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THE SUBGRID MODELING FOR MAXWELL'S EQUATIONS WITH MULTISCALE ISOTROPIC RANDOM CONDUCTIVITY AND PERMITTIVITY

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Abstract—The effective coefficients for Maxwell's equations in the frequency domain are calculated for a multiscale isotropic medium by using a subgrid modeling approach. The correlated fields of conductivity and permeability are approximated by Kolmogorov's multiplicative continuous cascades with a lognormal probability distribution. The wavelength is assumed to be large as compared with the scale of heterogeneities of the medium. The permittivity $\varepsilon(\mathbf{x})$ and the electric conductivity $\sigma(\mathbf{x})$ satisfy the condition $\sigma(\mathbf{x})/(\omega\varepsilon(\mathbf{x})) < 1$, where ω is the cyclic frequency. The theoretical results obtained in the paper are compared with the results from direct 3D numerical simulation.

1. INTRODUCTION

Wave propagation in complex inhomogeneous media is an urgent problem in many fields of research. In electromagnetics, these problems arise in such applications as estimation of soil water content, well logging methods, etc.. In order to compute the electromagnetic fields in an arbitrary medium, one must numerically solve Maxwell's equations. The large scale variations of coefficients as compared with wavelength are taken into account in these models with the help of some boundary conditions (see, for example, [1–3]). The numerical solution of the problem with variations of parameters on all the scales require high computational costs. The small scale heterogeneities are taken into account by the effective parameters. In this case, equations are

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found on the scales that can be numerically resolved. Methods of homogenization are often applied to Maxwell's equations. In these methods it is assumed that the medium is ε-periodic in the sense that it can be viewed as the union of a collection of disjoint open identical cubes with side length ε [4]. Such assumptions are correct for composite materials. This is hardly ever observed for rock or soil. If the spatial positions of the small scale heterogeneities are exactly known, one may apply some coarser grid methods (see, for example, [5]).

It has been experimentally shown that the irregularity of electric conductivity, permeability, porosity, density abruptly increases as the scale of measurement decreases. The spatial positions of the small-scale heterogeneities are very seldom exactly known. It is customary to assume the parameters with the small scale variations to be random fields characterized by the joint probability distribution functions. In this case, the solution of the effective equations must be close to the ensemble-averaged solution of the initial problem. For such problems, a well-known procedure of subgrid modeling [6–12] is often used. To apply the subgrid modeling method, we need a "scale regular" medium [10–12]. It has been experimentally shown that many natural media are "scale regular" in the sense that their parameters, for example, permeability, porosity, density, electric conductivity can be approximated by fractals and multiplicative cascades [12–15]. The effective coefficients in the quasi-steady Maxwell's equations for a multiscale isotropic medium are described in [16]. We considered only the conductivity coefficient in this paper. In the present paper, the electric conductivity and permittivity are approximated by a multiplicative continuous cascade. We obtain formulas of effective coefficients for Maxwell's equations in the frequency domain when the following condition $\sigma(\mathbf{x})/(\omega\varepsilon(\mathbf{x})) < 1$ is satisfied. Usually, this condition is valid for ε and σ in moist soil at high frequencies.

2. GOVERNING EQUATIONS AND APPROXIMATION OF A MEDIUM

We consider Maxwell's equations in the time-harmonic form with an impressed current source $\mathbf{F}$ in a 3D-medium, which are

$$\begin{aligned} \mathrm{rot}\mathbf{H}(\mathbf{x}) &= (-i\omega\varepsilon(\mathbf{x}) + \sigma(\mathbf{x}))\,\mathbf{E}(\mathbf{x}) + \mathbf{F}, \\ \mathrm{rot}\mathbf{E} &= i\omega\mu\mathbf{H}, \end{aligned} \tag{1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the vectors of electric and magnetic field strengths respectively; $\varepsilon(\mathbf{x})$ is the permittivity, μ is the magnetic permeability; $\sigma(\mathbf{x})$ is the electric conductivity; ω is the cyclic frequency; $\mathbf{x}$ is the vector of spatial coordinates. The magnetic permeability is assumed to be equal to the magnetic permeability of vacuum. The parameters ε and σ satisfy the inequality

$$\sigma(\mathbf{x})/(\omega\varepsilon(\mathbf{x})) < 1. \tag{2}$$

At infinity, the radiation conditions must be satisfied. The wavelength is assumed to be large as compared with the maximum scale of heterogeneities of the medium L.

For the approximation of the coefficients $\sigma(\mathbf{x})$, $\varepsilon(\mathbf{x})$, we use the approach described in [17]. Let, for example, the field of permittivity be known. This means that the field is measured on a small scale l_0 at each point $\mathbf{x}$, $\varepsilon(\mathbf{x})_{l_0} = \varepsilon(\mathbf{x})$. To pass to a coarser scale grid, it is not sufficient to smooth the field $\varepsilon(\mathbf{x})_{l_0}$ on a scale l, $l > l_0$. The field thus smoothed is not a physical parameter that can describe the physical process, governed by Equation (1), on the scales $(l,\, L)$. This is due to the fact that the fluctuations of permittivity on the scale interval $(l_0,\, l)$ correlate with the fluctuations of the electric field strength $\mathbf{E}$ induced by the permittivity. To find a permittivity that can describe an ensemble-averaged physical process on the scales $(l,\, L)$, system (1) will be used. Following Kolmogorov [18], consider a dimensionless field ψ, which is equal to the ratio of two fields obtained by smoothing the field $\varepsilon(\mathbf{x})_{l_0}$ on two different scales l', l. Let $\varepsilon(\mathbf{x})_l$ denote the parameter $\varepsilon(\mathbf{x})_{l_0}$ smoothed on the scale l. Then $\psi(\mathbf{x}, l, l') = \varepsilon(\mathbf{x})_{l'}/\varepsilon(\mathbf{x})_l$, $l' < l$. Expanding the field ψ into a power series in $l - l'$ and retaining first order terms of the series, at $l' \to l$, we obtain the equation:

$$\frac{\partial \ln \varepsilon(\mathbf{x})_l}{\partial \ln l} = \chi(\mathbf{x}, l), \tag{3}$$

where $\chi(\mathbf{x}, l') = (\partial\psi(\mathbf{x}, l', l'y)/\partial y)\mid_{y=1}$. The solution of Equation (3) is

$$\varepsilon(\mathbf{x})_{l_0} = \varepsilon_0 \exp\left(-\int_{l_0}^{L} \chi(\mathbf{x}, l_1)\frac{dl_1}{l_1}\right), \tag{4}$$

where ε_0 is a constant. The field χ determines the statistical properties of the permittivity. According to the limit theorem for sums of independent random variables [19] if the variance of $\chi(\mathbf{x}, l)$ is finite, the integral in (4) tends to a field with a normal distribution as the ratio L/l_0 increases. If the variance of $\chi(\mathbf{x}, l)$ is infinite and there exists a non-degenerate limit of the integral in (4), the integral tends to a field with a stable distribution. In this paper, it is assumed that the field $\chi(\mathbf{x}, l)$ is isotropic with a normal distribution and a statistically homogeneous correlation function:

$$\langle\chi(\mathbf{x}, l)\chi(\mathbf{y}, l')\rangle - \langle\chi(\mathbf{x}, l)\rangle\langle\chi(\mathbf{y}, l')\rangle = \Phi^{\chi\chi}(|\mathbf{x}-\mathbf{y}|, l, l')\delta(\ln l - \ln l'). \tag{5}$$

Here the angle brackets denote ensemble averaging. It follows from (5) that the fluctuations of $\chi(\mathbf{x}, l)$ on different scales do not correlate. This assumption is standard in the scaling models [18]. This is due to the fact that the statistical dependence is small if the scales of fluctuations are different. To derive subgrid formulas to calculate effective coefficients, this assumption may be ignored. However, this assumption is important for the numerical simulation of the field ε. We choose the correlation function in the form

$$\Phi^{\chi\chi}\left(|\mathbf{x}-\mathbf{y}|, l, l'\right) = \Phi_0^{\chi\chi} \exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{l^2}\right] \delta\left(\ln l - \ln l'\right) \qquad (6)$$

The correlation function (6) is "base function" in some sense, because the formula $R(r) = \int\limits_0^\infty \exp(-\eta r^2) dF(\eta)$, where F — the probability distribution, is describes all statistically homogeneous continuous correlation functions [20]. Farther, we denote $\Phi^{\chi\chi}(|\mathbf{x}-\mathbf{y}|, l) = \Phi_0^{\chi\chi} \exp[-\frac{(\mathbf{x}-\mathbf{y})^2}{l^2}]$. For simplicity, we use the same notation $\Phi^{\chi\chi}$ on left-hand side.

For a scale invariant medium, the following relation holds for any positive K

$$\Phi^{\chi\chi}(|\mathbf{x}-\mathbf{y}|, l, l') = \Phi^{\chi\chi}\left(K|\mathbf{x}-\mathbf{y}|, Kl, Kl'\right).$$

In a scale invariant medium, the correlation function does not depend on the scale at $\mathbf{x} = \mathbf{y}$, and the following estimation is obtained [17]:

$$l_0 < l_\eta < r < L \quad \langle \varepsilon(\mathbf{x})_{l_0} \varepsilon(\mathbf{x}+\mathbf{r})_{l_0} \rangle \sim C\left(\frac{r}{L}\right)^{-\Phi_0^{\chi\chi}}, \qquad (7)$$

where $C = \varepsilon_0^2 (L/l_0)^{-2\langle\chi\rangle} e^{-\Phi_0^{\chi\chi}\gamma/2}$, γ is the Euler constant. For $r \gg L$, we have

$$\langle \varepsilon(\mathbf{x})_{l_0} \varepsilon(\mathbf{x}+\mathbf{r})_{l_0} \rangle \to \varepsilon_0^2. \qquad (8)$$

If for any l the equality $\langle \varepsilon(\mathbf{x})_l \rangle = \varepsilon_0$ is valid then it follows from (4), (5) that

$$\Phi_0^{\chi\chi} = 2\langle\chi\rangle. \qquad (9)$$

As the minimum scale l_0 tends to zero, the permittivity field described in (4) becomes a multifractal and we obtain an irregular field on a Cantor-type set to be nonzero.

The conductivity coefficient $\sigma(\mathbf{x})$ is constructed by analogy with the permittivity coefficient:

$$\sigma(\mathbf{x})_{l_0} = \sigma_0 \exp\left(-\int_{l_0}^{L} \varphi\left(\mathbf{x}, l_1\right) \frac{dl_1}{l_1}\right). \qquad (10)$$

The function $\varphi(\mathbf{x}, l)$ is assumed to have the normal distribution and delta-correlated in the logarithm of the scale:

$$\Phi^{\varphi\varphi}\left(\mathbf{x}, \mathbf{y}, l, l'\right) = \Phi_0^{\varphi\varphi} \exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{l^2}\right] \delta\left(\ln l - \ln l'\right). \tag{11}$$

If for any l the equality $\langle\sigma(\mathbf{x})_l\rangle = \sigma_0$ is valid then it follows from (10), (11) that

$$\Phi_0^{\varphi\varphi} = 2\langle\varphi\rangle. \tag{12}$$

The correlation between the permittivity and conductivity fields is determined by the correlation of the fields $\chi(\mathbf{x}, l')$ and $\varphi(\mathbf{x}, l')$:

$$\begin{aligned}\Phi^{\varphi\chi}\left(\mathbf{x}, \mathbf{y}, l, l'\right) &= \left\langle\varphi(\mathbf{x}, l)\chi(\mathbf{y}, l')\right\rangle - \left\langle\varphi(\mathbf{x}, l)\right\rangle\left\langle\chi(\mathbf{y}, l')\right\rangle \\ &= \Phi_0^{\varphi\chi} \exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{l^2}\right] \delta\left(\ln l - \ln l'\right).\end{aligned} \tag{13}$$

3. SUBGRID MODEL

The electric conductivity and permittivity functions $\sigma(\mathbf{x}) = \sigma(\mathbf{x})_{l_0}$, $\varepsilon(\mathbf{x}) = \varepsilon(\mathbf{x})_{l_0}$ are divided into two components with respect to the scale l. The large-scale (ongrid) components $\sigma(\mathbf{x}, l)$, $\varepsilon(\mathbf{x}, l)$ are obtained, respectively, by statistical averaging over all $\varphi(\mathbf{x}, l_1)$ and $\chi(\mathbf{x}, l_1)$ with $l_0 < l_1 < l$, $l - l_0 = dl$, where dl is small. The small-scale (subgrid) components are equal to $\sigma'(\mathbf{x}) = \sigma(\mathbf{x}) - \sigma(\mathbf{x}, l)$, $\varepsilon'(\mathbf{x}) = \varepsilon(\mathbf{x}) - \varepsilon(\mathbf{x}, l)$:

$$\begin{aligned}&\varepsilon(\mathbf{x}, l) = \varepsilon_0 \exp\left[-\int_l^L \chi(\mathbf{x}, l_1)\frac{dl_1}{l_1}\right] \times \left\langle \exp\left[-\int_{l_0}^l \chi(\mathbf{x}, l_1)\frac{dl_1}{l_1}\right]\right\rangle \\ &\varepsilon'(\mathbf{x}) = \varepsilon(\mathbf{x}, l)\left[\frac{\exp\left[-\int\limits_{l_0}^{l} \chi(\mathbf{x}, l_1)\frac{dl_1}{l_1}\right]}{\left\langle\exp\left[-\int\limits_{l_0}^{l} \chi(\mathbf{x}, l_1)\frac{dl_1}{l_1}\right]\right\rangle} - 1\right], \\ &\left\langle\varepsilon'(\mathbf{x})\right\rangle = 0.\end{aligned} \tag{14}$$

The coefficients $\sigma(\mathbf{x}, l)$, $\sigma'(\mathbf{x})$ are calculated the same way. Hence

$$\begin{aligned}\varepsilon(\mathbf{x}, l) &\simeq \left[1 - \langle\chi\rangle\frac{dl}{l} + \frac{1}{2}\Phi_0^{\chi\chi}\frac{dl}{l}\right]\varepsilon(\mathbf{x})_l, \\ \sigma(\mathbf{x}, l) &\simeq \left[1 - \langle\varphi\rangle\frac{dl}{l} + \frac{1}{2}\Phi_0^{\varphi\varphi}\frac{dl}{l}\right]\sigma(\mathbf{x})_l.\end{aligned} \tag{15}$$

The large scale (ongrid) components of electric and magnetic field strengths $\mathbf{E}(\mathbf{x}, l)$, $\mathbf{H}(\mathbf{x}, l)$ are obtained by averaging the solutions to system (1), in which the large scale components of conductivity $\sigma(\mathbf{x}, l)$ and permittivity $\varepsilon(\mathbf{x}, l)$ are fixed and the small components $\sigma'(\mathbf{x})$, $\varepsilon'(\mathbf{x})$ are random variables. The subgrid components of the electric and magnetic field strengths are equal to $\mathbf{H}'(\mathbf{x}) = \mathbf{H}(\mathbf{x}) - \mathbf{H}(\mathbf{x}, l)$, $\mathbf{E}'(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - \mathbf{E}(\mathbf{x}, l)$. Substituting the relations for $\mathbf{E}(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$ and $\sigma(\mathbf{x})$, $\varepsilon(\mathbf{x})$ into system (1) and averaging over small scale components, we have

$$\begin{aligned} &\mathrm{rot}\mathbf{H}\left(\mathbf{x}, l\right) = \left(-i\omega\varepsilon\left(\mathbf{x}, l\right) + \sigma\left(\mathbf{x}, l\right)\right)\mathbf{E}\left(\mathbf{x}, l\right) + \left\langle\left(-i\omega\varepsilon' + \sigma'\right)\mathbf{E}'\right\rangle + \mathbf{F}, \\ &\mathrm{rot}\mathbf{E}\left(\mathbf{x}, l\right) = \mu i\omega\mathbf{H}\left(\mathbf{x}, l\right). \end{aligned} \tag{16}$$

The subgrid term $\langle(-i\omega\varepsilon' + \sigma')\mathbf{E}'\rangle$ in system (16) is unknown. This term cannot be neglected without preliminary estimation, since the correlation between the field $-i\omega\varepsilon' + \sigma'$ and the electric field strength $\mathbf{E}'$ may be significant. The form of this term in (16) determines a subgrid model. The subgrid term is estimated using the perturbation theory. Subtracting system (16) from system (1) and taking into account only the first order terms, we obtain the subgrid equations:

$$\begin{aligned} &\mathrm{rot}\mathbf{H}'(\mathbf{x}) = (-i\omega\varepsilon(\mathbf{x}, l) + \sigma(\mathbf{x}, l))\mathbf{E}'(\mathbf{x}) + \left(-i\omega\varepsilon'(\mathbf{x}) + \sigma'(\mathbf{x})\right)\mathbf{E}(\mathbf{x}, l), \\ &\mathrm{rot}\mathbf{E}'\left(\mathbf{x}\right) = \mu i\omega\mathbf{H}'\left(\mathbf{x}\right). \end{aligned} \tag{17}$$

The variable $\mathbf{E}(\mathbf{x}, l)$ on the right-hand side of (17) is assumed to be known. Using "frozen-coefficients" method, as a first approximation we can write down solution of system (17) for the components of the electric field strength [21]:

$$\begin{aligned} E'_\alpha \approx\ & \Omega \int_{-\infty}^{\infty} \frac{e^{ikr}}{r}\left(-i\omega\varepsilon'\left(\mathbf{x}'\right) + \sigma'\left(\mathbf{x}'\right)\right) E_\alpha\left(\mathbf{x}', l\right) d\mathbf{x}' \\ & + \Omega_1 \int_{-\infty}^{\infty} \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\frac{e^{ikr}}{r}\left(-i\omega\varepsilon'\left(\mathbf{x}'\right) + \sigma'\left(\mathbf{x}'\right)\right) E_\beta\left(\mathbf{x}', l\right) d\mathbf{x}', \end{aligned} \tag{18}$$

where $\Omega = i\omega\mu/(4\pi)$, $\Omega_1 = 1/(4\pi(-i\omega\varepsilon(\mathbf{x}, l) + \sigma(\mathbf{x}, l)))$, $r = |\mathbf{x} - \mathbf{x}'|$, $k^2 = \omega\mu(\omega\varepsilon(\mathbf{x}, l) + i\sigma(\mathbf{x}, l))$. Here the summation of repeated indices is implied. We take the square root such that $\mathrm{Re}k > 0$, $\mathrm{Im}k > 0$. Using (18) and the formula $\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\frac{1}{r}e^{ikr} = \frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\frac{1}{r}e^{ikr}$, the subgrid term can be written down as

$$\begin{aligned} &\left\langle\left(-i\omega\varepsilon'\left(\mathbf{x}\right) + \sigma'\left(\mathbf{x}\right)\right) E'_\alpha\left(\mathbf{x}\right)\right\rangle \approx \Omega \int_{-\infty}^{\infty} \frac{1}{r} e^{ikr} \\ &\times \left\langle\left(-i\omega\varepsilon'\left(\mathbf{x}\right) + \sigma'\left(\mathbf{x}\right)\right)\left(-i\omega\varepsilon'\left(\mathbf{x}'\right) + \sigma'\left(\mathbf{x}'\right)\right)\right\rangle E_\alpha\left(\mathbf{x}', l\right) d\mathbf{x}' \end{aligned}$$

$$+\Omega_1 \int_{-\infty}^{\infty} \frac{\partial}{\partial x'_\alpha} \frac{\partial}{\partial x'_\beta} \frac{1}{r} e^{ikr} \times \left\langle \left(-i\omega\varepsilon'(\mathbf{x}) + \sigma'(\mathbf{x})\right)\right.$$
$$\left.\left(-i\omega\varepsilon'\left(\mathbf{x}'\right) + \sigma'\left(\mathbf{x}'\right)\right)\right\rangle E_\beta\left(\mathbf{x}', l\right) d\mathbf{x}'. \tag{19}$$

As follows from formulas (7), (13), (14) for a lognormal probability distribution of σ and ε at small dl, we have

$$\begin{aligned}
&\left\langle \sigma'(\mathbf{x})\, \sigma'\left(\mathbf{x}'\right)\right\rangle \approx \sigma^2(\mathbf{x}, l)\Phi^{\varphi\varphi}(r, l)\, \frac{dl}{l} \\
&\left\langle \varepsilon'(\mathbf{x})\, \varepsilon'\left(\mathbf{x}'\right)\right\rangle \approx \varepsilon^2(\mathbf{x}, l)\Phi^{\chi\chi}(r, l)\, \frac{dl}{l} \\
&\left\langle \sigma'(\mathbf{x})\, \varepsilon'\left(\mathbf{x}'\right)\right\rangle \approx \varepsilon(\mathbf{x}, l)\sigma(\mathbf{x}, l)\Phi^{\chi\varphi}(r, l)\, \frac{dl}{l} \\
&\left\langle \left(-i\omega\varepsilon'(\mathbf{x}) + \sigma'(\mathbf{x})\right)\left(-i\omega\varepsilon'\left(\mathbf{x}'\right) + \sigma'\left(\mathbf{x}'\right)\right)\right\rangle \\
&\approx -\omega^2\varepsilon^2(\mathbf{x}, l)\left[\Phi^{\chi\chi}(r, l) - 2i\nu\Phi^{\chi\varphi}(r, l) - \nu^2\Phi^{\varphi\varphi}(r, l)\right]\frac{dl}{l},
\end{aligned} \tag{20}$$

where $\nu = \sigma(\mathbf{x}, l)/(\omega\varepsilon(\mathbf{x}, l))$. The wavelength is assumed to be large as compared with the maximum scale of heterogeneities of the medium L, $l < L$. Taking into account formulas (6), (13) correlation radiuses are approximately equal to l. The components $E_j(\mathbf{x}', l)$ are changing slightly in the domain $|\mathbf{x} - \mathbf{x}'| < l$. So, these components can be factored outside the integral sign in (19). Using (20) the first integral in (19) can be written down as

$$\begin{aligned}
I_1 =& \frac{i\omega\mu}{4\pi} \int_{-\infty}^{\infty} \frac{e^{ikr}}{r} \left\langle \left(-i\omega\varepsilon'(\mathbf{x}) + \sigma'(\mathbf{x})\right)\left(-i\omega\varepsilon'\left(\mathbf{x}'\right) + \sigma'\left(\mathbf{x}'\right)\right)\right\rangle d\mathbf{x}' E_\alpha(\mathbf{x}, l) \\
\approx& -\frac{1}{4\pi}\mu\omega^2\varepsilon(\mathbf{x}, l) \int_{-\infty}^{\infty} \frac{1}{r} e^{ikr}\Phi^{\chi\chi}(r, l)\, d\mathbf{x}' \frac{dl}{l} i\omega\varepsilon(\mathbf{x}, l)E_\alpha(\mathbf{x}, l) \\
&- \frac{1}{2\pi} i\omega\mu\sigma(\mathbf{x}, l) \int_{-\infty}^{\infty} \frac{1}{r} e^{ikr}\Phi^{\chi\varphi}(r, l)\, d\mathbf{x}' \frac{dl}{l} i\omega\varepsilon(\mathbf{x}, l)E_\alpha(\mathbf{x}, l) \\
&+ \frac{1}{4\pi} i\omega\mu\sigma(\mathbf{x}, l) \int_{-\infty}^{\infty} \frac{1}{r} e^{ikr}\Phi^{\varphi\varphi}(r, l)\, d\mathbf{x}' \frac{dl}{l} \sigma(\mathbf{x}, l)E_\alpha(\mathbf{x}, l).
\end{aligned} \tag{21}$$

Changing the Cartesian coordinates for spherical coordinates in (21) we have

$$I_1 \approx -\mu\omega^2\varepsilon(\mathbf{x},l)\int_0^\infty re^{ikr}\Phi^{\chi\chi}(r,l)\,d\mathbf{x}'\frac{dl}{l}i\omega\varepsilon(\mathbf{x},l)E_\alpha(\mathbf{x},l)$$
$$+2\mu\omega^2\varepsilon(\mathbf{x},l)\int_0^\infty re^{ikr}\Phi^{\chi\varphi}(r,l)\,d\mathbf{x}'\frac{dl}{l}\sigma(\mathbf{x},l)E_\alpha(\mathbf{x},l)$$
$$+i\omega\mu\sigma(\mathbf{x},l)\int_0^\infty re^{ikr}\Phi^{\varphi\varphi}(r,l)\,d\mathbf{x}'\frac{dl}{l}\sigma(\mathbf{x},l)E_\alpha(\mathbf{x},l)\,. \qquad (22)$$

Using the formula $\frac{1}{[4\pi(-i\omega\varepsilon(\mathbf{x},l)+\sigma(\mathbf{x},l))]} \approx -\frac{1}{4\pi i\omega\varepsilon(\mathbf{x},l)}(1-\frac{i\sigma(\mathbf{x},l)}{\omega\varepsilon(\mathbf{x},l)})$ and formula (20), neglecting the terms of second order of smallness of $\sigma(\mathbf{x},l)/\omega\varepsilon(\mathbf{x},l)$ we evaluate the second integral in (19):

$$I_2 \approx -\frac{i\omega\varepsilon(\mathbf{x},l)}{4\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\left(\frac{1}{r}e^{ikr}\right)\left(1-\frac{i\sigma(\mathbf{x},l)}{\omega\varepsilon(\mathbf{x},l)}\right)$$
$$\times\left(\Phi^{\chi\chi}(r,l)+2i\frac{\sigma(\mathbf{x},l)}{\omega\varepsilon(\mathbf{x},l)}\Phi^{\chi\varphi}(r,l)-\frac{\sigma^2(\mathbf{x},l)\Phi^{\varphi\varphi}(r,l)}{\omega^2\varepsilon^2(\mathbf{x},l)}\right)d\mathbf{x}'\frac{dl}{l}E_\beta(\mathbf{x},l)$$
$$=-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\left(\frac{1}{r}e^{ikr}\right)\Phi^{\chi\chi}(r,l)\,d\mathbf{x}'\frac{dl}{l}i\omega\varepsilon(\mathbf{x},l)E_\beta(\mathbf{x},l)$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\left(\frac{1}{r}e^{ikr}\right)\Phi^{\chi\varphi}(r,l)\,d\mathbf{x}'\frac{dl}{l}\sigma(\mathbf{x},l)E_\beta(\mathbf{x},l)$$
$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\left(\frac{1}{r}e^{ikr}\right)\Phi^{\chi\chi}(r,l)\,d\mathbf{x}'\frac{dl}{l}\sigma(\mathbf{x},l)\,E_\beta(\mathbf{x},l)\,. \qquad (23)$$

In formula (23), the Cartesian coordinates are changed for spherical coordinates. Integrating $n_j n_m$, where $n_m = x_m/r$, over the complete solid angle, we arrive at the formula $\int n_j n_m d\vartheta = \frac{4\pi}{3}\delta_{jm}$. Using this formula and integrating (23) by parts, we arrive at

$$I_2 \approx \frac{1}{3}\left(\Phi_0^{\chi\chi}+k^2\int_0^\infty re^{ikr}\Phi^{\chi\chi}(r,l)\,dr\right)\frac{dl}{l}i\omega\varepsilon(\mathbf{x},l)E_\alpha(\mathbf{x},l)$$

$$-\frac{2}{3}\left(\Phi_0^{\varphi\chi}+k^2\int\limits_0^\infty re^{ikr}\Phi^{\chi\varphi}\left(r,l\right)dr\right)\frac{dl}{l}\sigma(\mathbf{x},l)E_\alpha\left(\mathbf{x},l\right)$$
$$+\frac{1}{3}\left(\Phi_0^{\chi\chi}+k^2\int\limits_0^\infty re^{ikr}\Phi^{\chi\chi}\left(r,l\right)dr\right)\frac{dl}{l}\sigma\left(\mathbf{x},l\right)E_\alpha\left(\mathbf{x},l\right). \qquad (24)$$

To evaluate the integrals in (22), (24) we consider the correlation function (6): We obtain

$$\int\limits_0^\infty re^{ikr}\exp\left(-\frac{r^2}{l^2}\right)dr = l^2\times\left\{\frac{\sqrt{\pi}l}{4}\left[\cos\left(\frac{ik^2l^2}{4}\right)\left(1-erf\left(-\frac{ikl}{2}\right)\right)\right.\right.$$
$$\left.\left.+\sin\left(\frac{ik^2l^2}{4}\right)\left(1-erf\left(\frac{kl}{2}\right)\right)\right]+\frac{1}{2}\right\}$$

The term in braces is bounded. This integral has l^2-order of smallness. So if $\omega\mu L^2|(i\omega\varepsilon(\mathbf{x},l)+\sigma(\mathbf{x},l))| \ll 1$, the integrals in (22), (24) are small. This inequality is not restrictive for the problems of electromagnetic waves if L is much smaller than the wavelength. Hence, the integrals in (24) can be neglected. Hence estimation of the subgrid term in (16) is equal to

$$\left\langle -i\omega\varepsilon'(\mathbf{x})E'_\alpha(\mathbf{x})\right\rangle+\left\langle\sigma'(\mathbf{x})E'_\alpha(\mathbf{x})\right\rangle\approx-\frac{1}{3}\Phi_0^{\chi\chi}\left(-i\omega\varepsilon(\mathbf{x},l)E_\alpha(\mathbf{x},l)\right)\frac{dl}{l}$$
$$-\left(\frac{2}{3}\Phi_0^{\chi\varphi}-\frac{1}{3}\Phi_0^{\chi\chi}\right)\frac{dl}{l}\sigma(\mathbf{x},l)E_\alpha\left(\mathbf{x},l\right). \qquad (25)$$

Substituting (25) into (16), we have:

$$\begin{aligned}\mathrm{rot}\mathbf{H}\left(\mathbf{x},l\right) &= -i\omega\varepsilon_{l0}\exp\left[-\int_l^L\chi(\mathbf{x},l_1)\frac{dl_1}{l_1}\right]\mathbf{E}\left(\mathbf{x},l\right)\\ &\quad+\sigma_{l0}\exp\left[-\int_l^L\varphi(\mathbf{x},l_1)\frac{dl_1}{l_1}\right]\mathbf{E}\left(\mathbf{x},l\right),\\ \mathrm{rot}\mathbf{E}\left(\mathbf{x},l\right) &= i\omega\mu\mathbf{H}\left(\mathbf{x},l\right), \qquad (26)\\ \varepsilon_{l0} &= \left(1-\frac{\Phi_0^{\chi\chi}}{3}\frac{dl}{l}\right)\left[1+\left(\frac{\Phi_0^{\chi\chi}}{2}-\langle\chi\rangle\right)\frac{dl}{l}\right]\varepsilon_0,\\ \sigma_{l0} &= \left[1-\left(\frac{2}{3}\Phi_0^{\chi\varphi}-\frac{1}{3}\Phi_0^{\chi\chi}\right)\frac{dl}{l}\right]\times\left[1+\left(\frac{\Phi_0^{\varphi\varphi}}{2}-\langle\varphi\rangle\right)\frac{dl}{l}\right]\sigma_0.\end{aligned}$$

It follows from (26) that the new coefficients σ_{l0} and ε_{l0} with second order of accuracy in dl/l are equal to:

$$\varepsilon_{l0} = \varepsilon_0 + \left(\frac{\Phi_0^{\chi\chi}}{6} - \langle\chi\rangle\right)\varepsilon_0\frac{dl}{l},$$
$$\sigma_{l0} = \sigma_0 + \left(-\frac{2}{3}\Phi_0^{\chi\varphi} + \frac{1}{3}\Phi_0^{\chi\chi} + \frac{1}{2}\Phi_0^{\varphi\varphi} - \langle\varphi\rangle\right)\sigma_0\frac{dl}{l}.$$

As $dl \to 0$, we obtain the equation

$$\begin{aligned}\frac{d\ln\varepsilon_{0l}}{d\ln l} &= \frac{1}{6}\Phi_0^{\chi\chi} - \langle\chi\rangle\,,\\ \frac{d\ln\sigma_{0l}}{d\ln l} &= -\frac{2}{3}\Phi_0^{\chi\varphi} + \frac{1}{3}\Phi_0^{\chi\chi} + \frac{1}{2}\Phi_0^{\varphi\varphi} - \langle\varphi\rangle\,.\end{aligned} \tag{27}$$

The effective equations have the following form:

$$\begin{aligned}\mathrm{rot}\mathbf{H}(\mathbf{x},l) =& -i\omega\left(\frac{l}{L}\right)^{\langle\chi\rangle-\Phi_0^{\chi\chi}/6}\varepsilon_l(\mathbf{x})\,\mathbf{E}(\mathbf{x},l)\\ &+\left(\frac{l}{L}\right)^{\langle\varphi\rangle+\frac{2}{3}\Phi_0^{\chi\varphi}-\frac{1}{3}\Phi_0^{\chi\chi}-\frac{1}{2}\Phi_0^{\varphi\varphi}}\sigma_l(\mathbf{x})\,\mathbf{E}(\mathbf{x},l)+\mathbf{F},\\ \mathrm{rot}\mathbf{E}(\mathbf{x},l) =& \, i\omega\mu\mathbf{H}(\mathbf{x},l)\,.\end{aligned} \tag{28}$$

4. NUMERICAL SIMULATION

The following numerical problem was solved in order to verify the formulas obtained above. Equation (1) is solved in a cube with edge L_0. The following dimensionless variables are used: $\hat{\mathbf{x}} = \mathbf{x}/L_0$, $\hat{\sigma} = \sigma/\sigma_0$, $\hat{\mathbf{H}} = \mathbf{H}/H_0$, $\hat{\mathbf{E}} = \frac{L_0\sigma_0}{k_1 H_0}\mathbf{E}$, $k_1 = L_0\sqrt{\sigma_0\mu\omega}$, $k = k_1\sqrt{\hat{\sigma} - i\kappa\hat{\varepsilon}}$, $\kappa = \frac{\omega\varepsilon_0}{\sigma_0}$. In the calculation, the parameter κ is equal to 5. This corresponds to $\frac{\sigma_0}{\omega\varepsilon_0} = 0.2$. Thus, the problem is solved in a unit cube, with $\sigma_0 = 1$, $\varepsilon_0 = 1$, $k_1 = 4\sqrt{2}$. Equation (1) in the dimensionless form is written as

$$\begin{aligned}\mathrm{rot}\hat{\mathbf{H}} &= (\hat{\sigma} - i\kappa\hat{\varepsilon})\,k_1\hat{\mathbf{E}} + \mathbf{F}\\ \mathrm{rot}\hat{\mathbf{E}} &= ik_1\hat{\mathbf{H}}.\end{aligned} \tag{29}$$

The domain of integration is divided into three parts. To satisfy the radiation conditions at infinity, the perfectly matched layers [22] are located in the domain $0 \le \hat{x}_i < 0.1$, $1.5 < \hat{x}_i \le 1.6$. Therefore, a change of variables is used $s_{x_i}(\hat{x}_i) = \frac{d\xi_i}{d\hat{x}_i} = 1 + i\eta(\hat{x}_i)$, which gives an exponential decrease of waves into the perfectly matched layers.

According to [22], we choose $\eta(\hat{x}_i)$ as the ramp-like function:

$$\eta\left(\hat{x}_i\right)=\begin{cases}3.5\left(\frac{0.1-\hat{x}_i}{0.1}\right)^2, & 0\le\hat{x}_i<0.1,\\ 0, & 0.1\le\hat{x}_i<1.5,\\ 3.5\left(\frac{1.5-\hat{x}_i}{0.1}\right)^2, & 1.5\le\hat{x}_i<1.6.\end{cases}$$

In the domain $0.1 \le \hat{x}_i < 0.3$, $1.3 \le \hat{x}_i < 1.5$, the coefficients $\hat{\varepsilon}\hat{\sigma}$ are equal to 1. The current source $F_{\hat{x}_1} = 0$, $F_{\hat{x}_2} = 0$, $F_{\hat{x}_3} = 0.5\exp\left(-q^2(\hat{x}_3-0.2)^2\right)$, $q = 60$ is located at the point (0, 0, 0.2). In the domain $0.3 \le \hat{x}_i < 1.3$, the conductivity and the permittivity are simulated by multiplicative cascades. The integrals in (4), (10) are approximated by finite difference formulas. A $256 \times 256 \times 256$ grid is used for the spatial variables in the domain $0.3 \le \hat{x}_i < 1.3$. In these formulas, it is convenient to pass to a logarithm to base 2:

$$\begin{aligned}\sigma\left(\hat{\mathbf{x}}\right)_{l_0} &\approx 2^{-\sum\limits_{i=-8}^{0}\varphi(\hat{\mathbf{x}},\tau_i)\Delta\tau},\\ \varepsilon\left(\hat{\mathbf{x}}\right)_{l_0} &\approx 2^{-\sum\limits_{i=-8}^{0}\chi(\hat{\mathbf{x}},\tau_i)\Delta\tau},\end{aligned}\tag{30}$$

where $\langle\sigma(\hat{\mathbf{x}})_{l_0}\rangle = 1$, $\langle\varepsilon(\hat{\mathbf{x}})_{l_0}\rangle = 1$, $l = 2^\tau$, $\Delta\tau$ is the τ grid-size. In our calculations $\Delta\tau$ is taken to be one. For the random fields φ, χ, the following formulas are used for each τ_i:

$$\begin{aligned}\varphi(\hat{\mathbf{x}},\tau_k) &= \sqrt{\frac{\Phi_0^{\varphi\varphi}}{\ln 2}}\zeta_1(\hat{\mathbf{x}},\tau_k)+\frac{\Phi_0^{\varphi\varphi}}{2},\\ \chi(\hat{\mathbf{x}},\tau_k) &= \sqrt{\frac{\Phi_0^{\chi\chi}}{\ln 2}}\left(\rho\zeta_1(\hat{\mathbf{x}},\tau_k)+\sqrt{1-\rho^2}\zeta_2(\hat{\mathbf{x}},\tau_k)\right)+\frac{\Phi_0^{\chi\chi}}{2},\end{aligned}$$

where $\Phi_0^{\varphi\chi} = \rho\sqrt{\Phi_0^{\chi\chi}\Phi_0^{\varphi\varphi}}$, $-1 \le \rho \le 1$, ζ_1, ζ_2 are independent Gaussian random fields with unit variance, zero mean and the following correlation function

$$\langle\zeta_1(\hat{\mathbf{x}},\tau_i)\zeta_1(\hat{\mathbf{y}},\tau_j)\rangle = \langle\zeta_2(\hat{\mathbf{x}},\tau_i)\zeta_2(\hat{\mathbf{y}},\tau_j)\rangle = \exp\left[-\left(\mathbf{x}-\mathbf{y}\right)^2/2^{2\tau_i}\delta_{ij}\right].$$

The coefficients $\Phi_0^{\varphi\varphi} = 2\langle\varphi\rangle$, $\Phi_0^{\chi\chi} = 2\langle\chi\rangle$ are constants that should be taken from experimental data. In [13] approximate values for Φ_0 are given for some natural media. To numerically simulate the Gaussian fields ζ_1, ζ_2, we use the algorithm "along rows and columns" [23]. The delta-correlation in the scale logarithm means that the fields φ, χ are independently generated for each scale l_i. The number of terms in (30) is chosen so that probabilistic averaging can be replaced by volume averaging on the largest fluctuation scale. The smallest fluctuations scale is chosen in such a way as to approximate (29) by a difference scheme with good accuracy on the all scales. The fields in the exponents of (30) are generated as the sum of two scales: $i = -5$, -4. The minimum scale is $l_0 = 1/32$, the maximum scale is $L = 1/16$. Figure 1 shows the conductivity field for these two scales at a cross-section $x_3 = 1/2$ calculated by formula (30).

To numerically solve system (29), a method based on a finite difference scheme proposed in [24] and a decomposition method from [25] are used. We use a square $412 \times 412 \times 412$ grid with a constant spacing. The cube is divided into two subdomains, P and R. The subdomain P contains the points with even sums of indices (three even numbers or one even number and two odd ones). The subdomain R contains the remaining points. Denote the functions defined on P and R by the upper indices P and R, respectively. Then both subdomains P and R are divided into four independent subdomains in which Equation (29) is solved independently. The derivatives with respect to x_1 are approximated by

$$\begin{aligned} \left(f_{x_1}^P\right) &= \frac{f_{m+1,n,j}^R - f_{m-1,n,j}^R}{h}, \\ \left(f_{x_1}^R\right) &= \frac{f_{m+1,n,j}^P - f_{m-1,n,j}^P}{h}. \end{aligned} \tag{31}$$

The derivatives with respect to x_2 and x_3 are approximated in a similar way. For each grid point, twelve values are calculated: **H** and **E** vectors, each having three real and three imaginary components. The problem is solved eight times on a $206 \times 206 \times 206$ grid. Then we combine the complete solution. To solve the linear systems obtained

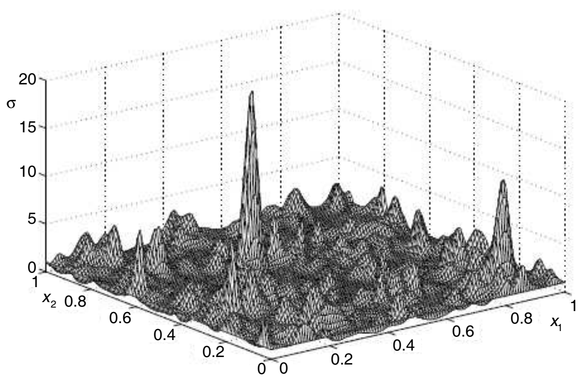

Figure 1. Electric conductivity field at cross-section $x_3 = 1/2$ calculated by (30) on two scales $i = -5$, -4.

by discretization in the grid subdomains, an iterative method for a self-adjoint operator with an arbitrary spectrum, SYMMLQ, is used. The algorithm is described in [26].

The theoretical formulas obtained in the previous section can be verified by solving the above initial problem and performing probabilistic averaging over the small-scale fluctuations. The solution thus obtained can be compared with the solution of effective Equation (28). However a different procedure is used in the present paper. The mean characteristics of the current density and the electric and magnetic field strengths are calculated on the scales $(l_0,\, L)$. At each x_3, these fields are averaged over the planes $(x_1,\, x_2)$. Then these fields are additionally averaged over the Gibbs ensemble. Equation (29) is solved 48 times. The fields thus obtained are compared with the solution to effective Equation (28). In the calculations we use:

$$\Phi_0^{\varphi\varphi} = \Phi_0^{\chi\chi} = 0.4,\ \langle\varphi\rangle = \langle\chi\rangle = 0.2,\ \rho = 1 \text{ or } \rho = -1,$$
$$\Phi_0^{\varphi\chi} = \frac{\rho}{\ln 2}\sqrt{\Phi_0^{\varphi\varphi}\Phi_0^{\chi\chi}}.$$

Figures 2–3 show a comparison between the mean fields obtained by the numerical method described above, the effective fields obtained by Equation (28) and by the fields obtained by Equation (29) with the coefficients $\sigma = \langle\sigma(\mathbf{x})\rangle = 1$, $\varepsilon = \langle\varepsilon(\mathbf{x})\rangle = 1$. The deviations between curves 4, 5 (numerical results) in Figures 2, 3 and curve 1 (with the coefficients $\sigma = \langle\sigma(\mathbf{x})\rangle = 1$, $\varepsilon = \langle\varepsilon(\mathbf{x})\rangle = 1$) depend on coefficients $\Phi_0^{\chi\chi}$, $\Phi_0^{\varphi\varphi}$, $\Phi_0^{\varphi\varphi}$ in correlation functions. In theory of steady filtration such approach gives a good accuracy for estimating the mean

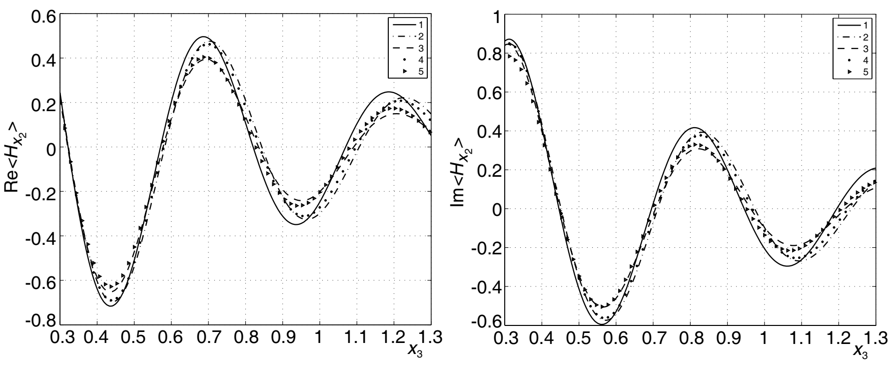

Figure 2. Real and imaginary parts of H_{x_2} obtained by: 1 — system (29) at $\sigma = 1$, $\varepsilon = 1$; 2 — effective system at $\rho = 1$; 3 — effective system with $\rho = -1$; 4 — numerical method at $\rho = 1$; 5 — numerical method at $\rho = -1$.

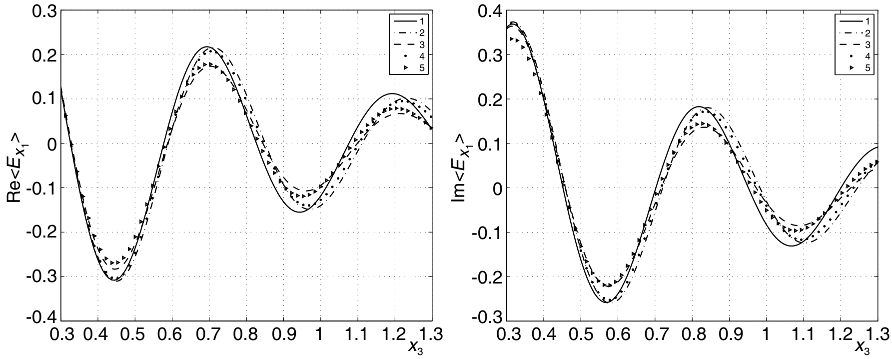

Figure 3. Real and imaginary parts of E_{x_1} obtained by: 1 — system (29) at $\sigma = 1$, $\varepsilon = 1$; 2 — effective system at $\rho = 1$; 3 — effective system with $\rho = -1$; 4 — numerical method at $\rho = 1$; 5 — numerical method at $\rho = -1$.

filtration velocity at $D < 2$, where D is the variance of the logarithm of hydraulic conductivity [9]. In our problem the numerical method used at $N\Phi_0^{\chi\chi} \geq 1.2$, where N is the number of of scales, does not converge. So, we have not carried out computations for coefficients greater than 0.4. Although curves 4, 5 in Figures 2–3 small differ in magnitude from curve 1, curves 4, 5 decay faster than curve 1 for one wavelength. Such deviations will have an influence over a distance containing many wavelengths.

5. CONCLUSIONS

We have presented the effective coefficients for Maxwell's equations if parameters in equations are described by extremely irregular fields that are close to continuous multifractals. The continuous multifractals can be obtained if the minimum scale l_0 in formulas (4), (10) tends to zero. To approximate the medium, we have started from the modified Kolmogorov theory in terms of ratios of smoothed fields. As the minimum scale is finite, then any singularities are absent, therefore we use only the theory of differential equations and the theory of stochastic processes. For a scale-invariant medium, the effective coefficients are power functions of the smoothing scale. The exponents of these functions have been calculated. The results of numerical testing have shown that the approach proposed to estimate the impact of small-scale medium heterogeneities on the means E, H at $\omega\mu L^2|(i\omega\varepsilon(\mathbf{x}, l) + \sigma(\mathbf{x}, l))| < 1$, $\sigma(\mathbf{x})/(\omega\varepsilon(\mathbf{x})) < 1$ is effective. If the dielectric permittivity and the conductivity depend on frequency and the above inequalities are satisfied, the effective coefficients also satisfy Equation (27). For some problems of mathematical modeling electromagnetic wave in moisture soil at the high-frequencies we have $\sigma(\mathbf{x})/(\omega\varepsilon(\mathbf{x})) < 0.1$. In this case the coefficient of electric conductivity in Maxwell's equations may be ignored and the effective coefficient for permittivity can be calculated using the first equation in (27).

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