4 \to n+1$, by successively solving

$$\text{sub-step 1:}\quad \partial\vec{u}/\partial t = 4\cdot C_x\cdot[A]/2\cdot\vec{u} \quad n \to n+1/4 \tag{3a}$$
$$\text{sub-step 2:}\quad \partial\vec{u}/\partial t = 4\cdot C_y\cdot[B]/2\cdot\vec{u} \quad n+1/4 \to n+2/4 \tag{3b}$$
$$\text{sub-step 3:}\quad \partial\vec{u}/\partial t = 4\cdot C_x\cdot[A]/2\cdot\vec{u} \quad n+2/4 \to n+3/4 \tag{3c}$$
$$\text{sub-step 4:}\quad \partial\vec{u}/\partial t = 4\cdot C_y\cdot[B]/2\cdot\vec{u}. \quad n+3/4 \to n+1 \tag{3d}$$

Moreover, the right side of the above equations can be approximated by using the Crank-Nicolson scheme. Subsequently, four sub-procedures are generated as follows

$$([I]-\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n+1/4} = ([I]+\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n} \tag{4a}$$
$$([I]-\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+2/4} = ([I]+\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+1/4} \tag{4b}$$
$$([I]-\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n+3/4} = ([I]+\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n+2/4} \tag{4c}$$
$$([I]-\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+1} = ([I]+\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+3/4}. \tag{4d}$$

where $[I]$ is a 3×3 identity matrix. (4a)–(4d) are the formulations of the efficient 4-stages SS-FDTD_1 method. The number of sub-steps of this proposed method is four, same as that of the 4-stages SS-FDTD_1 method in [22], except those of controlling parameters C_x and C_y. Specifically, when $C_x = C_y = 1$, (4a)–(4d) can simply be reduced to the formulation of 4-stages SS-FDTD_1 method. In other words, the execution procedure of this proposed method is similar to the 4-stages SS-FDTD_1 method, which means that there is no extra computational complexity involved. For instance, for the sub-step 1, after a series of manipulation, (4a) can be expressed as

$$\begin{aligned}
&\left[1+\frac{(C_x\Delta t)^2}{8\mu\varepsilon(\Delta x)^2}\right]E_z\Big|_{i,j}^{n+1/4}-\left[\frac{(C_x\Delta t)^2}{16\mu\varepsilon(\Delta x)^2}\right]\left(E_z\Big|_{i+1,j}^{n+1/4}+E_z\Big|_{i-1,j}^{n+1/4}\right)\\
=&\left[1-\frac{(C_x\Delta t)^2}{8\mu\varepsilon(\Delta x)^2}\right]E_z\Big|_{i,j}^{n}+\left[\frac{(C_x\Delta t)^2}{16\mu\varepsilon(\Delta x)^2}\right]\left(E_z\Big|_{i+1,j}^{n}+E_z\Big|_{i-1,j}^{n}\right)\\
&+\frac{C_x\Delta t}{2\varepsilon\Delta x}\left(H_y\Big|_{i+1/2,j}^{n}-H_y\Big|_{i-1/2,j}^{n}\right) \qquad (5a)
\end{aligned}$$

$$H_x\Big|_{i,j+1/2}^{n+1/4} = H_x\Big|_{i,j+1/2}^{n} \qquad (5b)$$

$$\begin{aligned}
H_y\Big|_{i+1/2,j}^{n+1/4} = {}& H_y\Big|_{i+1/2,j}^{n}+\frac{C_x\Delta t}{4\mu\Delta x}\left(E_z\Big|_{i+1,j}^{n+1/4}-E_z\Big|_{i,j}^{n+1/4}\right)\\
&+\frac{C_x\Delta t}{4\mu\Delta x}\left(E_z\Big|_{i+1,j}^{n}-E_z\Big|_{i,j}^{n}\right). \qquad (5c)
\end{aligned}$$

It can be found that only (5a) and (5c) need to be solved in this sub-step. (5a) is a linear system with a tri-diagonal coefficient matrix; it can be solved efficiently with special numerical packages. In addition, (5c) is an explicit equation that can be computed directly. Similar update equations can be achieved for other sub-steps.

2.2. The Efficient 4-stages SS-FDTD_2 Method

Adjusting the sequence of the sub-matrices in the form of $[A]/2$, $[B]/2$, $[B]/2$, $[A]/2$, which is different from the sequence in the first proposed method, a series of operations are taken, which are similar to the efficient 4-stages SS-FDTD_1 method, four sub procedures are acquired as follows

$$([I]-\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n+1/4} = ([I]+\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n} \qquad (6a)$$

$$([I]-\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+2/4} = ([I]+\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+1/4} \qquad (6b)$$

$$([I]-\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+3/4} = ([I]+\Delta t/4\cdot C_y\cdot[B])\,\vec{u}^{n+2/4} \qquad (6c)$$

$$([I]-\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n+1} = ([I]+\Delta t/4\cdot C_x\cdot[A])\,\vec{u}^{n+3/4}. \qquad (6d)$$

Then, the formulation of the efficient 4-stages SS-FDTD_2 method is generated. A similar manipulation is adopted for the above equations, which is mentioned in the efficient 4-stages SS-FDTD_1 method. Therefore, two proposed methods have similar formulations. Moreover, the number of sub-steps of the second proposed method is four, and it is the same as that of the 4-stages SS-FDTD_2 method in [23], except those of controlling parameters C_x and C_y. Specifically, when $C_x = C_y = 1$, (6a)–(6d) can simply be reduced to the formulation of the 4-stages SS-FDTD_2 method.

Note that in the above formulations of (5a)–(5c), there are one implicit equation and one explicit equation; therefore, there are eight equations to be computed in total for four sub-steps (i.e., a whole time step). However, in the 2-D ADI-FDTD method [3] and 2-D LOD-FDTD method [28], six and four equations to be computed in two sub-steps, respectively. In the 2-D 3-stages SS-FDTD method [19] and 2-D 6-stages SS-FDTD method [21], six and twelve equations to be

Table 1. Number of arithmetic operations and tri-diagonal matrices.

Method		ADI-FDTD	LOD-FDTD	3-stages SS-FDTD
Number of tri-Diagonal matrices		2	2	3
Implicit	M/D	4	6	9
	A/S	8	8	12
Explicit	M/D	4	2	3
	A/S	8	8	12
Total	M/D	8	8	12
	A/S	16	16	24

Method		6-stages SS-FDTD	4-stages SS-FDTD_1/ first proposed	4-stages SS-FDTD_2/ second proposed
Number of tri-Diagonal matrices		6	4	4
Implicit	M/D	18	12	12
	A/S	24	16	16
Explicit	M/D	6	4	4
	A/S	24	16	16
Total	M/D	24	16	16
	A/S	48	32	32

computed in a full time step, respectively.

In order to investigate the computational requirements of two proposed methods and some previously published FDTD methods, the number of arithmetic operations and tri-diagonal matrices is shown in Table 1, in which M/D and A/S indicate multiplication/division and addition/subtraction, respectively. From Table 1, at each time step, more arithmetic operations and tri-diagonal matrices are involved in the computations of the proposed methods. Therefore, the computational requirement of the proposed methods is then larger than the ADI-FDTD, LOD-FDTD and 3-stages SS-FDTD methods at each time step. However, the proposed methods have higher order accuracy, a larger time step and a coarser mesh can be used. Therefore, the total number of iterations required by the proposed methods can be reduced. Consequently, the computational requirement of the proposed methods is lesser.

3. NUMERICAL STABILITY ANALYSIS

To analyze the stability condition of the proposed methods, the Fourier method is employed, which has been employed to prove the unconditionally stable ADI-FDTD method [4, 18], 3-stages SS-FDTD method [19], 4-stages SS-FDTD method [25], and LOD-FDTD method [32, 34]. With the method, the amplification matrix of the proposed methods is first obtained through projection of (5) into the spatial domain with Fourier transformation applied in space. Then modules of all the eigenvalues of the amplification matrix are examined: if every one of them is not larger than unity in magnitude, the method is considered unconditionally stable. In this section, the Fourier method is applied to prove the unconditional stability of the proposed methods described in this paper.

3.1. The Efficient 4-stages SS-FDTD_1 Method

By using the Fourier method, assume that a wave propagating at angle ϕ is in the spherical coordinate system. Then, $k_x = k\cos\phi$, $k_y = k\sin\phi$, the field components in spectral domain at the nth time step can be denoted as

$$U\big|_{I,J}^{n} = U^n e^{-j(k_x I\Delta x + k_y J\Delta y)}. \tag{7}$$

Equations (4a)–(4d) in each sub-step can be represented as the following matrices form

$$\text{sub-step 1:} \quad U^{n+1/4} = [\Lambda_A]\, U^n \tag{8a}$$

$$\text{sub-step 2:} \quad U^{n+2/4} = [\Lambda_B]\, U^{n+1/4} \tag{8b}$$

$$\text{sub-step 3:}\quad U^{n+3/4} = [\Lambda_A]\, U^{n+2/4} \tag{8c}$$

$$\text{sub-step 4:}\quad U^{n+1} = [\Lambda_B]\, U^{n+3/4}. \tag{8d}$$

$$[\Lambda_A] = \begin{bmatrix} \frac{B_x}{A_x} & 0 & \frac{4jbC_xP_x}{A_x} \\ 0 & 1 & 0 \\ \frac{4jdC_xP_x}{A_x} & 0 & \frac{B_x}{A_x} \end{bmatrix} [\Lambda_B] = \begin{bmatrix} \frac{B_y}{A_y} & -\frac{4jbC_yP_y}{A_y} & 0 \\ -\frac{4jdC_yP_y}{A_y} & \frac{B_y}{A_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $b = \Delta t/2\varepsilon)$, $d = \Delta t/(2\mu)$, $P_\alpha = -2\sin(k_\alpha \Delta\alpha)/\Delta\alpha$, $B_\alpha = 4 - 4bdC_\alpha^2 P_\alpha^2$, $A_\alpha = 4 + 4bdC_\alpha^2 P_\alpha^2$, $\alpha = x$ or y.

Substituting (8a)–(8c) into (8d), the matrix form in one whole time step is generated as

$$U^{n+1} = [\Lambda_B]\,[\Lambda_A]\,[\Lambda_B]\,[\Lambda_A]\, U^n = [\Lambda_1]\, U^n. \tag{9}$$

where $[\Lambda_1]$ is the amplification matrix of the first proposed method.

By using Maple 9.0, the eigenvalues of $[\Lambda_1]$ can be found, as

$$\lambda_{1,1} = 1, \quad \lambda_{1,2} = \lambda_{1,3}^* = \xi_1 + j\sqrt{1-\xi_1^2}. \tag{10}$$

where $\xi_1 = R_1/S_1$, and

$$R_1 = 256 - 384bd\left(C_x^2P_x^2 + C_y^2P_y^2\right) + 16b^2d^2\left(C_x^4P_x^4 + C_y^4P_y^4 - 4C_x^2C_y^2P_x^2P_y^2\right)$$
$$+ 8b^3d^3\left(C_x^2C_y^4P_x^2P_y^4 + C_x^4C_y^2P_x^4P_y^2\right) + b^4d^4C_x^4C_y^4P_x^4P_y^4 \tag{11a}$$

$$S_1 = 256 + 128bd\left(C_x^2P_x^2 + C_y^2P_y^2\right) + 16b^2d^2\left(C_x^4P_x^4 + C_y^4P_y^4 + 4C_x^2C_y^2P_x^2P_y^2\right)$$
$$+ 8b^3d^3\left(C_x^2C_y^4P_x^2P_y^4 + C_x^4C_y^2P_x^4P_y^2\right) + b^4d^4C_x^4C_y^4P_x^4P_y^4. \tag{11b}$$

Since $|\lambda_{1,1}| = |\lambda_{1,2}| = |\lambda_{1,3}| = 1$, we can conclude that the efficient 4-stages SS-FDTD_1 method is unconditionally stable for all cases.

3.2. The Efficient 4-stages SS-FDTD_2 Method

By using the Fourier method, a series of operations are taken, which are similar to the efficient 4-stages SS-FDTD_1 method in Section 3.1, substituting (7) into (6a)–(6d), the matrix form in one whole time step is generated as

$$U^{n+1} = [\Lambda_A]\,[\Lambda_B]\,[\Lambda_B]\,[\Lambda_A]\, U^n = [\Lambda_2]\, U^n. \tag{12}$$

where $[\Lambda_2]$ is the amplification matrix of the second proposed method.

By using Maple 9.0, the eigenvalues of $[\Lambda_2]$ also can be found, as

$$\lambda_{2,1} = 1, \quad \lambda_{2,2} = \lambda_{2,3}^* = \xi_2 + j\sqrt{1-\xi_2^2}. \tag{13}$$

where $\xi_2 = R_2/S_2$, and

$$R_2 = 256 - 384bd\left(C_x^2P_x^2 + C_y^2P_y^2\right) + 16b^2d^2\left(C_x^4P_x^4 + C_y^4P_y^4 + 4C_x^2C_y^2P_x^2P_y^2\right)$$

$$-24b^3d^3\left(C_x^2C_y^4P_x^2P_y^4+C_x^4C_y^2P_x^4P_y^2\right)+b^4d^4C_x^4C_y^4P_x^4P_y^4 \tag{14a}$$

$$S_2=256+128bd\left(C_x^2P_x^2+C_y^2P_y^2\right)+16b^2d^2\left(C_x^4P_x^4+C_y^4P_y^4+4C_x^2C_y^2P_x^2P_y^2\right)$$
$$+8b^3d^3\left(C_x^2C_y^4P_x^2P_y^4+C_x^4C_y^2P_x^4P_y^2\right)+b^4d^4C_x^4C_y^4P_x^4P_y^4. \tag{14b}$$

Similar to the efficient 4-stages SS-FDTD_1 method, since $|\lambda_{2,1}| = |\lambda_{2,2}| = |\lambda_{2,3}| = 1$, the efficient 4-stages SS-FDTD_2 method is also unconditionally stable for all cases.

4. NUMERICAL DISPERSION ANALYSIS

In this section, the dispersion relations of the proposed methods are derived by following a similar procedure described in [5].

4.1. The Efficient 4-stages SS-FDTD_1 Method

To analyze the dispersion characteristic, the field is assumed to be a monochromatic wave with angular frequency ω. Then, the field components become

$$E_z^n = E_z e^{j\omega\Delta tn}, \quad H_\alpha^n = H_\alpha e^{j\omega\Delta tn}, \quad \alpha = x,\, y. \tag{15}$$

Then, (9) can be expressed as

$$\left(e^{j\omega\Delta t}[I]-[\Lambda_1]\right)U^n = 0. \tag{16}$$

where U^n is related to the initial field vector U^0 and defined by

$$U^n = U^0 e^{j\omega\Delta tn}. \tag{17}$$

For a nontrivial solution of (16), the determinant of the coefficient matrix should be zero as follows

$$\det\left(e^{j\omega\Delta t}[I]-[\Lambda_1]\right) = 0. \tag{18}$$

With reference to the eigenvalues of $[\Lambda_1]$ above, the dispersion relationship of the efficient 4-stages SS-FDTD_1 method can be deduced in (19).

$$\tan^2(\omega\Delta t/2) = P_1/Q_1. \tag{19}$$

where

$$P_1 = 256bd\left(C_x^2P_x^2+C_y^2P_y^2\right)+64b^2d^2C_x^2C_y^2P_x^2P_y^2 \tag{20a}$$

$$Q_1 = 256-128bd\left(C_x^2P_x^2+C_y^2P_y^2\right)+16b^2d^2\left(C_x^4P_x^4+C_y^4P_y^4\right)$$
$$+8b^3d^3\left(C_x^2C_y^4P_x^2P_y^4+C_x^4C_y^2P_x^4P_y^2\right)+b^4d^4C_x^4C_y^4P_x^4P_y^4. \tag{20b}$$

when $C_x = C_y = 1$, (19) can simply be reduced to the numerical dispersion expression of the initial 4-stages SS-FDTD_1 method in [22].

4.2. The Efficient 4-stages SS-FDTD_2 Method

The numerical dispersion relation of the efficient 4-stages SS-FDTD_2 method is studied in this subsection. A series of operations are taken, which are similar to the efficient 4-stages SS-FDTD_1 method in Section 4.1, (15) is introduced into (12), and then (12) can be expressed as

$$\left(e^{j\omega\Delta t}[I]-[\Lambda_2]\right)U^n=0. \tag{21}$$

For a nontrivial solution of (21), the determinant of the coefficient matrix should be zero as follows

$$\det\left(e^{j\omega\Delta t}[I]-[\Lambda_2]\right)=0. \tag{22}$$

With reference to the eigenvalues of $[\Lambda_2]$ above, the dispersion relationship of the efficient 4-stages SS-FDTD_2 method can be deduced in (23).

$$\tan^2\left(\omega\Delta t/2\right)=P_2/Q_2. \tag{23}$$

where

$$P_2=256bd\left(C_x^2P_x^2+C_y^2P_y^2\right)+16b^3d^3\left(C_x^4C_y^2P_x^4P_y^2+C_x^2C_y^4P_x^2P_y^4\right) \tag{24a}$$

$$\begin{aligned}Q_2=&\,256-128bd\left(C_x^2P_x^2+C_y^2P_y^2\right)+16b^2d^2\left(C_x^4P_x^4+C_y^4P_y^4+4C_x^2C_y^2P_x^2P_y^2\right)\\&-8b^3d^3\left(C_x^2C_y^4P_x^2P_y^4+C_x^4C_y^2P_x^4P_y^2\right)+b^4d^4C_x^4C_y^4P_x^4P_y^4.\end{aligned} \tag{24b}$$

when $C_x=C_y=1$, (23) can simply be reduced to the numerical dispersion expression of the initial 4-stages SS-FDTD_2 method in [23].

5. DETERMINATION OF CONTROLLING PARAMETERS

In this section, our strategy is to optimize the controlling parameters such that the normalized numerical phase velocity $A(\phi)=\widetilde{v}_p(\phi)/v$ closes to 1 in all propagation directions, where $\widetilde{v}_p=\omega/\widetilde{k}$ is the numerical phase velocity, and v is the speed of light in the medium. We start by determining the initial parameter values C_{x0} and C_{y0} that yield $A=1$ along axial directions. By sweeping the wave propagation angle ϕ, we can find the maximum value $A_{\max}$ at ϕ_m. Thus, the maximum deviation of A from 1 is $Q=(A_{\max}-1)$. Setting $A'=1-Q/2$ along axial directions, which can be ensured that the corrected normalized phase velocity has its minimum in all propagation directions and the corrected controlling parameter values C_x and C_y are obtained. The efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method have the similar processes of controlling parameters. For simplify, only the detailed processes of controlling parameters of the efficient 4-stages SS-FDTD_1 method are given in this paper, which is shown as follows.

Before the descriptions, several notations are introduced for clarity. The normalized numerical phase velocity error (NNPVE) is defined as $|\widetilde{v}_p(\phi)/v - 1| \times 100\%$. Here, in the entire range of ϕ, the maximum value of the NNPVE is denoted as the maximum NNPVE. For clarity, CFLN is used: it is defined as the ratio between the time step taken and the maximum CFL limit of the explicit FDTD method originally proposed in [2]. In addition, the cell per wavelength (CPW): $\lambda/\Delta x$, where λ is the wavelength with no numerical anisotropy. For simplicity, uniform cells are considered here ($\Delta x = \Delta y$).

(a) Determination of the initial parameter values C_{x0} and C_{y0}.

Firstly, assume $A = 1$ along axial directions. Let $\phi = 0°$, then $k_x = k$, $k_y = 0$, $P_x = -2 \cdot (1/\Delta x) \cdot \sin(k_x \Delta x/2)$, $P_y = 0$. Therefore, (19) can be simplified as

$$\tan(\omega\Delta t/2) = -4\sqrt{bd}C_x P_x / \left|bdC_x^2 P_x^2 - 4\right|. \tag{25}$$

Then, we can obtain

$$A = \frac{\widetilde{v}_p}{v} = \frac{\sqrt{2}\text{CPW}}{\pi\text{CFLN}} a\tan\left(\frac{-4\sqrt{bd}C_x P_x}{|bdC_x^2 P_x^2 - 4|}\right). \tag{26}$$

Since the initial parameter of A is equals to 1, and then the initial parameter value C_{x0} can be obtained, as shown in (27).

$$C_{x0} = \frac{-2+2\sqrt{1+\tan^2\left[(1\cdot\pi\text{CFLN})/\left(\sqrt{2}\text{CPW}\right)\right]}}{\tan\left[(1\cdot\pi\text{CFLN})/\left(\sqrt{2}\text{CPW}\right)\right]\left(\text{CFLN}/\sqrt{2}\right)\cdot\sin(\pi/\text{CPW})}. \tag{27}$$

Secondly, let $\phi = 90°$, then $k_x = 0$, $k_y = k$, $P_x = 0$, $P_y = -2 \cdot (1/\Delta y) \cdot \sin(k_y \Delta y/2)$. Then, (19) can be simplified as

$$\tan(\omega\Delta t/2) = -4C_y\sqrt{bd}P_y / \left|bdC_y^2 P_y^2 - 4\right|. \tag{28}$$

Therefore, we can obtain

$$A = \frac{\widetilde{v}_p}{v} = \frac{\sqrt{2}\text{CPW}}{\pi\text{CFLN}} a\tan\left(\frac{-4\sqrt{bd}C_y P_y}{\left|bdC_y^2 P_y^2 - 4\right|}\right). \tag{29}$$

As the initial parameter of A is equals to 1, and then the initial parameter value C_{y0} can be obtained, as shown in (30).

$$C_{y0} = \frac{-2+2\sqrt{1+\tan^2\left[(1\cdot\pi\text{CFLN})/\left(\sqrt{2}\text{CPW}\right)\right]}}{\tan\left[(1\cdot\pi\text{CFLN})/\left(\sqrt{2}\text{CPW}\right)\right]\cdot\left(\text{CFLN}/\sqrt{2}\right)\cdot\sin(\pi/\text{CPW})}. \tag{30}$$

(b) By sweeping the wave propagation angle ϕ from 0° to 90°, the maximum value $A_{\max}$ at ϕ_m can be generated.

(c) The maximum deviation of A from 1 is $Q = (A_{\max} - 1)$.
(d) Setting $A' = 1 - Q/2$ along axial directions, which can be ensured that the corrected normalized phase velocity has its minimum in all propagation directions.
(e) The corrected controlling parameter values C_x and C_y are obtained, as shown in (31).

$$C_{x,y} = \frac{-2+2\sqrt{1+\tan^2\left[(A'\pi\mathrm{CFLN})/\left(\sqrt{2}\mathrm{CPW}\right)\right]}}{\tan\left[(A'\pi\mathrm{CFLN})/\left(\sqrt{2}\mathrm{CPW}\right)\right]\cdot\left(\mathrm{CFLN}/\sqrt{2}\right)\cdot\sin\left(\pi/\mathrm{CPW}\right)}. \quad (31)$$

When CFLN = 5 and CPW = 30, the processes of controlling parameters of the efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method are shown in Table 2. It can be seen that the values of C_x and C_y are equal for the efficient 4-stages SS-FDTD_1 method or the efficient 4-stages SS-FDTD_2 method. This is because uniform cells are used.

6. NUMERICAL DISPERSION CHARACTERISTICS

In this section, to verify the superiority of two proposed methods, the numerical dispersion characteristics of two proposed methods are

Table 2. The process of the controlling parameters with CFLN = 5 and CPW = 30.

	Efficient 4-stages SS-FDTD_1	Efficient 4-stages SS-FDTD_2
Initial value A_0	1	1
Initial values $C_{x0} = C_{y0}$	1.013433	1.013433
$A_{\max}$	1.005179	1.000743
Corrected value A'	0.997411	0.999629
Corrected values $C_x = C_y$	1.010749	1.013048

Table 3. The information of the controlling parameters with CPW = 20.

Method	CFLN	C_x	C_y
Efficient 4-stages SS-FDTD_1	1	1.003927	1.003927
	5	1.024461	1.024461
Efficient 4-stages SS-FDTD_2	1	1.004122	1.004122
	5	1.030121	1.030121

investigated, and compared with other unconditionally stable FDTD methods, i.e., the ADI-FDTD method, LOD-FDTD method, 3-stages SS-FDTD method, 4-stages SS-FDTD_1 method, 4-stages SS-FDTD_2 method, and 6-stages SS-FDTD method. Since the accuracy of the LOD-FDTD method is similar to that of the ADI-FDTD method, in order to have a clear view, the results of comparison with the LOD-FDTD method are omitted in this paper.

6.1. Normalized Numerical Phase Velocity Versus Propagation Angle

When CPW = 20 and CFLN = 1, 5, the information on the controlling parameters of the efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method are shown in Table 3. From Table 3, it can be seen that the values of C_x and C_y of the efficient 4-stages SS-FDTD methods with CFLN = 5 are larger than those of the efficient 4-stages SS-FDTD methods with CFLN = 1. It can be explained by a simple physical argument. Compared with the efficient 4-stages SS-FDTD methods with CFLN = 1, the efficient 4-stages SS-FDTD methods with CFLN = 5 have more serious numerical dispersion errors. Therefore, the efficient 4-stages SS-FDTD methods with CFLN = 5 need larger values of controlling parameters to speed up the wave propagation velocity further.

Figures 1 and 2 show the normalized numerical phase velocity and normalized numerical phase velocity error (NNPVE) versus ϕ with CPW = 20 and CFLN = 1, 5 for seven kinds of FDTD methods, respectively. As can be seen from Figure 1(a), for seven kinds of

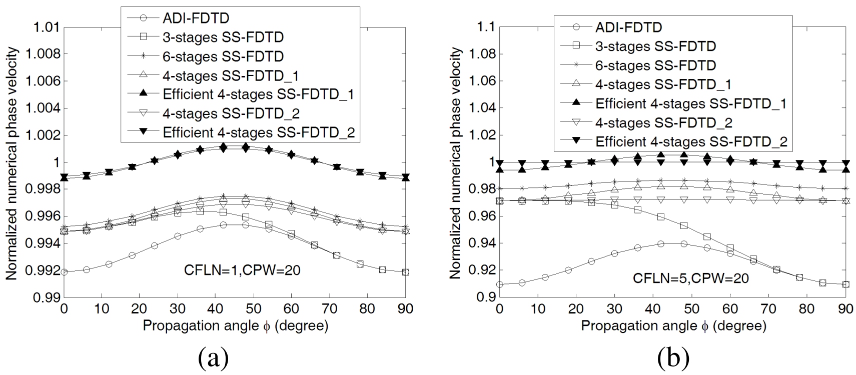


Figure 1. Normalized numerical phase velocity versus ϕ with CPW = 20 for seven kinds of FDTD methods. (a) CFLN = 1; (b) CFLN = 5.

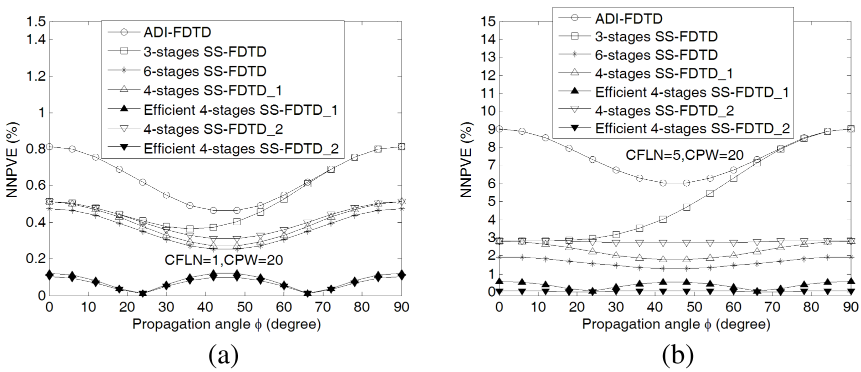


Figure 2. Normalized numerical phase velocity error (NNPVE) versus ϕ with CPW = 20 for seven kinds of FDTD methods. (a) CFLN = 1; (b) CFLN = 5.

FDTD methods, except that of the 3-stages SS-FDTD method, the normalized numerical phase velocity reaches minimum at $\phi = 0^\circ$, 90° and maximum at $\phi = 45^\circ$. Specifically, for the 3-stages SS-FDTD method, the normalized numerical phase velocity reaches minimum at $\phi = 90^\circ$ and maximum at $\phi = 45^\circ$. On the other hand, the normalized numerical phase velocities of the proposed methods are higher than other five kinds of FDTD methods. From Figure 1(b), it can be seen that Figure 1(b) is similar to Figure 1(a), except that the normalized numerical phase velocities of the efficient 4-stages SS-FDTD_2 method and the 4-stages SS-FDTD_2 method almost have not change as the propagation angle increases.

As can be seen from Figure 2(a), the NNPVE of the proposed methods are lower than other five kinds of FDTD methods. For instance, the values of the maximum NNPVE for the ADI-FDTD method, the 3-stages SS-FDTD method, 6-stages SS-FDTD method, the 4-stages SS-FDTD_1 method, the 4-stages SS-FDTD_2 method, the efficient 4-stages SS-FDTD_1 method, and the efficient 4-stages SS-FDTD_2 method are 0.8%, 0.8%, 0.50%, 0.50%, 0.47%, 0.12%, and 0.10%, respectively. On the other hand, the NNPVE of the ADI-FDTD method, the 6-stages SS-FDTD method, the 4-stages SS-FDTD_1 method, and the 4-stages SS-FDTD_2 method reaches minimum at $\phi = 45^\circ$ and maximum at $\phi = 0^\circ$, 90°. However, the NNPVE of the proposed methods reaches minimum at $\phi = 22.5^\circ$, 67.5° and maximum at $\phi = 0^\circ$, 45°, 90°. Figure 2(b) is similar to Figure 2(a) except that the NNPVE of the efficient 4-stages SS-FDTD_2 method is almost zero for all the propagation angles, which has the better numerical dispersion characteristics.

6.2. Maximum NNPVE Versus CFLN

Figure 3 presents the maximum NNPVE of seven kinds of FDTD methods versus CFLN with CPW = 40. As can be seen from Figure 3, the maximum NNPVE of seven kinds of FDTD methods increases as CFLN increases. However, the increase of the maximum NNPVE of the proposed methods is much less pronounced than other five kinds of FDTD methods. Specially, the values of the maximum NNPVE of the proposed methods are lower than other five kinds of FDTD methods. In addition, the ADI-FDTD method and 3-stages SS-FDTD method have the same value of maximum NNPVE, and the initial 4-stages SS-FDTD methods have the same value of maximum NNPVE. Moreover, the value of the maximum NNPVE of the efficient 4-stages SS-FDTD_2 method is almost zero for CFLN = 1 ∼ 10. The efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method show a significant reduction of the maximum NNPVE when compared with the initial 4-stages SS-FDTD methods at a larger time step.

Subsequently, when CFLN = 10, the values of the maximum NNPVE of the ADI-FDTD method, 3-stages SS-FDTD method, 6-stages SS-FDTD method, 4-stages SS-FDTD_1 method, and 4-stages SS-FDTD_2 method are 8.8%, 8.8%, 1.6%, 2.5%, and 2.5%, respectively. However, the values of the maximum NNPVE of the efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method are 0.5% and 0.05%, respectively, which are lower than other five kinds of FDTD methods. On the other hand, when CFLN = 5, the values of the maximum NNPVE of the 6-stages SS-FDTD method, the 4-stages SS-FDTD_1 method, and the 4-stages

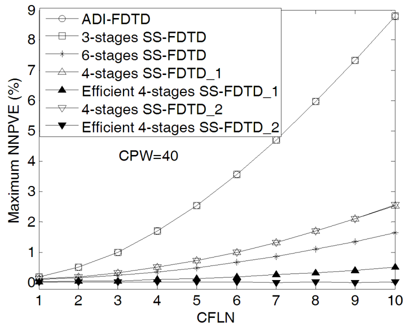


Figure 3. Maximum NNPVE versus CFLN with CPW = 40 for seven kinds of FDTD methods.

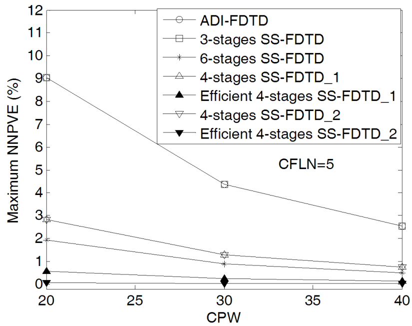


Figure 4. Maximum NNPVE versus CPW with CFLN = 5 for seven kinds of FDTD methods.

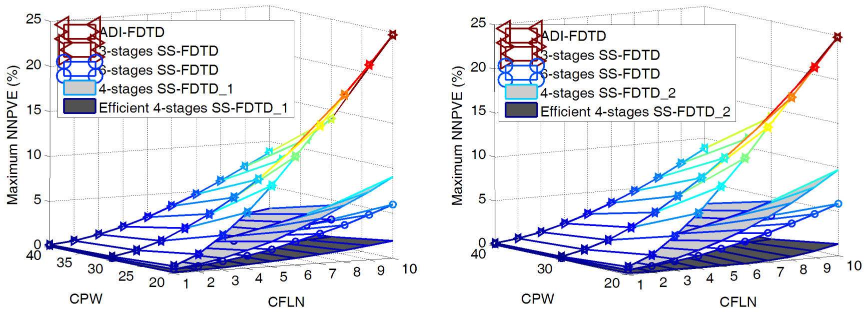
SS-FDTD_2 method are 0.5%, 0.75%, and 0.75%, respectively, which are similar to the efficient 4-stages SS-FDTD_1 method and efficient 4-stages SS-FDTD_2 method with CFLN = 10. Consequently, it is concluded that the proposed methods with the larger CFLN value lead to the better accuracy in comparisons with the 4-stages SS-FDTD_1 method and the 4-stages SS-FDTD_2 method with the smaller CFLN value. Such an improvement leads to other advantages, such as higher computational efficiency.

6.3. Maximum NNPVE Versus CPW

Figure 4 presents the maximum NNPVE of seven kinds of FDTD methods versus CPW with CFLN = 5. It can be seen that the maximum NNPVE of seven kinds of FDTD methods reduces as CPW increases. However, the reduction of the maximum NNPVE of the proposed methods is much more pronounced than other five kinds of FDTD methods.

Specially, when CPW = 20, the values of the maximum NNPVE of the ADI-FDTD method, 3-stages SS-FDTD method, 6-stages SS-FDTD method, 4-stages SS-FDTD_1 method, and 4-stages SS-FDTD_2 method are 9.0%, 9.0%, 1.9%, 2.8%, and 2.8%, respectively. However, the values of the maximum NNPVE of the efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method are 0.6% and 0.1%, respectively, which are lower than other five kinds of FDTD methods. On the other hand, when CPW = 40, the values of the maximum NNPVE of the ADI-FDTD method, 3-stages SS-FDTD method, the 6-stages SS-FDTD method, the 4-stages SS-FDTD_1 method, and the 4-stages SS-FDTD_2 method are 2.5%, 2.5%, 0.5%, 0.75%, and 0.75%, respectively. The results are also similar to the proposed methods with CPW = 20. Therefore, it can be concluded that the proposed methods with the coarser mesh lead to the same lever of accuracy compared with other five kinds of FDTD methods with the finer mesh. This improvement can reduce the memory requirement.

For completeness, the maximum NNPVE versus CPW and CFLN for seven kinds of FDTD methods is illustrated in Figure 5. As can be seen from Figure 5, for seven kinds of unconditionally-stable FDTD methods, the maximum NNPVE becomes worse while CFLN increases, whereas the maximum NNPVE becomes better as CPW increases. In addition, the best performance is reached when both CFLN and CPW are close to 1 and 40. Moreover, for the same CFLN value and CPW value, the values of the maximum NNPVE of the proposed methods are lower than other five kinds of FDTD methods. In one word, compared with other five kinds of FDTD methods, the efficient 4-stages SS-FDTD methods show better accuracy, isotropy, and can be used to model



(a) (b)

Figure 5. Maximum NNPVE versus CPW and CFLN for seven kinds of FDTD methods. (a) The 4-stages SS-FDTD_1 methods; (b) The 4-stages SS-FDTD_2 methods.

electromagnetic wave propagation on a coarser mesh or with a larger time step.

7. NUMERICAL EXPERIMENTS

In order to verify the properties of the proposed methods, the FDTD method, ADI-FDTD method, LOD-FDTD method, 3-stages SS-FDTD method, 6-stages SS-FDTD method, 4-stages SS-FDTD_1 method, efficient 4-stages SS-FDTD_1 method, 4-stages SS-FDTD_2 method, and efficient 4-stages SS-FDTD_2 method are utilized to simulate a structure of $10\,\text{cm} \times 10\,\text{cm}$ in size. Moreover, the structure is filled with air and terminated with perfect electric conducting (PEC) boundaries. Furthermore, a Gaussian pulse of $\exp[-(t-t_0)^2/T^2]$ is used as the excitation source at the centre of the structure, where $T = 0.1\,\text{ns}$, $t_0 = 3 \times T$, and the E_z component is sampled at the middle point between the source and PEC along the centre horizontal line. The mesh size is chosen as $\Delta x = \Delta y = 5\,\text{mm}$, leading to the mesh number of 20×20. The analytical resonant frequencies of TM_{11} mode and TM_{31} mode are 2.1213 GHz and 4.7434 GHz, respectively. $\Delta t_{\text{CFL\_FDTD}} = 11.793\,\text{ps}$ is the maximum time step size to satisfy the limitation of the 2D CFL condition in the conventional FDTD method. For the FDTD method, CFLN $= 1$, and the time number is 100000, and the total simulation time is selected to be 1179.3 ns. The simulations are performed on a computer of Pentium IV with 4 GB RAM, and the computer program is developed with C++.

The controlling parameters are optimized for 2.1213 GHz and

4.7434 GHz, respectively. In addition, the information on controlling parameters for CFLN = 3, 5 is shown in Table 4. In order to verify the accuracy of the proposed methods, the relative error is used: it is defined as $f - f_0/f_0 \times 100\%$, where f is the resonant frequency of TM_{11} mode or TM_{31} mode computed by the unconditionally-stable FDTD methods, and f_0 is the analytical resonant frequency of TM_{11} mode or TM_{31} mode. Moreover, Table 5 shows the comparisons of results of nine FDTD methods.

Table 4. Information on the controlling parameters.

Method	Analytical frequency (GHz)	CFLN	$C_x = C_y$
Efficient 4-stages SS-FDTD_1	f_1 (2.1213)	3	1.005429
		5	1.012355
	f_2 (4.7434)	3	1.025516
		5	1.059607
Efficient 4-stages SS-FDTD_2	f_1 (2.1213)	3	1.006344
		5	1.015024
	f_2 (4.7434)	3	1.030299
		5	1.076072

Table 5. Comparisons of results with nine FDTD methods.

Scheme	CFLN	Step number	Result (GHz) TM_{11} (2.1213)	Relative error (%)	Result (GHz) TM_{31} (4.7434)	Relative error (%)	CPU time (s)	Memory (MB)
FDTD	1	100000	2.1220	0.0330	4.7290	0.3036	71	0.018
ADI-FDTD	3	33333	2.0950	1.2398	4.3800	7.6612	27	0.024
	5	20000	2.0540	3.1726	3.9590	16.5367	16	0.024
LOD-FDTD	3	33333	2.0950	1.2398	4.3800	7.6612	26	0.024
	5	20000	2.0540	3.1726	3.9590	16.5367	14	0.024
3-stages SS-FDTD	3	33333	2.1020	0.9098	4.5970	3.0864	27	0.024
	5	20000	2.0730	2.2769	4.4220	6.7757	16	0.024
4-stages SS-FDTD_1	3	33333	2.1130	0.3913	4.6145	2.7175	28	0.024
	5	20000	2.1024	0.8910	4.4700	5.7638	17	0.024
Efficient 4-stages SS-FDTD_1	3	33333	2.1245	0.1509	4.7275	0.3352	30	0.025
	5	20000	2.1280	0.3158	4.7089	0.7273	19	0.025
4-stages SS-FDTD_2	3	33333	2.1093	0.5657	4.5997	3.0295	28	0.024
	5	20000	2.0922	1.3718	4.4290	6.6282	17	0.024
Efficient 4-stages SS-FDTD_2	3	33333	2.1226	0.0613	4.7326	0.2277	30	0.025
	5	20000	2.1228	0.0707	4.7240	0.4090	19	0.025
6-stages SS-FDTD	3	33333	2.1150	0.2970	4.6460	2.0534	29	0.024
	5	20000	2.1070	0.6741	4.5500	4.0772	18	0.024

From Table 5, for TM_{11} mode, the relative errors of the ADI-FDTD method, LOD-FDTD method, 3-stages SS-FDTD method, the 6-stages SS-FDTD method, 4-stages SS-FDTD_1 method, and 4-stages SS-FDTD_2 method with CFLN = 3 are 1.2398%, 1.2398%, 0.9098%, 0.2970%, 0.3913%, and 0.5657%, respectively. Nevertheless, the relative errors of the efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method with CFLN = 5 are 0.3158%, 0.0707%, respectively, which are lower than other six kinds of FDTD methods. On the other hand, for the TM_{31} mode, the relative errors of the ADI-FDTD method, LOD-FDTD method, 3-stages SS-FDTD method, 6-stages SS-FDTD method, 4-stages SS-FDTD_1 method, and 4-stages SS-FDTD_2 method with CFLN = 3 are 7.6612%, 7.6612%, 3.0864%, 2.0534%, 2.7175%, 3.0295%, respectively. Nevertheless, the relative errors of the efficient 4-stages SS-FDTD_1 method and the efficient 4-stages SS-FDTD_2 method with CFLN = 5 are 0.7273%, 0.4090%, respectively, which are also lower than other six kinds of FDTD methods.

In addition, the ADI-FDTD method, LOD-FDTD method, 3-stages SS-FDTD method, 6-stages SS-FDTD method, 4-stages SS-FDTD_1 method, and 4-stages SS-FDTD_2 method with CFLN = 3 require the CPU time of 27 s, 26 s, 27 s, 29 s, 28 s, 28 s, respectively. However, the efficient 4-stages SS-FDTD methods with CFLN = 5 require CPU time of 19 s. Consequently, the reduction in the CPU time of the efficient 4-stages SS-FDTD methods can be 29.6%, 26.9%, 29.6%, 34.5%, 32.1%, and 32.1% in comparisons with other six kinds of FDTD methods. The increasing in the memory requirement of the efficient 4-stages SS-FDTD methods (0.025 MB) is 4.2% in comparisons with other six kinds of FDTD methods (0.024 MB). The reason for the phenomenon is that adding the controlling parameters for the efficient 4-stages SS-FDTD methods, it is necessary for extra storage. However, the increasing of the storage is very little compared with decreasing of the relative errors. Consequently, the better accuracy and efficiency of the proposed methods are achieved.

8. CONCLUSION

Two efficient four-stages split-step unconditionally-stable FDTD methods based on the controlling parameters have been proposed, which have low numerical dispersion. In the proposed methods, the Maxwell's matrix has been split into four sub-matrices. Simultaneously, controlling parameters are added to decrease the numerical dispersion error. In addition, the dispersion relation and the process of obtaining the controlling parameters have been shown.

Moreover, the numerical dispersion characteristics of the proposed methods have also been analyzed. Specifically, the NNPVE and maximum NNPVE of the proposed methods are lower than those of the 4-stages SS-FDTD methods. Furthermore, numerical results have been presented. The relative errors of the proposed methods can be lower than those of the 4-stages SS-FDTD methods. Therefore, the better efficiency of the proposed methods has been achieved.

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