Progress In Electromagnetics Research B, Vol. 62, 29–47, 2015

Scattered Field in Random Dielectric Inhomogeneous Media: A Random Resolvent Approach

Ellaheh Barzegar[1, *], Stef van Eijndhoven[2], and Martijn van Beurden[1]

Abstract—In modeling electromagnetic phenomena randomness of the propagation medium and of the dielectric object should be taken up. The usually applied Monte-Carlo based methods reveal true characteristics of the random electromagnetic field at the expense of large computation time and computer memory. Use of expansion based methods and their resulting algorithm is an efficient alternative. In this paper the focus is on characteristics of electromagnetic fields that satisfy integral equations where the integral kernel has a random component, typically, electromagnetic fields that describe scattering due to dielectric objects with an inhomogeneous random contrast field. The assumption is that the contrast is affinely related to a random variable. The integral equation is of second kind Fredholm type so that its solutions are determined by the resolvent, a random operator field. The key idea is to expand that operator field with respect to orthogonal polynomials defined by the probability measure on the underlying sample space and to derive the properties of the solution from that expansion. Two types of illustration are presented: an inhomogeneous dielectric slab and a 2D dielectric grating with 1D periodicity.

1. INTRODUCTION

In many engineering applications, natural variations and randomness of dielectric structures and random inhomogeneous media need to be taken up in electromagnetic (EM) models. When exposed to an incident EM wave, such structures and media cause random scattering. For a one-dimensional medium the scattering effect is quantified by random reflection and transmission coefficients. For multidimensional scattering the random power scattered by the object is a quantifying concept. In the context of applications, the two main questions are: (1) Given a measured reflected or transmitted EM-wave, what are the dielectric properties of the random structure? (2) Assuming a type of randomness of the dielectric object, what random properties does the scattered EM-field have? As applications addressing questions (1) we mention the references [1] for soil layer analysis, and [2] for human tissue analysis; as references addressing question (2) we mention the research papers [3–5] on wave propagation in random media.

Monte-Carlo method [6, 7], perturbation analysis [8, 9], maximum-entropy [10, 11], and polynomial chaos [12, 13] are commonly used to estimate random characteristics such as average field and the Karhunen-Loève uncorrelated statistics. These methods are used in the analysis of EM wave propagation in media with random inhomogeneities like the atmosphere [4, 8], in random dielectric structures such as human tissue [14, 15], in layered soil [1, 16], and in dielectric silicon layers [17, 18].

The focus of this paper is on methods to derive characteristics of random electric fields, so that questions of type (1) and (2) can be addressed. The fields are generated by an incident electromagnetic plane wave that is scattered by an inhomogeneous dielectric structure with random permittivity. The scattered field can be obtained from the electric field inside the dielectric object that satisfies a Fredholm

Received 13 November 2014, Accepted 11 January 2015, Scheduled 19 January 2015
* Corresponding author: Ellaheh Barzegar (e.barzegar@tue.nl).
[1] Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands. [2] Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands.

integral equation of the second kind. The permittivity is determined by a random contrast field defined on a one-dimensional sample space such that the spatial variable and the random variable are separated. The solution of the Fredholm integral equation is determined by the random resolvent [19] of the related integral operator. We show how the resolvent determines the characteristics of the random electric field inside the dielectric object and the one scattered by that object. We introduce three methods based on polynomial expansion and the spectral properties of the integral operator by which the random nature of that resolvent can be fully explored. By doing so, we find readily computable expressions for the average electric field inside the dielectric object, its variation and autocorrelation. Random variables such as transmission and reflection coefficients are expressed in terms of these expansions and the Karhunen-Loève uncorrelated statistics are derived. In the polynomial expansion of the resolvent we use the monic orthogonal polynomials uniquely related to the probability measure on the probability space. The corresponding operator-valued expansion coefficients satisfy a three-term recurrence relation that is very similar to the one introduced in Miller's algorithm [20, (p. 114)]. We present algorithms that solve the recurrence relation after truncation. Random characteristics of the random operator field are expressed in terms of these expansion coefficients. We illustrate the application of the methods by two examples: a one-dimensional inhomogeneous dielectric slab and a periodic 2D dielectric structure in TE and TM polarization.

The outline of this paper is as follows. In Section 2 we discuss the mathematical setting of the problem. The section describes the mathematical context and concepts so that expressions of characteristics of a random resolvent can be derived in a structured way. This is possible by applying a Monte-Carlo based approach only at the expense of large computation times and memory storage. In Section 3 we introduce two methods, a spectral method and an algebraic method to expand the random resolvent as a series with respect to the unique sequence of monic orthogonal polynomials associated to the probability measure. We also present extensions and generic applications of the introduced methods. Section 4 contains two illustrations: a dielectric slab with a contrast function that depends affinely on the random variable and a periodic structure where the reflection coefficient is a random variable that depends linearly on the unknown electric field. For a slab the scattering is completely determined by the reflection coefficient. We present characteristics such as mean, variation, and autocorrelation for the electric field inside the slab and for the reflection and transmission coefficients. For the periodic structure we show mean and uncorrelated statistics of the real and imaginary parts of the reflection coefficient. Finally in Section 5 we draw conclusions and give a short outlook of the proposed methods.

2. MATHEMATICAL SETTING

We consider a random inhomogeneous dielectric object that occupies a domain V in physical space. The field $\mathbf{E}^{sc}$ scattered by the object satisfies an integral relation of the form

$$\mathbf{E}^{sc}\left(\mathbf{x},\omega;\,\alpha\right)=\frac{\omega^2}{c^2}V\int\limits_V\mathfrak{G}\left(\mathbf{x},\mathbf{x}',\omega\right)\chi\left(\mathbf{x}';\,\alpha\right)\mathbf{E}\left(\mathbf{x}',\omega;\,\alpha\right)\mathbf{d}\mathbf{x}'. \tag{1}$$

Here $\mathbf{E}(\mathbf{x},\omega;\,\alpha)$ denotes the total electric field, $\chi(\mathbf{x},\alpha)$ the contrast function, $\mathfrak{G}(\mathbf{x},\mathbf{x}';\,\omega)$ the Green tensor, ω the angular frequency, $\mathbf{x}$ the spatial variable, and α the sample variable. The contrast function describes the randomness and the inhomogeneity of the dielectric object. The total electric field is the superposition of the scattered field and the incident field, i.e., $\mathbf{E}(\mathbf{x},\omega;\,\alpha)=\mathbf{E}^{\mathbf{sc}}(\mathbf{x},\omega;\,\alpha)+\mathbf{E}^{\mathbf{in}}(\mathbf{x},\omega)$. Thus the following integral equation for the field $\mathbf{E}^{do}=\mathbf{E}|_V$ inside the dielectric object

$$\mathbf{E}^{do}\left(\mathbf{x},\omega;\,\alpha\right)=\mathbf{E}^{in}\left(\mathbf{x},\omega\right)+\frac{\omega^2}{c^2}\int\limits_V\mathfrak{G}\left(\mathbf{x},\mathbf{x}';\,\omega\right)\chi\left(\mathbf{x}',\alpha\right)\mathbf{E}^{do}\left(\mathbf{x}',\omega;\,\alpha\right)d\mathbf{x}', \tag{2}$$

emerges. By solving this integral equation on V, Relation (1) yields the field scattered by the dielectric object. We assume that the contrast function specifies the random nature of the dielectric object according to

$$\chi\left(\mathbf{x},\alpha\right)=\chi_0\left(\mathbf{x}\right)+f\left(\alpha\right)\chi_1\left(\mathbf{x}\right) \tag{3}$$

where f is a bounded random variable for the probability space $(\mathbb{R},\mathfrak{B},\mu)$ with $\mathfrak{B}$ the Borel algebra of $\mathbb{R}$ and μ the probability measure [21]. Since there are $a<b$ such that $f(\mathbb{R})\subset[a,b]$, there is a measure μ_f

on the interval $[a, b]$ such that $\mu_f(A) = \mu(f^{-1}(A))$, with $f^{-1}(A) = \{\alpha \in \mathbb{R} \,|\, f(\alpha) \in A\}$. By dilatation and translation we obtain the scaled sample variable $\hat{\alpha} \in [-1, 1]$,

$$\hat{\alpha} = \frac{2}{b-a}\left(f(\alpha) - \frac{a+b}{2}\right) = \hat{f}(\alpha), \tag{4}$$

a probability measure $\hat{\mu}$ on the interval $J = [-1, 1]$, and the decomposition of the random contrast field,

$$\chi(\mathbf{x}, \hat{\alpha}) = \left[\chi_0(\mathbf{x}) + \frac{b+a}{b-a}\chi_1(\mathbf{x})\right] + \hat{\alpha}\left[\frac{b-a}{2}\chi_1(\mathbf{x})\right]. \tag{5}$$

Thus, without damaging generality, we assume a probability space $(J, \mathfrak{B}, \mu)$ with the identity function $f(\alpha) = \alpha$ as the random variable and the random contrast field given by

$$\chi(\mathbf{x}, \alpha) = \chi_0(\mathbf{x}) + \alpha\chi_1(\mathbf{x}). \tag{6}$$

We can write the integral equation for the field $\mathbf{E}^{do}$ in operator form

$$\left(I - G_0^{do} - \alpha G_1^{do}\right)\mathbf{E}^{do} = \mathbf{E}^{in}, \tag{7}$$

where G_0^{do} and G_1^{do} denote the integral operators,

$$\left(G_r^{do}\mathbf{u}\right)(\mathbf{x}) = \frac{\omega^2}{c^2}\int_V \mathfrak{G}(\mathbf{x}, \mathbf{x}'; \omega)\,\chi_r(\mathbf{x}')\,\mathbf{u}(\mathbf{x}')\,d\mathbf{x}', \quad r = 1, 2,\ \mathbf{x} \in V \tag{8}$$

on the Hilbert space $\mathcal{L}_2(V, d\mathbf{x})$. The scattered field is the result of the application of the integral operator $G_0^{sc} + \alpha G_1^{sc}$ defined by

$$\left(G_r^{sc}\mathbf{u}\right)(\mathbf{x}) = \frac{\omega^2}{c^2}\int_V \mathfrak{G}(\mathbf{x}, \mathbf{x}'; \omega)\,\chi_r(\mathbf{x}')\,\mathbf{u}(\mathbf{x}')\,d\mathbf{x}', \quad r = 1, 2,\ \mathbf{x} \notin V \tag{9}$$

to the electric field $\mathbf{E}^{do}$,

$$\mathbf{E}^{sc} = \left(G_0^{sc} + \alpha G_1^{sc}\right)\mathbf{E}^{do}. \tag{10}$$

In an application, the typical stochastic characteristics are: (1) the average electric field inside the object, $\mathbf{E} = \mathbf{E}^{do}$, and of the scattered field, $\mathbf{E} = \mathbf{E}^{sc}$,

$$\langle \mathbf{E}(\mathbf{x}) \rangle = \int_J \mathbf{E}(\mathbf{x}, \alpha)\,d\mu(\alpha), \tag{11}$$

(2) the autocorrelation kernel

$$C_{pq}(\mathbf{x}, \xi) = \int_J \tilde{\mathbf{E}}_p(\mathbf{x}, \alpha)\,\tilde{\mathbf{E}}_q(\xi, \alpha)^* d\mu(\alpha), \tag{12}$$

where p and q indicate the components of the electric field and where $\tilde{\mathbf{E}}(\mathbf{x}, \alpha) = \mathbf{E}(\mathbf{x}, \alpha) - \langle \mathbf{E}(\mathbf{x}) \rangle$, and (3) the absolute variation defined by

$$\Gamma(\mathbf{x}, \omega) = \left\langle |\mathbf{E}(\mathbf{x}, \omega)|^2 \right\rangle - \left|\left\langle \mathbf{E}(\mathbf{x}, \omega) \right\rangle\right|^2, \tag{13}$$

which is a measure of incoherence due to randomness.

In a mathematically generic formulation, we consider bounded operators G_0 and G_1 on a Hilbert space $\mathcal{H}$. The inner product of $\mathcal{H}$ is denoted by $[u, v]_{\mathcal{H}}$ and corresponding to that inner product A^* denotes the adjoint of the bounded linear operator A, such that $[Au, v]_{\mathcal{H}} = [u, A^* v]_{\mathcal{H}}$. The random resolvent field

$$\alpha(I - G_0 - \alpha G_1)^{-1}, \tag{14}$$

with I the identity operator, is associated to the following linear equation in $\mathcal{H}$,

$$(I - G_0 - \alpha G_1)\,u = e. \tag{15}$$

A straightforward solution method to solve Equation (15) is the following. We write $G = (I - G_0)^{-1} G_1$ and $e_0 = (I - G_0)^{-1} e$. Then the equation becomes

$$(I - \alpha G)\, u = e_0. \tag{16}$$

The solution $u = (I - \alpha G)^{-1} e_0$ can be represented by the (Neumann) series

$$u = \sum_{n=0}^{\infty} \alpha^n G^n e_0. \tag{17}$$

for α with $|\alpha| < r(G)^{-1}$ where $r(G) = \overline{\lim}_{n\to\infty} \|G^n\|^{\frac{1}{n}}$ is the spectral radius of G. Thus, if $r(G) < 1$, then the series converges absolutely and uniformly on the interval J. For details we refer to [22].

For notational convenience we write $R(G, \alpha) = (I - \alpha G)^{-1}$, the random resolvent. We define the *resolvent mean* $\langle R(G) \rangle$ and the *resolvent variation* $\Sigma(G)$ by

$$R_0(G) = \big\langle R(G) \big\rangle = \int_J R\,(G, \alpha)\, d\mu(\alpha), \tag{18}$$

and

$$\Sigma(G) = \big\langle R(G)^* R(G) \big\rangle - R_0(G)^* R_0(G) = \int_J \left(R\,(G, \alpha) - R_0(G)\right)^* \left(R\,(G, \alpha) - R_0(G)\right) d\mu\,(\alpha)\,. \tag{19}$$

Then for given e_0, e_1 and $e_2 \in \mathcal{H}$, and a bounded operator G, the characteristic averages of the integral equation satisfy

$$\langle R(G) e_0 \rangle = R_0(G) e_0, \tag{20}$$

$$\langle [e_1, R(G) e_0]_{\mathcal{H}} \rangle = [e_1, R_0(G) e_0]_{\mathcal{H}}\,, \tag{21}$$

$$\langle [(R(G) - R_0(G))\, e_1,\, (R(G) - R_0(G))\, e_2]_{\mathcal{H}} \rangle = [e_1, \Sigma(G) e_2]_{\mathcal{H}}\,. \tag{22}$$

If $r(G) < 1$, we can apply the Neumann series and obtain the following series expansions

$$R_0(G) \;=\; \sum_{n=0}^{\infty} m_n G^n, \tag{23}$$

$$\Sigma(G) \;=\; \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(m_n - m_k m_{n-k}\right) (G^*)^{n-k} G^k. \tag{24}$$

We note that the moments $m_n = \int_J \alpha^n d\mu(\alpha)$ of the probability measure satisfy,

$$|m_n| = \int_J |\alpha|^n d\mu\,(\alpha) \leqslant 1. \tag{25}$$

which guarantees the convergence of both expansions.

The relationship between the above mathematical formulation and Equations (7) and (10) is given by

$$\mathbf{E}^{do} = \left(I - G_0^{do} - \alpha G_1^{do}\right)^{-1} \mathbf{E}^{in} = R\left(G^{do}, \alpha\right) \mathbf{E}_0^{do}, \tag{26}$$

where $\mathbf{E}_0^{do} = \left(I - G_0^{do}\right)^{-1} \mathbf{E}^{in}$, and

$$\mathbf{E}^{sc} = \left(G_0^{sc} + \alpha G_1^{sc}\right) R\left(G^{do}, \alpha\right) \mathbf{E}_0^{do}. \tag{27}$$

Remark 1

In case of a random operator field $G(\alpha)$, $|\alpha - \alpha_0| \leq \Delta\alpha$, instead of $G_0 + \alpha G_1$, $-1 \leq \alpha \leq 1$, we can linearize the expression according to

$$G(\alpha) = G\left(\alpha_0 + \hat{\alpha}\Delta\alpha\right) \cong G_0 + \hat{\alpha} G_1, \tag{28}$$

where $G_0 = G(\alpha_0)$ and $G_1 = \frac{1}{2\Delta\alpha}(G(\alpha_0 + \Delta\alpha) - (G(\alpha_0 - \Delta\alpha))$.

In the next section we elaborate on the expansion of the resolvent $\alpha \to R(G, \alpha)$ in terms of the unique sequence of monic orthogonal polynomials corresponding to the probability measure μ. We present two algorithms to compute the corresponding operator coefficients: one based on the collection of eigenvalues (spectrum) of the integral operator, the other one based on an algebraic factorization related to the recurrence relations satisfied by the orthogonal polynomials.

3. ORTHOGONAL POLYNOMIAL RESOLVENT EXPANSION

As a general reference to this section we refer to the monograph of Gautschi [20]. For the probability measure μ on the interval J, there is a sequence of polynomials $p_n(\alpha)$, $n = 0, 1, 2, \ldots$ uniquely defined by

(i) p_n is a polynomial of degree n that is monic, i.e., $p_n(\alpha) = \alpha^n + \ldots$

(ii) For all n and all m, $0 \leq m < n$,

$$\int_J \alpha^m p_n d\mu = 0, \tag{29}$$

i.e., $p_n p_m = 0$, $m \neq n$ following the definition in Equation (18). We introduce the normalizations,

$$\nu_n = (p_n)^2 = \int_J (p_n)^2 d\mu, \tag{30}$$

and

$$\sigma_n = \alpha(p_n)^2 = \int_J \alpha(p_n)^2 d\mu. \tag{31}$$

The polynomials satisfy a three-term recurrence relation of the form

$$\alpha p_n(\alpha) = p_{n+1}(\alpha) + a_n p_n(\alpha) + b_n p_{n-1}(\alpha), \quad n = 1, 2, \ldots \tag{32}$$

with initialization $p_0(\alpha) = 1$ and $p_1(\alpha) = \alpha - m_1$. The coefficients (a_n) and (b_n) are called the Jacobi-Szegö parameters [23]. We have

$$a_n = \frac{\sigma_n}{\nu_n},\ n = 1, \ldots \quad \text{and} \quad b_n = \frac{\nu_n}{\nu_{n-1}},\ n = 1, 2, \ldots \tag{33}$$

We note that $a_0 = m_1$. To initiate the spectral approach, we first briefly explore the scalar case. For each λ with $\lambda \notin (-\infty, 1] \cup [1, \infty)$ the function $\alpha \to (1 - \alpha\lambda)^{-1}$ is square integrable and thus can be expanded with respect to the polynomials p_n,

$$(1 - \alpha\lambda)^{-1} = \sum_{n=0}^{\infty} \rho_n(\lambda) p_n(\alpha), \tag{34}$$

where $\rho_n(\lambda) = \frac{1}{\nu_n} \int_J \frac{p_n(\alpha)}{1-\alpha\lambda} d\mu(\alpha)$. The functions $\rho_n(\lambda)$ are related to the associated functions $q_n(z)$ that are analytic on the complex plane cut by the segment J, see [20, p. 18]. For the classical polynomials the associated functions are known explicitly. In the general case, a backward recurrence scheme (c.f. Miller's algorithm) is applied to calculate the associated functions. In both spectral and algebraic approaches, we use the polynomial expansion of the resolvent with respect to the monic orthogonal polynomials $p_n(\alpha)$, i.e.,

$$R(G, \alpha) = \sum_{n=0}^{\infty} p_n(\alpha) R_n(G). \tag{35}$$

For such expansions we obtain, see (18) and (19)

$$\langle R(G)\rangle = R_0(G), \tag{36}$$

and

$$\Sigma(G) = \langle R(G)^* R(G)\rangle - R_0(G)^* R_0(G) = \sum_{n=0}^{\infty} \nu_n R_n(G)^* R_n(G). \tag{37}$$

3.1. Spectral Approach

In the spectral approach, for a matrix approximation $G^{[M]}$ of the integral operator G diagonalized according to $G^{[M]} = S^{[M]}\Lambda^{[M]}(S^{[M]})^{-1}$ with $\Lambda^{[M]} = \mathrm{diag}(\lambda_1, \ldots, \lambda_M)$ a diagonal matrix, we approximate the operator expansion coefficients $R_n(G)$ in (35) by

$$R_n(G) \approx R_n\left(G^{[M]}\right) = S^{[M]}\mathrm{diag}\left(\rho_n(\lambda_1), \ldots, \rho_n(\lambda_M)\right)\left(S^{[M]}\right)^{-1}. \tag{38}$$

We note that

$$R_n\left(\Lambda^{[M]}\right) = \mathrm{diag}\left(\rho_n(\lambda_1), \ldots, \rho_n(\lambda_M)\right). \tag{39}$$

If the functions $\rho_n(\lambda)$ are not known explicitly, we can calculate them from the recurrence relation (see (32))

$$(1 - a_0\lambda)\,\rho_0 - b_1\lambda\rho_1 = 1, \tag{40a}$$

and for all $n = 1, 2, \ldots$,

$$(1 - a_n\lambda)\,\rho_n - \lambda\rho_{n-1} - b_{n+1}\lambda\rho_{n+1} = 0. \tag{40b}$$

We assume convergence of the series and hence for some N that $\rho_{N+1} \cong 0$ and write the corresponding solution as $\rho_n^{[N]}$. With $\rho_{N+1} = 0$ we get the N-th order approximation $\rho_n^{[N]}$ of ρ_n. According to Miller's algorithm we express $\rho_{n+1}^{[N]}$ as

$$\rho_{n+1}^{[N]}(\lambda) = \frac{v_{N-n}^{[N]}(\lambda)}{w_{N-n}^{[N]}(\lambda)}\rho_n^{[N]}(\lambda), \quad n = N-1, \ldots, 0. \tag{41}$$

The polynomials $v_k^{[N]}(\lambda)$ and $w_k^{[N]}(\lambda)$ are solutions of the two-term recurrence relations

$$w_{k+1}^{[N]}(\lambda) = (1 - \lambda a_{N-k})\, w_k^{[N]}(\lambda) - \lambda b_{N-k+1} v_k^{[N]}(\lambda) \tag{42}$$

$$v_{k+1}^{[N]}(\lambda) = \lambda w_k^{[N]}(\lambda), \tag{43}$$

where $k = 1, \ldots, N-1$ and $v_1^{[N]}(\lambda) = \lambda$ and $w_1^{[N]}(\lambda) = 1 - \lambda a_N$.

For the functions $\rho_n^{[N]}(\lambda)$ we derive the relation

$$\rho_0^{[N]}(\lambda) = \frac{w_N^{[N]}(\lambda)}{w_{N+1}^{[N]}(\lambda)}, \tag{44}$$

and for $n = 0, \ldots, N-1$

$$\rho_{n+1}^{[N]}(\lambda) = \frac{\lambda^n v_{N-n}^{[N]}(\lambda)}{w_{N+1}^{[N]}(\lambda)}. \tag{45}$$

3.2. Algebraic Approach

In the algebraic approach we start from the series expansion (35). By multiplying both sides of (35) by $I - \alpha G$ we deduce that

$$\begin{aligned} I &= \sum_{n=0}^{\infty} p_n(\alpha)\,(I-\alpha G)\,R_n = \sum_{n=0}^{\infty} p_n(\alpha)R_n - \alpha p_n(\alpha)GR_n \\ &= \sum_{n=0}^{\infty} p_n(\alpha)\,(I-a_nG) - p_{n+1}(\alpha)GR_n - b_np_{n-1}(\alpha)GR_n. \end{aligned} \tag{46}$$

We introduce $b_0 = 0$ and obtain the recurrence relations

$$(I-a_0G)\,R_0 - b_1GR_1 = I, \tag{47a}$$

and for all $n = 1, 2, \ldots,$

$$(I-a_nG)\,R_n - GR_{n-1} - b_{n+1}GR_{n+1} = 0. \tag{47b}$$

If R_{N+1} is known for some N, then the recurrence allows to calculate all other R_n with $n = 0, \ldots, N$, from that one. In the N-th order approximation we take $R_{N+1}^{[N]} = \mathrm{O}$. We introduce the operators

$$Z_n = (I-a_nG)^{-1}G, \quad n = 0, 1, \ldots \tag{48}$$

Then the (backward) recurrence system of equations is written as

$$R_N^{[N]} - Z_NR_{N-1}^{[N]} = 0 \tag{49a}$$

$$R_n^{[N]} - Z_nR_{n-1}^{[N]} - b_{n+1}Z_nR_{n+1}^{[N]} = 0 \tag{49b}$$

$$R_0 - b_1Z_0R_1 = (I-a_0G)^{-1} \tag{49c}$$

from which the unknown $R_n^{[N]}$ can be obtained recursively.

In the algorithm described by (49) at each step of the recursive procedure an inverse should be taken. In the algorithm described by the formulas in (50) only one inverse should be taken. Inspired by Miller's algorithm (see (41)), we see that the operator polynomials $w_k^{[N]}(G)$ and $v_k^{[N]}(G)$ satisfy $w_{N-n}^{[N]}(G)R_{n+1}^{[N]} = v_{N-n}^{[N]}(G)R_n^{[N]}$, $n = N-1, \ldots, 0$. Thus we obtain

$$R_0\,(G) = \left\{w_{N+1}^{[N]}\,(G)\right\}^{-1} w_N^{[N]}\,(G)\,, \tag{50a}$$

and for $n = 1, \ldots, N-1$

$$R_n\,(G) = \left\{w_{N+1}^{[N]}\,(G)\right\}^{-1} G^{n-1} v_{N-n+1}^{[N]}\,(G)\,. \tag{50b}$$

We note that the operator polynomials $w_k^{[N]}(G)$ and $v_k^{[N]}(G)$ satisfy

$$w_{k+1}^{[N]}(G) = (I-a_{N-k}G)\,w_k^{[N]}(G) - b_{N-k+1}Gv_k^{[N]}(G), \tag{51a}$$

$$v_{k+1}^{[N]}(G) = Gw_k^{[N]}(G). \tag{51b}$$

Remark 2

Let the random vector field $u : [-1, 1] \to \mathcal{H}$ be defined by $u(\alpha) = (I-\alpha G)^{-1}e_0$. If $\mathcal{L}$ is a complex-valued linear functional on the Hilbert space $\mathcal{H}$, then $L_u(\alpha) = \mathcal{L}(u(\alpha))$ is a random variable and

$$L_u(\alpha) = \sum_{n=0}^{\infty} L_{u,n}p_n(\alpha) \tag{52}$$

where $L_{u,n} = \mathcal{L}(R_n(G)e_0)$. We write

$$L_u(\alpha) = L_{u,0} + \sum_{n=1}^{\infty} L_{u,n}p_n(\alpha) = L_{u,0} + \tilde{L}_u(\alpha) \tag{53}$$

with $L_{u,0}$ the average of the complex random variable L_u. The random variable $L_u(\alpha)$ describes a curve in the complex plane.

Remark 3

As a side result of the polynomial expansion of a random variable we derive the Karhunen-Loève expansion. Let K denote the 2×2-autocorrelation matrix,

$$K = \sum_{n=1}^{\infty} \nu_n \begin{pmatrix} \mathrm{Re}(L_{u,n})^2 & \mathrm{Re}\,(L_{u,n})\,\mathrm{Im}\,(L_{u,n}) \\ \mathrm{Re}\,(L_{u,n})\,\mathrm{Im}\,(L_{u,n}) & \mathrm{Im}(L_{u,n})^2 \end{pmatrix} \tag{54}$$

Then we find the uncorrelated statistics in the form

$$L_u(\alpha) = L_{u,0} + [\xi_1\ \psi_1(\alpha) + \xi_1\psi_2(\alpha)] \tag{55}$$

where $\xi_1 = \xi_{11} + j\xi_{21}$ and $\xi_2 = \xi_{12} + j\xi_{22}$ and where $\xi_1 = \begin{bmatrix}\xi_{11}\\ \xi_{21}\end{bmatrix}$ and $\xi_2 = \begin{bmatrix}\xi_{12}\\ \xi_{22}\end{bmatrix}$ are the orthonormal eigenvectors of the matrix K with positive eigenvalues λ_1 and λ_2. Further, with $\mathbf{L}_{u,n} = [\mathrm{Re}\,(L_{u,n})\,, \mathrm{Im}\,(L_{u,n})]^T$ we calculate the $\psi_r(\alpha)$ according to

$$\psi_r\,(\alpha) = \sum_{n=1}^{\infty} [\xi_r, \mathbf{L}_{u,n}]_{\mathcal{H}} p_n\,(\alpha)\,, \quad r = 1, 2 \tag{56}$$

Then $\langle \psi_r \rangle = 0$ and $\langle \psi_r \psi_{r'} \rangle = \lambda_r \delta_{rr'}$. Thus the variation of the random variable L_u is given by

$$\left\langle |L_u|^2 \right\rangle - \left| \left\langle L_u \right\rangle \right|^2 = \lambda_1 + \lambda_2. \tag{57}$$

4. ILLUSTRATIONS

We continue by illustrating the methods presented in the previous section for two examples:

(i) an inhomogeneous dielectric slab with random permittivity,

(ii) a two-dimensional periodic dielectric object with random permittivity.

In a true application, the integral equation should be approximated by a matrix equation. This is an additional complication that we do not address in this paper. Therefore, we use illustrations with known discretizations. We apply our methods to an inhomogeneous dielectric slab with a contrast function that depends linearly on the spatial variable, as well as on the random variable. We determine the average and the variation of the random electric field inside the slab. We show the dependence of these characteristics on the selected probability measures. The second illustration refers to a well-established model that describes the electromagnetic scattering effects of TE and TM polarizations for a two-dimensional dielectric object with one-dimensional periodicity, see [24]. We provide random characteristics of the reflection coefficient such as average and variation for different ranges of the dielectric permittivity for both TM and TE polarizations. With reference to (7) and (8), in our numerical implementations, G_0^{do} and G_1^{do} are replaced by size-M matrices $G_0^{do,[M]}$ and $G_1^{do,[M]}$, the incident field $\mathbf{E}^{in}(\cdot)$ by the vector $\mathbf{E}^{in,[M]}$, the field $E_0^{do} = (I - G_0^{do})^{-1} E^{in}$ by the vector $E_0^{do,[M]} = (I - G_0^{do,[M]})^{-1} E^{in,[M]}$, and the operator $G^{do} = (I - G_0^{do})^{-1} G_1^{do}$ by the matrix $G^{do,[M]} = (I - G_0^{do,[M]})^{-1} G_1^{do,[M]}$. Thus in the N-th approximation the random resolvent $R(G^{do}, \alpha)$ is replaced by

$$R^{[N]}\left(G^{do,[M]}, \alpha\right) = \sum_{n=0}^{\infty} R_n^{[N]}\left(G^{do,[M]}\right) p_n(\alpha). \tag{58}$$

For the scattered field we discretized the integral operator $G_0^{sc} + \alpha G_1^{sc}$ and obtain the matrices $G_0^{sc,[M]} + \alpha G_1^{sc,[M]}$. The corresponding scattered field is generally described in terms of the random reflection and transmission coefficients.

4.1. Illustration 1: Dielectric Slab

The slab thickness d is fixed and described by the spatial variable z, $0 < z < d$. The inhomogeneity is determined by the contrast function that linearly depends on the variable z. The randomness of the

contrast function is according to $\chi = \chi_0 + \alpha\chi_1$, where

$$\chi_0(z) = (\bar{\varepsilon}_r - 1)\frac{z}{d} \quad \text{and} \quad \chi_1(z) = \Delta\varepsilon_r \frac{z}{d}. \tag{59}$$

In (59), the parameters $\bar{\varepsilon}_r$ and $\Delta\varepsilon_r$, with $\bar{\varepsilon}_r > 1$ and $\Delta\varepsilon_r > 0$ determine the relative permittivity range $[\bar{\varepsilon}_r - \Delta\varepsilon_r, \bar{\varepsilon}_r + \Delta\varepsilon_r]$ on which a probability distribution is defined. The sample variable $\alpha \in [-1, 1]$ rescales the permittivity range and with that the probability distribution. The contrast function is assumed independent of the frequency of the incident field. In the lossless slab this assumption is valid in the frequency range that we consider.

We scale the spatial variable z by d and the wave number by $\frac{1}{d}$, to obtain the dimensionless number $\kappa = \omega d\sqrt{\varepsilon_0\mu_0}$. The scaled field E^{do} satisfies the following integral equation

$$E^{do}(z) - \frac{j\kappa}{2}\int_0^1 e^{j\kappa|z-z'|}\chi(z')\,E^{do}(z')\,dz' = e^{j\kappa z} \quad 0 < z < 1 \tag{60}$$

with time convention $e^{-j\omega t}$. Thus the integral operators G_0 and G_1 are defined on the Hilbert space $\mathcal{L}_2([0,1])$ according to

$$(G_r u)(z) = \frac{j\kappa}{2}\int_0^1 e^{j\kappa|z-z'|}\chi_r(z')\,u(z')\,dz', \quad r = 0, 1 \tag{61}$$

In our illustration we take κ in the range from 0.2π to 20π. We can vary κ by changing frequency or slab thickness.

The normalized reflected and transmitted fields, E^{re} and E^{tr} are determined by the relation

$$E^{re}(z) = e^{j\kappa z} + L^{re}(E)e^{-j\kappa z}, \quad z < 0 \tag{62}$$

and

$$E^{tr}(z) = \left(1 + L^{tr}(E)\right)e^{j\kappa z}, \quad z > 1. \tag{63}$$

Here the linear functionals L^{re} and L^{tr} are defined by

$$L^{re}(u) = \frac{j\kappa}{2}\int_0^1 e^{j\kappa z'}\chi\left(z'\right)u(z')\,dz', \tag{64}$$

$$L^{tr}(u) = \frac{j\kappa}{2}\int_0^1 e^{-j\kappa z'}\chi\left(z'\right)u(z')\,dz'. \tag{65}$$

Then reflection coefficient can be written as

$$L^{re}(E(\alpha)) = L_0^{re}(E(\alpha)) + \alpha L_1^{re}(E(\alpha)) \tag{66}$$

with $L_r^{re}(u) = \frac{j\kappa}{2}\int_0^1 e^{j\kappa z'}\chi_r(z')u(z')dz'$, $r = 0, 1$. Since $E(\alpha) = \sum_{n=0}^{\infty}(R_n(G)E_0)p_n(\alpha)$ by applying the recurrence relation for $\alpha p_n(\alpha)$ we obtain the following expansion for the random reflection coefficient C^{re}

$$C^{re}(\alpha) = L^{re}(E(\alpha)) = \sum_{n=0}^{\infty}\ell_n^{re}(E_0)\,p_n(\alpha) \tag{67}$$

with

$$\ell_n^{re}(E_0) = L_0^{re}(R_n(G)E_0) + L_1^{re}(S_n(G)E_0) \tag{68}$$

and

$$S_0(G) = m_1R_0(G) + b_1R_1(G) \tag{69a}$$

$$S_n(G) = R_{n-1}(G) + a_nR_n(G) + b_{n+1}R_{n+1}(G), \quad n \geq 1. \tag{69b}$$

In the discretization z is replaced by the discretized variable $z_l = \frac{l-1}{M}$, $l = 1, M+1$. We equidistantly discretize the integral equation based on the formulation in [25, p. 163] and replace the integral operators G_0 and G_1 by the matrices $G_0^{[M]}$ and $G_1^{[M]}$ defined by

$$G_r^{[M]} = \frac{j\kappa}{2M}\left[e^{j\kappa\frac{|m-l|}{M}}\chi_r\left(\frac{l-1}{M}\right)\right]_{m,l}, \quad r = 1, 2 \tag{70}$$

In the illustrations we take $M = 200$.

For the reflection coefficient in the N-th order approximation and after discretization we obtain the random variable

$$C^{re,[M]}(\alpha) = \sum_{n=0}^{N} \ell_n^{re,[N]}\left(E_0^{[M]}\right) p_n(\alpha), \tag{71}$$

with

$$\ell_n^{re,[N]}\left(E_0^{[M]}\right) = \left(f_0^{[M]}\right)^T R_n^{[N]}\left(G^{[M]}\right) E_0^{[M]} + \left(f_1^{[M]}\right)^T S_n^{[N]}\left(G^{[M]}\right) E_0^{[M]}, \tag{72}$$

$$f_r^{[M]} = \frac{j\kappa}{2M}\left(e^{j\kappa\frac{l-1}{M}}\chi_r\left(\frac{l-1}{M}\right)\right) \in \mathbb{C}^{M+1\times 1}. \tag{73}$$

The following characteristics are calculated:

- Average field vector

$$\left\langle E^{do,[M]}\right\rangle = R_0^{[N]}\left(G^{do,[M]}\right) E_0^{do,[M]}, \tag{74}$$

with absolute value A_m and phase ϕ_m components

$$\left\langle E_m^{do,[M]}\right\rangle = A_m e^{j\varphi_m}, \quad m = 1, \ldots, M+1. \tag{75}$$

- Absolute variation vector $\Gamma^{do,[M]}$ defined by

$$\Gamma_m^{do,[M]} = \sum_{n=1}^{N} \nu_n \left|\left(R_n^{[N]}\left(G^{do,[M]}\right) E_0^{do,[M]}\right)_m\right|^2, \tag{76}$$

- With mean $\bar{\Gamma}^{do,[M]}$

$$\bar{\Gamma}^{do,[M]} = \frac{1}{M+1}\sum_{m=1}^{M+1} \Gamma_m^{do,[M]}. \tag{77}$$

Similarly, for the reflection coefficient the average $\ell_0^{re,[N]}(E_0^{[M]})$, the variation $\sum_{n=1}^{N} \nu_n |L_n^{[N,re]}(E_0^{[M]})|^2$, and the Karhunen-Loève expansion (see (55)) can be computed.

First we address the Neumann series approach as suggested in Section 2. If the spectrum of the matrix $G^{[M]}$ contains values with a modulus larger than 1, the series

$$\sum_{n=0}^{\infty} m_{2n}\left(G^{do,[M]}\right)^{2n} \tag{78}$$

is not convergent to $R_0(G^{do,[M]})$. We use the uniform distribution on the interval $[-1, 1]$ with nonzero moments $m_{2n} = \frac{1}{2n+1}$.

Figure 1(a) shows that the absolute value of the average electric field obtained by the spectral, algebraic and Neumann series approaches for $\kappa = 2.8$ are the same, where we took $N = 35$ to cut-off the series. We note that all the eigenvalues of $G^{do,[M]}$ are within the unit circle. In Figure 1(b) we compare the solution obtained from these three approaches for $\kappa = 28$ on logarithmic scale. We see that the solution obtained from the Neumann series is not equal to the solutions from the other two approaches. The reason is that for $\kappa = 28$ the matrix $G^{do,[M]}$ has two eigenvalues with an absolute value larger than one.

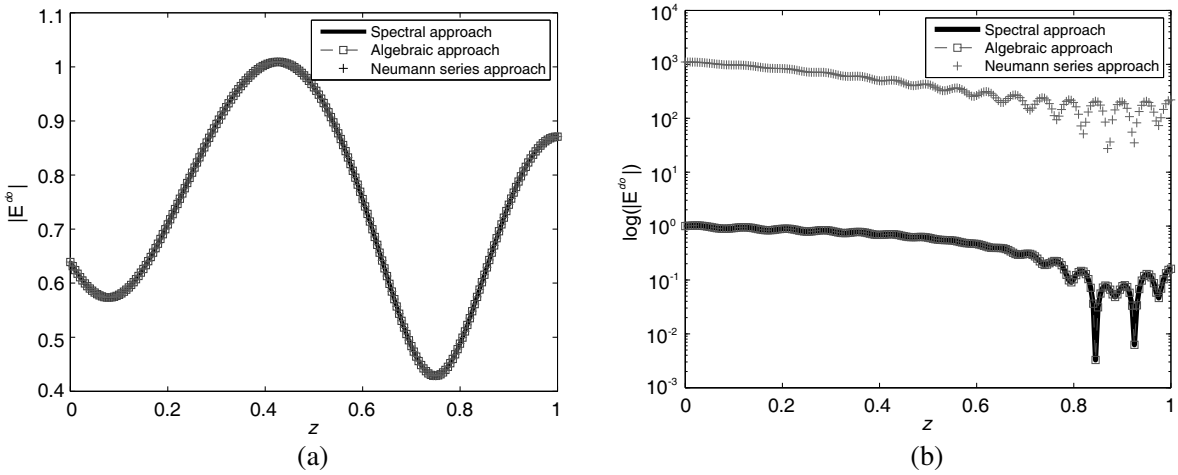

Figure 1. Absolute average electric field inside the slab for (a) $\kappa = 2.8$ and (b) $\kappa = 28$. For $\kappa = 2.8$ all eigenvalues have absolute value smaller than one. For $\kappa = 28$ two eigenvalues have absolute value larger than one.

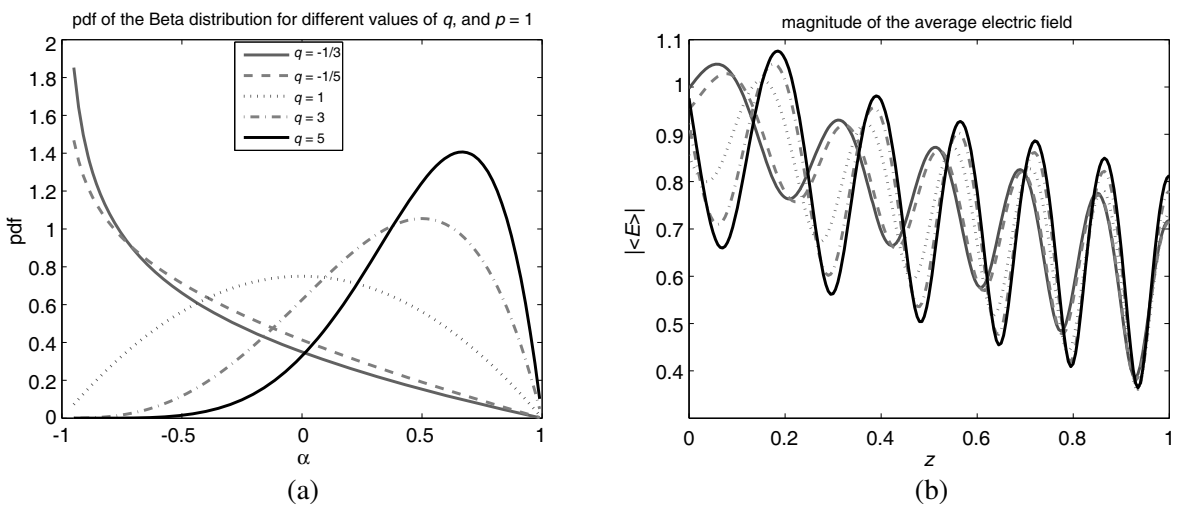

Figure 2. (a) Pdf-s of the Beta distribution for $p = 1$ and $q = -\frac{1}{3}$, $-\frac{1}{5}$, 1, 3, 5, (b) magnitude of the average field inside the slab for the distributions depicted in (a) where $\kappa = 10$, $\bar{\varepsilon} = 5$, and $\Delta\varepsilon_r = 1$. The legend for both plots is the same.

Figure 2 shows the probability density functions $w^{(p,q)}$ (see (A14)) for the Beta probability measure with $p = 1$ and $q = -\frac{1}{3}$, $-\frac{1}{5}$, 1, 3 and 5. Figure 2(b) shows the magnitude of the average field $|\langle E^{do}\rangle|$ in the slab. The effect of the choice of the probability measure is clearly visible. Globally we see a linear decrease in magnitude with an oscillation on top. The decrease is sharper for the negative values of q where the interval $[-1, 0]$ is more probable than the interval $[0, 1]$. Also amplitude of the oscillations increases with increasing q.

Figure 3(a) presents the magnitude and relative variation of the average field for different values of κ, $\kappa = 1$, 5, 10, 20 and 40 for the uniform probability measure where the mean relative permittivity is $\bar{\varepsilon} = 5$ and $\Delta\varepsilon = 1$. We see that the number of oscillations increases proportionally with κ and there is an overall decrease with a rate that increases with κ except for $\kappa = 40$, where a node appears at $z = 0.73$. Figure 3(b) shows the absolute variations of the random electric field for $\kappa = 10$ and for various values of the range $[5 - \Delta\varepsilon_r, 5 + \Delta\varepsilon_r]$, with $\Delta\varepsilon_r = 0.25$, 0.50, 0.75, 1.00, 1.25 and 1.50. The variation increases as $\Delta\varepsilon_r$ increases.

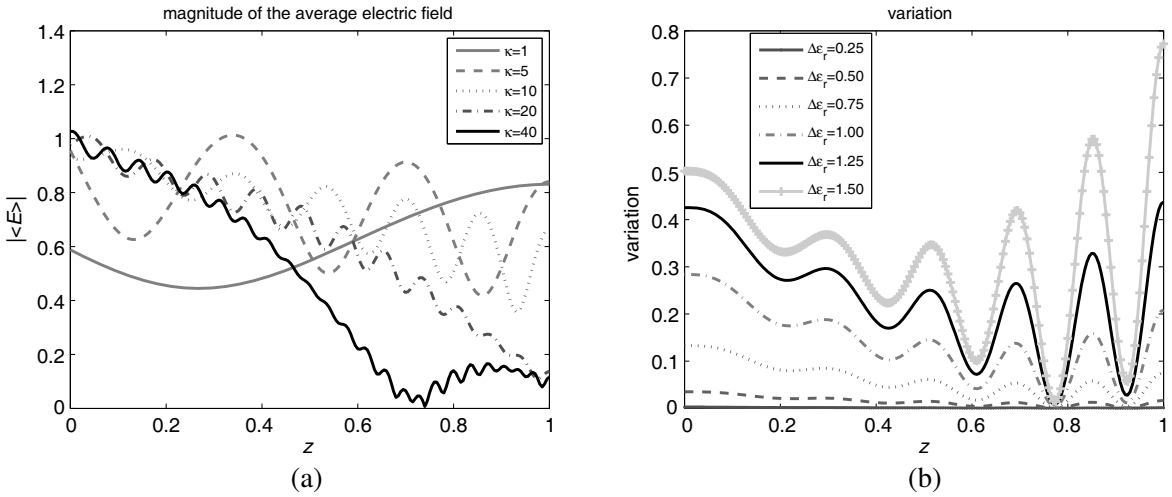

Figure 3. (a) Magnitude of the average field for $\kappa = 1$, 5, 10, 20 and 40, with $\bar{\varepsilon}_r = 5$ and $\Delta\varepsilon_r = 1$, (b) variation of the random electric field for $\kappa = 10$, $\bar{\varepsilon}_r = 5$, $\Delta\varepsilon_r = 0.25$ to 1.5. Both (a) and (b) with uniform probability distribution.

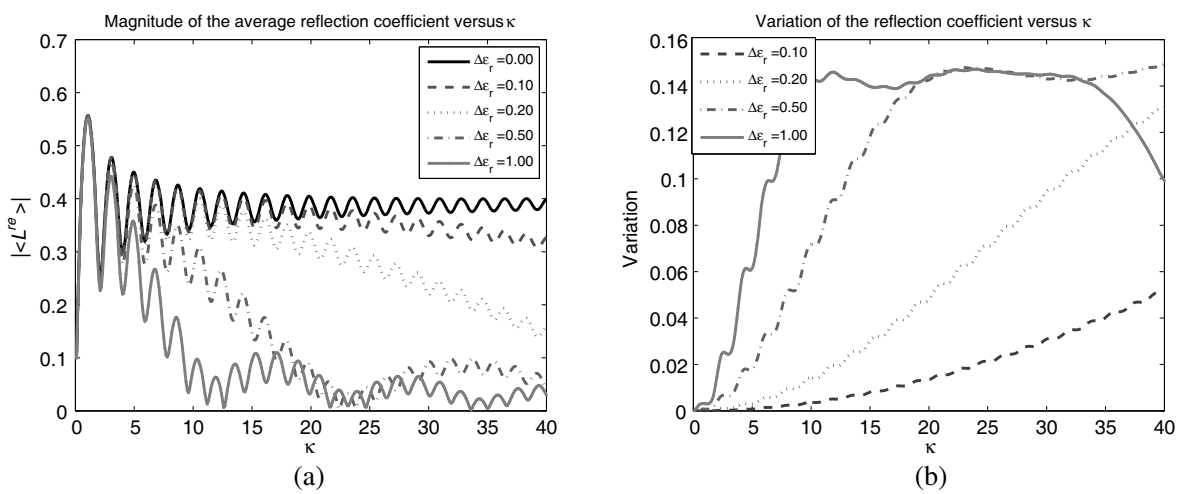

Figure 4. (a) Magnitude of the average reflection coefficient as function of κ for different values of $\Delta\varepsilon_r$, (b) the variation of the reflection coefficient as function of κ for different values of $\Delta\varepsilon_r$.

Figure 4(a) shows the magnitude of the average reflection coefficient as function of κ where $\Delta\varepsilon_r = 0.1$, 0.2, 0.5 and 1.0. We see that for $\kappa < 2$ there is hardly any effect of the randomness of the permittivity on the average reflection coefficient. This observation is confirmed in Figure 4(b) where the variation of the reflection coefficient is shown; for values of $\kappa < 2$ the variation is negligible. For $\kappa > 2$ we see that κ really distinguishes between the different values of $\Delta\varepsilon_r$. In Figure 4(a) we observe a decrease with a higher rate for higher $\Delta\varepsilon_r$ values. Also this observation is confirmed by Figure 4(b) where we see a sharp increase of the variation as function of κ for larger values of $\Delta\varepsilon_r$. For the comparison between a standard Monte-Carlo method and the presented polynomial expansion we refer to [26], where it is shown that the convergence in the polynomial expansion method is exponential whereas the convergence rate of the Monte-Carlo method is in the order of one over the number of samples.

4.2. Illustration 2: Periodic Structure

In the papers [24] and [27], the 2-dimensional scattering problem of a 1-dimensionally periodic dielectric grating, embedded in a stratified medium is formulated as a spectral domain integral equation. We briefly outline both the TE and TM polarization. For TE polarization, the integral equation is given by

$$\left(I - g^{\mathrm{TE}} M\left(\varepsilon_r\right)\right) E^{do} = E^{in} \tag{79}$$

where, as in Section 2, E^{in} indicates the (scalar) electric field incident on the structure and where E^{do} represents the total electric field inside the structure. By g^{TE} we denote the integral operator related to the Green function of TE polarization. It depends on the stratified medium, the period Λ, the free-space wavelength λ_0, and the angle of incidence θ_i of the incident field. By $M(\varepsilon_r)$ we denote the contrast operator of the dielectric rectangular object (the grating) depicted in Figure 5 that is affinely related to its permittivity.

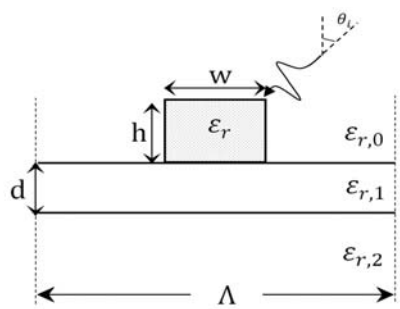

Figure 5. Schematic of the 2D periodic structure. We assume that the permittivity of the grating ε_r is random.

For the case of TM polarization, the formulation is more involved, due to the application of the inverse rule, see [28], to properly handle discontinuous material interfaces in the spectral domain. The integral equation in terms of the total electric field is then given by

$$\left(I - g^{\mathrm{TM}} M\left(\varepsilon_r\right)\left(L\left(\varepsilon_r\right)\right)^{-1}\right) E^{do} = E^{in} \tag{80}$$

where g^{TM} is the integral operator related to the TM polarization Green function, and E^{in} and E^{d0} are the (vectorial) TM-polarized incident field and total electric field inside the structure, respectively. The expression $M(\varepsilon_r)(L(\varepsilon_r))^{-1}$ denotes the contrast operator that involves the inverse rule. The operator $M(\varepsilon_r)$ depends affinely on ε_r, whereas $(L(\varepsilon_r))^{-1}$ depends linearly on ε_r^{-1}, the inverse permittivity of the grating. Thus, in general, $M(\varepsilon_r)(L(\varepsilon_r))^{-1}$ does not depend linearly on ε_r. In summary, the above integral equations for the scattering by a 2D dielectric periodic structure can be described as,

$$\left(I - G\left(\varepsilon_r\right)\right) E^{do} = E^{in} \tag{81}$$

where E^{in} indicates the electric field incident on the structure and where E^{do} represents the electric field inside the structure. By $G(\varepsilon_r)$ we denote domain integral operator (or its discretized version). We suppose that $\varepsilon_r \in [\varepsilon_{r,\min}, \varepsilon_{r,\max}]$. The solution of Equation (79) is the random electric field $E(\varepsilon_r) \equiv E(z, \varepsilon_r)$ where z refers to a spatial variable that fixes the structure.

For the TE- and TM-polarizations, we have $G(\varepsilon_r) = G^{\mathrm{TE}}(\varepsilon_r) = g^{\mathrm{TE}} M(\varepsilon_r)$ and $G(\varepsilon_r) = G^{\mathrm{TM}}(\varepsilon_r) = g^{\mathrm{TM}} M(\varepsilon_r)(L(\varepsilon_r))^{-1}$, respectively. Since $M(\varepsilon_r)$ affinely depends on ε_r we can write $G^{\mathrm{TE}}(\varepsilon_r) = G^{\mathrm{TE}}(\bar{\varepsilon}_r + \alpha\Delta\varepsilon_r) = G_0^{\mathrm{TE}} + \alpha G_1^{\mathrm{TE}}$ with $G_0^{\mathrm{TE}} = g^{\mathrm{TE}} M_0 + \bar{\varepsilon}_r g^{\mathrm{TE}} M_1$ and $G_1^{\mathrm{TE}} = \Delta\varepsilon_r g^{\mathrm{TE}} M_1$. So for the TE polarization we can use the methods introduced in Section 3 to calculate the random characteristics of Equation (79). To use the methods for the TM polarization we linearize the operator field $\varepsilon_r \to G^{\mathrm{TM}}(\varepsilon_r)$, $\varepsilon_r \in [\varepsilon_{r,\min}, \varepsilon_{r,\max}]$,

$$G^{\mathrm{TM}}\left(\varepsilon_r\right) = G^{\mathrm{TM}}\left(\bar{\varepsilon}_r + \alpha\varepsilon_r\right) \cong G_0^{\mathrm{TM}} + \alpha G_1^{\mathrm{TM}}. \tag{82}$$

See Remark 1 at the end of Section 2. Again we assume that the sample interval $[-1, 1]$ is endowed with a probability measure μ.

We apply formulas (52) and (53) to the random reflection coefficient C^{re} that is linearly related to the electric field. For the example of a layered medium with a setup depicted in Figure 5 and values according to Table 1, we assume a uniform distribution on the sample interval $[-1, 1]$.

Table 1. Values of the parameters shown in Figure 5.

Parameter	Symbol	Value
grating width	w	$0.5\lambda_0$
grating height	h	$1\lambda_0$
first layer thickness	d	$0.8\lambda_0$
period	Λ	λ_0
permittivity of upper halfspace	$\varepsilon_{r,0}$	1
permittivity of the first layer	$\varepsilon_{r,1}$	3
permittivity of the lower halfspace	$\varepsilon_{r,2}$	18.4+0.4j
permittivity of the grating	ε_r	$5 \pm \Delta\varepsilon$
angle of incidence	θ_i	$30°$

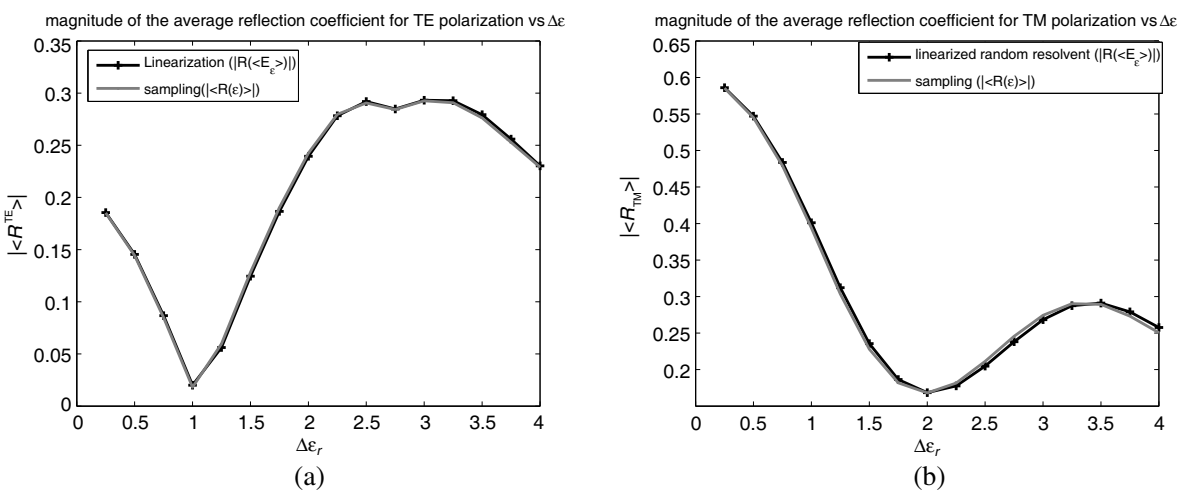

Figure 6. Magnitude of the average reflection coefficient. (a) $\langle R_{\mathrm{TE}} \rangle$ for TE polarization, (b) $\langle R_{\mathrm{TM}} \rangle$ for TM polarization versus $\Delta\varepsilon_r$ are shown where $\overline{\varepsilon}_r = 5$.

In all numerical calculations we used $\overline{\varepsilon}_r = 5$. In Figure 6, we show the magnitude of the average reflection coefficient as function of the sample range $\Delta\varepsilon_r$ for (a) the TE polarization and (b) the TM polarization. Calculations are based on Monte-Carlo sampling (for 100 samples) and the method of random resolvent described above. We observe that the TE polarization reflection coefficient calculated on basis of the Monte-Carlo method (gray) and the polynomial expansion method (black), respectively, match very well. In TM polarization also, despite of the nonlinear relation between the reflection coefficient and $\Delta\varepsilon_r$, the result of the linearized random resolvent method is remarkably good. The results (see Figure 6(b)) show that linearization gives an accurate approximation of the reflection coefficient of the structure.

In Figure 7 we show results of the Karhunen-Loève uncorrelated decomposition as applied to the TM and TE polarization reflection coefficients. In Figure 7(a) we show the variation of these reflection coefficients as a function of $\Delta\varepsilon_r$. This variation is equal to the sum of the eigenvalues $\xi_1(\Delta\varepsilon_r)$ and

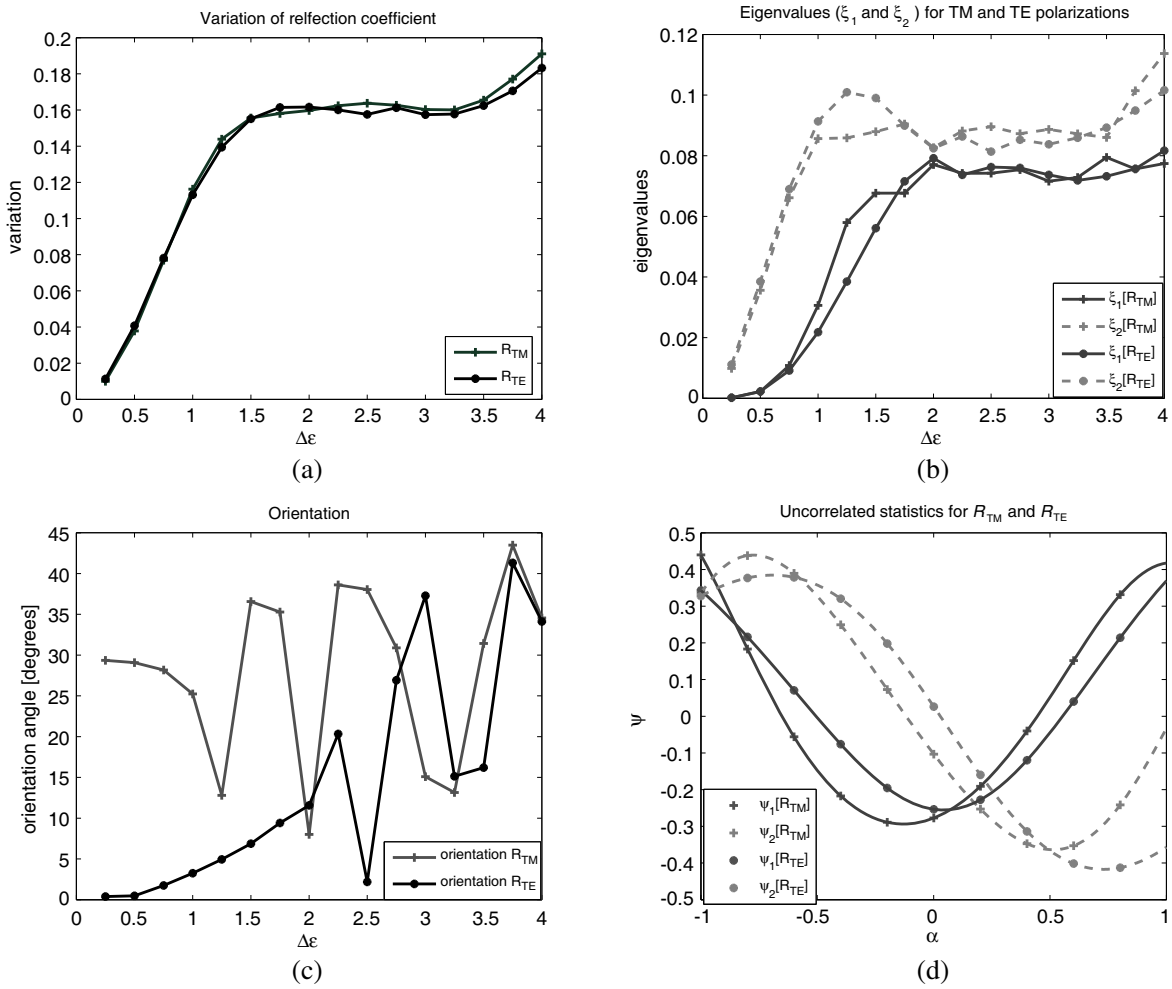

Figure 7. (a) Variation of the reflection coefficient for TM and TE polarizations as function of $\Delta\varepsilon_r$, (b) two eigenvalues (ξ_1 and ξ_2) for both TE and TM polarization, (c) the orientation of the eigenvector for TM and TE polarization, (d) uncorrelated statistics ψ_1 and ψ_2 based on Karhunen-Loève decomposition for $\Delta\varepsilon_r = 1.25$.

$\xi_2(\Delta\varepsilon_r)$ of the autocorrelation matrix (see (54)). The eigenvalues as function of $\Delta\varepsilon_r$ are displayed in Figure 7(b). We observe that the variation of both reflection coefficients is about the same. The eigenvalues are not; there is a clear difference observable for $0.75 < \Delta\varepsilon_r < 1.5$. We define the orientation of the basis of the eigenvectors (principle components) to be the smallest angle between each of these vectors and the x-axis. This orientation shown in Figure 7(c) shows a clearly distinguishable difference between TE and TM polarizations. The conclusion is that the difference in the statistics of the TM and the TE reflection coefficients is clearly observable in the orientation of the orthonormal basis. In Figure 7(d) we present the uncorrelated random variables ψ_1 and ψ_2 according to the Karhunen-Loève decomposition (see (55)) with $\Delta\varepsilon_r = 1.25$ for both polarizations. The qualitative behavior is the same, but there is a quantitative difference.

5. CONCLUSION

We introduced the concept of the random resolvent in relation to domain integral equations that describe the electric field inside a dielectric object with inhomogeneous random dielectric permittivity illuminated by an incident electric field. This concept is fruitfully applied in a novel method to solve the domain integral equation if the integral kernel depends affinely on the random variable or is well approximated

by its linearization with respect to that variable. We employ an expansion with respect to the unique set of monic orthogonal polynomials related to the probability measure on the sample interval that characterizes the randomness of the dielectric. In terms of this expansion, we describe mean, variation, and other properties of the random electric field or of its scattering characteristics. Thus we obtain the true characteristics of the random electric field at low computational costs in comparison to cost of methods based on a Monte-Carlo sampling strategy. The performance of the proposed method was verified on the basis of two cases: (1) computation of the mean and variation of the electric field inside a random inhomogeneous dielectric slab, and the associated reflection coefficient for different ranges of dielectric permittivity, different probability measures, and different frequencies; (2) computation of the mean and variation of the scattering coefficients for the TE and TM polarizations of a 2D dielectric periodic structure, and their uncorrelated statistics.

ACKNOWLEDGMENT

This research has been performed within the BESTCOM framework as part of the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy Office. The first author was partially funded by this programme.

APPENDIX A.

For the probability measure μ, the corresponding orthogonal polynomials satisfy a three-term recurrence relation

$$\alpha p_n(\alpha) = p_{n+1}(\alpha) + a_n p_n(\alpha) + b_n p_{n-1}(\alpha), \quad n = 1, 2, \ldots \tag{A1}$$

with initialization $p_0(\alpha) = 1$ and $p_1(\alpha) = \alpha - m_1$. We define the μ-associated functions by

$$q_n(z) = \int_J \frac{1}{z-\alpha} p_n(\alpha) d\mu(\alpha), \quad n = 0, 1, 2, \ldots \tag{A2}$$

where $z \in \mathbb{C}$, $z \notin J$, $J = [-1, 1]$. The functions are analytic on the complex plane cut by the segment J. We observe that

$$q_n(z) = p_n(z) q_0(z) - r_n(z) \tag{A3}$$

with the polynomial r_n of degree $n-1$ given by

$$r_n(z) = \int_J \frac{p_n(z) - p_n(\alpha)}{z-\alpha} d\mu(\alpha). \tag{A4}$$

In [20] the polynomials r_n are called the μ-associated polynomials. The μ-associated functions satisfy the same recurrence relation as the orthogonal polynomials p_n,

$$z q_n(z) = q_{n+1}(z) + a_n q_n(z) + b_n q_{n-1}(z) \tag{A5}$$

with $n \geq 1$ and $z q_0(z) = q_1(z) + m_1 q_0(z) + 1$. It follows that $q_1(z) = p_1(z) q_0(z) - 1$. We conclude that we can derive the polynomials r_n from the recurrence relation

$$z r_n(z) = r_{n+1}(z) + a_n r_n(z) + b_n r_{n-1}(z), \quad n = 1, 2, \ldots \tag{A6}$$

initialized by $r_0(z) = 0$ and $r_1(z) = 1$. Since

$$\frac{1}{1-\alpha\lambda} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\nu_n} q_n\left(\frac{1}{\lambda}\right) p_n(\alpha) \tag{A7}$$

we derive that $\rho_n(\lambda) = \frac{1}{\lambda \nu_n} q_n(\frac{1}{\lambda})$.

A.1. Legendre Polynomials, P_n

In this case we have $d\mu(\alpha) = \frac{1}{2}da$ with moments $m_{2k+1} = 0$, $m_{2k} = \frac{1}{2k+1}$ and the recurrence relations (see [29, Eq. (1.8.60)])

$$\alpha P_n(\alpha) = P_{n+1}(\alpha) + \frac{n^2}{4n^2-1}P_{n-1}(\alpha) \tag{A8a}$$

$$P_0(\alpha) = 1, \quad P_1(\alpha) = \alpha \tag{A8b}$$

The corresponding normalizations are given by

$$\nu_n = \frac{1}{2}\int_{-1}^{1} P_n(\alpha)^2 d\alpha = \frac{\pi\Gamma(n+1)\Gamma(n+1)}{2^{2n+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)} \tag{A9}$$

and the σ_n are 0. The μ-associated function $Q_0(z)$ can be calculated straightforwardly,

$$Q_0(z) = \frac{1}{2}\int_{-1}^{1} \frac{1}{z-\alpha}d\alpha = \frac{1}{2}\log\frac{z+1}{z-1}, \quad z \in C \setminus [-1,1] \tag{A10}$$

And thus for the μ-associated functions $Q_n(z)$ we have

$$Q_n(z) = \frac{1}{2}P_n(z)\log\frac{z-1}{z+1} - r_n(z), \quad z \in C \setminus [-1,1] \tag{A11}$$

with $r_0(z) = 0$ and $r_1(z) = 1$, and

$$zr_n(z) = r_{n+1}(z) + \frac{n^2}{4n^2-1}r_{n-1}(z), \quad n \geq 1 \tag{A12}$$

Consequently for $\lambda \in \mathbb{C} \setminus \{-1,1\}$

$$\frac{1}{2}\int_{-1}^{1} \frac{P_n(\alpha)}{1-\alpha\lambda}d\alpha = \frac{1}{\lambda}Q_n\left(\frac{1}{\lambda}\right) \tag{A13}$$

with the above interpretation.

A.2. Jacobi Polynomials, $P_n^{(p\cdot q)}$

The Jacobi polynomials are the system of orthogonal polynomials related to the measure $\mu^{(p,q)}$ on the interval $[-1,1]$ defined by the density function

$$w^{(p,q)}(\alpha) = \frac{1}{\omega_{(p,q)}}(1-\alpha)^p(1+\alpha)^q \tag{A14}$$

where p, $q > -1$ and where

$$\omega^{(p,q)} = 2^{p+q+1}B(p+1, q+1) \tag{A15}$$

The probability measures related to this density function are called beta distributions. Here $B(.,.)$ denotes the Beta function. The recursive relation satisfied by the monic Jacobi polynomials is quite involved. The Jacobi-Szegö coefficients are given by:

$$a_n^{(p,q)} = \frac{(q-p)(q+p)}{(2n+q+p)(2n+q+p+2)} \tag{A16a}$$

$$b_n^{(p,q)} = \begin{cases} \dfrac{4(1+p)(1+q)}{(2+q+p)^2(3+q+p)}, & n = 1 \\ \dfrac{4n(n+p)(n+q)(n+q+p)}{(2n+q+p-1)(2n+q+p)^2(2n+q+p+1)}, & n > 1 \end{cases} \tag{A16b}$$

The normalizations are obtained from $\nu_0^{(p,q)} = 1$, and

$$\nu_n^{(p,q)} = b_n^{(p,q)}\nu_{n-1}^{(p,q)}, \quad n = 1, 2, \ldots \tag{A17}$$

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