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Utilization of Riemann-Silberstein Vectors in Electromagnetics

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Abstract—Electromagnetic field modal expansion is traditionally an effective technique for solving Maxwell's Equations for numerous high-frequency engineering problems. In this paper, an alternative form of electromagnetic field representation is described. It is based on the Riemann-Silberstein vectors, which are a linear combination of the electric and magnetic field vectors. Utilizing such combination in homogeneous space, Maxwell's Equations are converted into a system of two independent equations. Under these circumstances, each vector describes the total electromagnetic field of an ideal circular polarization. Electromagnetic fields are simply expressed in the form of the Riemann-Silberstein vectors using the helical coordinate system and special functions, which form a set of generalized spherical harmonics. The new representation of vector spherical harmonics differs in simplicity and symmetry while having a more physically apparent expression. The amount of computational work is reduced due to the initial independence of the Riemann-Silberstein vectors. The purpose of this paper is to show the efficiency of a new approach that is based on Riemann-Silberstein vector field representation and spherical wave expansion.

1. INTRODUCTION

Maxwell's Equations in the steady state can be split into two independent equations for electric and magnetic fields. This property is used when constructing the quasi-static low-frequency asymptotic solution of Maxwell's Equations. Significantly less common in electromagnetic theory is the decomposition of Maxwell's Equations into two independent equations, which is valid at all frequencies including HF. It is based on the invariance of equations using linear combinations of the electric and magnetic fields, which we call the Riemann-Silberstein vectors (the RS vectors). Similar to the static asymptotic solution, it is possible to simplify the electromagnetic field representation and obtain a system of independent equations describing the complete electromagnetic field. We derive a general solution of Maxwell's equations directly as a set of orthogonal vector spherical harmonics for the RS vector field representation together with utilizing helical coordinates. Thus all of the above gives advantages for engineering problems and simplifies the spherical representation.

The Riemann-Silberstein vector theory is related to the well-known Beltrami vector fields. These fields were introduced in 1889 by Beltrami in [1] when working on hydrodynamics. Beltrami researched fields satisfying the condition $\nabla \times \mathbf{F} + \lambda\mathbf{F} = 0$ and pointed similar properties to the ones of electromagnetic fields described by Maxwell's equations. It is now commonly accepted and acknowledged. Although for the first time these fields were researched by the Russian scientist Gromeka in his doctoral dissertation 8 years before Beltrami [2]. It was published in Russian and is not known. The formulation of the RS vectors appeared for the first time in 1907 in Silberstein's work [3]. Silberstein noticed that such a vector made its first appearance in the lectures on partial differential equations by Riemann edited and published in 1901 by Weber [4]. The name the Riemann-Silberstein vectors was

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introduced by Bialynicki-Birula in his works on quantum electrodynamics in [5, 6], and we will use this name as the most reasonable for our formulation.

After Silberstein's work the same idea has been remarked upon during the 20th century. Von Laue [7] and Minkowski [8] introduced similar considerations in their extensive works on the theory of relativity. Bateman utilized a bivector form of an electromagnetic field $\mathbf{M} = \mathbf{H} \pm i\mathbf{E}$ in his book [9] in conjunction with hypercomlex calculus. A similar approach was developed by Lewin in his work on waveguides [10]. In [11] Rumsey considered the solutions of Maxwell's equations satisfying the condition $\mathbf{E} = \pm\, i\eta\mathbf{H}$. Rumsey did not introduce a new symbol for the solutions. This theory was applied to frequency-independent antennas. Utilization of Maxwell fields as complex Beltrami fields was developed by Lakhtakia in [12, 13]. In these works, the new field vectors were introduced and the Beltrami field theory was derived. The author applied the theory to chiral media. Lakhtakia also obtained solutions for the spherical Beltrami problem and extended the theory to vector spherical wavefunctions for orthorhombic dielectric-magnetic material with gyrotropic-like magnetoelectric properties [14].

In this paper, we operate with the Riemann-Silberstein vectors, which have the property of Beltrami fields. Although the Beltrami field theory has been utilized in electromagnetics and certain analysis has been done, the theory is rarely used in calculations and antenna design. This is due to the unclear possibilities concerning the application of the representation for actual engineering calculations and it's seeming complexity and redundancy. Nevertheless, it will be shown that utilization of the RS vectors yields advantages. In particular, the independence of the RS vectors and the clear comprehensive spherical representation.

The modal expansion method is widely used for antenna field calculations, field transformations, impedance calculations, etc. We develop the Riemann-Silberstein vector theory and introduce the field representation in a helical coordinate system with the corresponding vector spherical multipole expansion. In this representation the generalized spherical harmonics are introduced. Hence, we expand the fields into vector spherical harmonics with generalized spherical functions and use Lorentz reciprocity for the coefficients determination. It is in this matter that we adapt the vectors for engineering calculations, which a novel approach. Taking into account all the properties of the new representation, the theory developed can be used in numerous high-frequency engineering problems. The efficiency of the method is implied as the amount of computational work required for the complete field representation comparing to the traditional TE and TM waves expansion. Under certain circumstances only one of the vectors is needed for the representation of an electromagnetic field, thus we reduce the number of integrals calculation.

2. DETERMINATION OF RIEMANN-SILBERSTEIN VECTORS

For time-harmonic fields Maxwell's Equations with respect to complex magnitudes

$$\begin{cases} \nabla\times\, \mathbf{E} = -i\omega\mu\mathbf{H} - \mathbf{j}^m, \\ \nabla\times\mathbf{H} = i\omega\varepsilon\mathbf{E} + \mathbf{j}^e. \end{cases} \tag{1}$$

Another formulation of Maxwell's Equations can be obtained using a linear combination of the electric and magnetic field vectors, the Riemann-Silberstein vectors. In a linear isotropic lossless medium with constant in time parameters the Riemann-Silberstein vectors are

$$\mathbf{F}_\pm = 0.5\left(1/\sqrt{z}\,\mathbf{E}{\mp}i\sqrt{z}\,\mathbf{H}\right), \tag{2}$$

where $z = \sqrt{\mu/\varepsilon}$ — the wave impedance. In homogeneous space (grad $z \equiv 0$) Maxwell's Equations are converted into a system of two independent equations with the corresponding excitation sources according to

$$\nabla\times\mathbf{F}_\pm \mp k\mathbf{F}_\pm = \mathbf{j}_\pm, \tag{3}$$

with the equivalent current densities

$$\mathbf{j}_\pm = -0.5\left(1/\sqrt{z}\,\mathbf{j}^m \pm i\sqrt{z}\,\mathbf{j}^e\right). \tag{4}$$

This is Maxwell's Equations in terms of the Riemann-Silberstein vectors. The RS vectors $\mathbf{F}_+$ and $\mathbf{F}_-$ propagate independently from each other. Each vector $\mathbf{F}_+$ and $\mathbf{F}_-$ satisfies a vector differential equation of the first order. In homogeneous space, each vector $\mathbf{F}_+$ and $\mathbf{F}_-$ describes the total electromagnetic field of the ideal circular polarization [15]. The source $\mathbf{j}_+$ excites the right-handed circularly polarized far field in all directions, the source $\mathbf{j}_-$ excites the left-handed circularly polarized far field. Representation of the electromagnetic field in the RS vectors has a more physically apparent expression than it does in linearly-polarized components and in TE and TM waves. The representation in linearly-polarized components and in TE and TM waves depend on a choice of directions of coordinate system axes and is possible only in particular coordinate systems.

Poynting vector is also represented by the RS vectors

$$\mathbf{P} = \frac{1}{2} \cdot (\mathbf{E} \times \mathbf{H}^*) = -\frac{i}{2} \cdot \left((\mathbf{F}_+ \times \mathbf{F}_+^*) - (\mathbf{F}_- \times \mathbf{F}_-^*) \right) + \frac{i}{2} \cdot \left((\mathbf{F}_+ \times \mathbf{F}_-^*) - (\mathbf{F}_+^* \times \mathbf{F}_-) \right). \tag{5}$$

The first part of the right side is always real, whereas the second is imaginary. A clear physical meaning associated with the radiating power has only the real (active) part of the complex Poynting vector. It represents a vector sum of the Riemann-Silberstein vectors of right- and left-handed circular polarization. Each vector $\mathbf{F}_+$ and $\mathbf{F}_-$ conveys its energy apart from the other. Thus, the vectors are independent, and the directed power is added according to the vector addition rule.

From Maxwell's Equation (3) a statement of energy conservation is derived, known as Poynting's theorem

$$\frac{i}{2} \left(\int_S \left(\mathbf{F}_+ \times \mathbf{F}_+^* \right) \cdot \mathbf{n} da + \int_S \left(\mathbf{F}_- \times \mathbf{F}_-^* \right) \cdot \mathbf{n} da \right) = -\int_V (\mathbf{j}_+ \cdot \mathbf{F}_+^*) dv - \int_V (\mathbf{j}_- \cdot \mathbf{F}_-^*) dv. \tag{6}$$

The surface integral on the left side of Eq. (6) is a flow of electromagnetic energy across the surface which bounds V. The right side is the energy expended by the flow of charge.

Field discontinuities on a boundary surface S between two mediums with the parameters $\varepsilon_1, \mu_1, \sigma_1$ and $\varepsilon_2, \mu_2, \sigma_2$ are described by the boundary condition. The boundary conditions for the normal RS vector components

$$\begin{aligned} &(1/\sqrt{\mu_2 \varepsilon_2}\, \mathbf{F}_{\pm 2} - 1/\sqrt{\mu_1 \varepsilon_1}\, \mathbf{F}_{\pm 1}) \cdot \mathbf{n} = \sigma_\pm , \\ &\sigma_\pm = \lim_{\Delta l \to 0} (\rho_\pm \Delta l) = \lim_{\Delta l \to 0} \left(-0.5 \left(1/\sqrt{z}\ \rho^m \pm i\sqrt{z}\ \rho^e \right) \Delta l \right). \end{aligned} \tag{7}$$

where $\mathbf{F}_{\pm 1}$ and $\mathbf{F}_{\pm 2}$ are the field vectors in mediums 1 and 2, respectively, and $\sigma_\pm$ is a surface charge density. The boundary conditions for the tangential RS vector components introducing a surface current density $\mathbf{K}_\pm$

$$\begin{aligned} &\mathbf{n} \times (\mathbf{F}_{\pm 2} - \mathbf{F}_{\pm 1}) = \mathbf{K}_\pm, \\ &\mathbf{K}_\pm = \lim_{\Delta l \to 0} (\mathbf{j}_\pm \Delta l). \end{aligned} \tag{8}$$

If conductivities of the contiguous media are finite, there is no surface current and the right side of Eq. (8) is zero.

3. COMMON ELECTROMAGNETIC FIELD EXPRESSIONS AND IDENTITIES

In free space electromagnetic fields satisfying Maxwell's Equation (3) are expressed through the vector potentials

$$\mathbf{F}_\pm = \pm k \mathbf{A}_\pm + \nabla \times \mathbf{A}_\pm \pm \frac{1}{k} \nabla (\nabla \cdot \mathbf{A}_\pm). \tag{9}$$

Using the vector potentials, the Helmholtz equation is derived

$$\nabla^2 \mathbf{A}_\pm + k^2 \mathbf{A}_\pm = -j_\pm. \tag{10}$$

The solution of the inhomogeneous Helmholtz equation is a convolution

$$\mathbf{A}_\pm(q) = \int_V G_{pq} \cdot \mathbf{j}_\pm(p) dv, \tag{11}$$

where q is a source point, p an observation point, and G_{pq} the Green's function.

3.1. Hertzian Dipole Field

Take into consideration the Hertzian dipole directed in $\vec{a}$ direction

$$\mathbf{j} = \mathbf{a}\delta(p-q), \quad \mathbf{A} = \mathbf{a}G_{pq}. \tag{12}$$

Plugging this expression into Eq. (9) and using vector calculus identities, we obtain

$$\mathbf{F}_{\pm} = \frac{\partial G_{pq}}{\partial r_{pq}}(\mathbf{r}_{\mathrm{pq}} \times \mathbf{a}) \pm \frac{1}{k}\frac{\partial^2 G_{pq}}{\partial r_{pq}^2}(\mathbf{r}_{\mathrm{pq}} \times (\mathbf{r}_{\mathrm{pq}} \times \mathbf{a})). \tag{13}$$

where $\mathbf{r}_{\mathrm{pq}}$ is a vector from the source point q to the point of observation p. This is the Hertzian dipole field expression in the RS vectors, applicable to any coordinate system.

3.2. Lorentz Reciprocity and Stratton-Chu Solution

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the Riemann-Silberstein vectors of the same circular polarization: LHCP or RHCP. The vectors of the opposite polarizations $\mathbf{F}_+$ and $\mathbf{F}_-$ satisfy different equations and propagate independently from each other. Then Lorentz reciprocity for any chosen vector is derived from (3) and written according to

$$\int_S (\mathbf{F}_1 \times \mathbf{F}_2) \cdot \mathbf{n} da = \int_V [(\mathbf{j}_1 \cdot \mathbf{F}_2) - (\mathbf{j}_2 \cdot \mathbf{F}_1)]\, dv. \tag{14}$$

This expression can be used for field expression outside of a closed surface through the tangential field components on the surface (Fig. 1). For this reason, we use Lorentz reciprocity, assuming that the field $\mathbf{F}_1$ has to be found in volume V, $\mathbf{F}_2 = \mathbf{F}_{\mathrm{D}}$ is the field of the Hertzian dipole

$$(\mathbf{a} \cdot \mathbf{F}(p)) = \int_S (\mathbf{F}_{\mathrm{D}} \cdot \mathbf{J}) da + \int_V (\mathbf{j} \cdot \mathbf{F}_{\mathrm{D}}) dv. \tag{15}$$

where $\mathbf{J} = \mathbf{F} \times \mathbf{n}$ — surface current density.

In the free source area the vector field in $\vec{a}$ direction

$$(\mathbf{F}(p) \cdot \mathbf{a}) = -\int_S \left(\left((\mathbf{r}_{\mathrm{pq}} \times \mathbf{a})\frac{\partial G_{pq}}{\partial r_{pq}} \pm \frac{1}{k}(\mathbf{r}_{\mathrm{pq}} \times (\mathbf{r}_{\mathrm{pq}} \times \mathbf{a}))\frac{\partial^2 G_{pq}}{\partial r_{pq}^2}\right) \cdot \mathbf{J}\right) ds. \tag{16}$$

In the form of Stratton-Chu solution, the field is expressed through the tangential and normal field components on the surface

$$(\mathbf{F}(p) \cdot \mathbf{a}) = \int_S ((\{\pm k G_{pq}\mathbf{a} - (\mathrm{grad}G_{pq} \times \mathbf{a})\} \cdot (\mathbf{n} \times \mathbf{F}))\; - (\mathrm{grad}G_{pq} \cdot \mathbf{a})\,(\mathbf{F} \cdot \mathbf{n}))\, ds. \tag{17}$$

This expression can be used for the calculation of a field directly from a given current distribution.

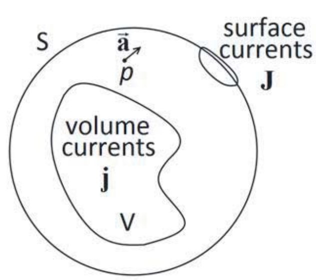


Figure 1. Field definition trough the volume and surface currents.

4. CYLINDRICAL WAVES AND HOLLOW WAVEGUIDE FIELDS

In cylindrical coordinates homogeneous Maxwell's Equations for the Riemann-Silberstein vectors

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial F_{z\pm}}{\partial \varphi} - \dfrac{\partial F_{\varphi\pm}}{\partial z} \mp kF_{r\pm} = 0, \\ \dfrac{\partial F_{r\pm}}{\partial z} - \dfrac{\partial F_{z\pm}}{\partial r} \mp kF_{\varphi\pm} = 0, \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(\rho F_{\varphi\pm}) - \dfrac{1}{r}\dfrac{\partial F_{r\pm}}{\partial \varphi} \mp kF_{z\pm} = 0. \end{cases} \tag{18}$$

Solution by variable separation

$$\begin{cases} F_r = A(r)e^{j(m\varphi-\beta z)}, \\ F_\varphi = B(r)e^{j(m\varphi-\beta z)}, \\ F_z = C(r)e^{j(m\varphi-\beta z)}. \end{cases} \tag{19}$$

Substituting Eq. (19) into Eq. (18) leads to the Bessel equation with respect to the function $C(r)$. Thus, for φ-dependence, the solution is exponential functions, and in the radial direction the solution is Bessel functions of the first, second or the third kind.

$$\begin{cases} F_{r\pm} = \dfrac{j}{g^2}\left(\pm\dfrac{km}{r}Z_m(gr) - g\beta Z'_m(gr)\right)e^{j(\pm m\varphi-\beta z)}, \\ F_{\varphi\pm} = \dfrac{1}{g^2}\left(\dfrac{\beta m}{r}Z_m(gr) \mp gkZ'_m(gr)\right)e^{j(\pm m\varphi-\beta z)}, \\ F_{z\pm} = Z_m(gr)e^{j(\pm m\varphi-\beta z)}. \end{cases} \tag{20}$$

Now we are able to find a solution for a wave propagating in a circular metallic waveguide for Riemann-Silberstein waves propagation (Fig. 2). We assume that the fields are time-harmonic, the medium is homogeneous and isotropic, no free sources are inside the waveguide, and the walls are PEC.

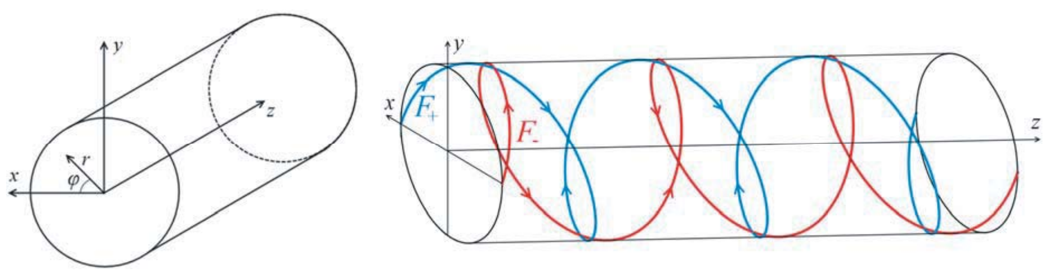


Figure 2. (a) Cylindrical hollow waveguide: 3D view; (b) propagation of the RS vectors — RHCP (F_+) and LHCP (F_-).

We solve Maxwell's Equations for the Riemann-Silberstein vectors to find eigenmodes in the circular waveguide. Assuming the wave propagates in Z direction $\mathbf{F} = \mathbf{F}_0(r,\varphi)e^{-j\beta z}$, we obtain the Helmholtz equation in cylindrical coordinates. The solution is written in the form of Eq. (20), choosing propagation waves as Bessel's functions of the first kind

$$\begin{cases} F_{r\pm} = C_{m\pm}\dfrac{j}{g^2}\left(\pm\dfrac{km}{r}J_m(gr) - g\beta J'_m(gr)\right)e^{j(\pm m\varphi-\beta z)}, \\ F_{\varphi\pm} = C_{m\pm}\dfrac{1}{g^2}\left(\dfrac{\beta m}{r}J_m(gr) \mp gkJ'_m(gr)\right)e^{j(\pm m\varphi-\beta z)}, \\ F_{z\pm} = C_{m\pm}J_m(gr)e^{j(\pm m\varphi-\beta z)}. \end{cases} \tag{21}$$

Consider the boundary conditions for the vectors $\mathbf{F}_+$ and $\mathbf{F}_-$. On PEC walls, the tangential component of the electric field must be zero, i.e., $E_\tau = 0 \longrightarrow\Rightarrow F_{+\tau} + F_{-\tau} = 0$. Therefore, reflecting from PEC walls, the field $\mathbf{F}_+$ is converted into $\mathbf{F}_-$.

Under certain circumstances, a corrugated circular waveguide with small longitudinal slots has a property of $E_\varphi = 0$, $H_\varphi = 0$. Vectors $\mathbf{F}_+$ and $\mathbf{F}_-$ propagate independently and a wave does not change its polarization — the waveguide propagating field has no cross polarization component.

The condition $E_\tau|_{r=a} = 0$ is valid for a smooth waveguide

$$\begin{cases} F_{\varphi+} + F_{\varphi-}|_{r=a} = 0, \\ F_{z+} + F_{z-}|_{r=a} = 0, \end{cases} \tag{22}$$

which leads to the dispersion equation

$$\begin{cases} C_{m+}\left[\dfrac{m\beta}{r}J_m(ga) - gkJ'_m(ga)\right] + C_{m-}\left[\dfrac{m\beta}{r}J_m(ga) + gkJ'_m(ga)\right] = 0, \\ C_{m+}J_m(ga) + C_{m-}J_m(ga) = 0. \end{cases} \tag{23}$$

Two roots of the dispersion equation are obtained when solving this system of equations. Consider two possible cases

$$\begin{cases} C_{m+} = -C_{m-} \\ C_{m+} = C_{m-} \end{cases} \Rightarrow \begin{cases} J_m(ga) = 0 \Rightarrow ga = \nu_{mn}, \\ J'_m(ga) = 0 \Rightarrow ga = \mu_{mn}. \end{cases} \tag{24}$$

From that result, we can derive the solution for standard TE and TM waves. For the TE waves $E_\mathrm{z} = 0$, and in order to satisfy a Neumann boundary condition, we choose the second root of Eq. (24). For the TM waves $H_\mathrm{z} = 0$, and in order to satisfy the Dirichlet boundary condition, we choose the first root of Eq. (24). Furthermore, the electric and magnetic field components can be found by applying reciprocal Equation (2). For example, the TE_{11} wave field

$$\begin{cases} H_r = C_{mn}j\dfrac{a\beta}{\sqrt{z}\mu_{mn}}J'_m(\dfrac{\mu_{mn}r}{a})e^{j(m\varphi-\beta z)} \\ H_\varphi = C_{mn}j\dfrac{a^2\beta m}{\sqrt{z}\mu_{mn}^2 r}J_m(\dfrac{\mu_{mn}r}{a})e^{j(m\varphi-\beta z)} \\ H_z = C_{mn}J_m\left(\dfrac{\mu_{mn}r}{a}\right)e^{j(m\varphi-\beta z)} \end{cases} \begin{cases} E_r = \dfrac{\omega\mu_0}{\beta}H_\varphi \\ E_\varphi = -\dfrac{\omega\mu_0}{\beta}H_r \\ E_z = 0 \end{cases}. \tag{25}$$

5. VECTOR SPHERICAL HARMONICS IN CIRCULAR POLARIZATION BASIS OF THE RIEMANN-SILBERSTEIN VECTORS

The modal expansion method is based on the fact that an arbitrary electromagnetic field pattern can be expressed as a linear combination of a set of orthogonal solutions in an appropriate coordinate system. This method is widely used for antenna field calculations, near-field far-field transformations and radiation pattern calculations, mutual impedance calculations and also in other high-frequency engineering problems. Traditional approaches, formulated by Mie and Debye in the early 20th century, utilize electromagnetic multipoles as its basis.

A more effective technique of spherical functions application was created by mathematicians in the theory of group representations [16, 17]. This theory, which was initiated by the needs of quantum physics, uses special functions which are invariantly defined relative to the group of rotations in order to represent an arbitrary vector and physical fields. Within this theory, electromagnetic fields are simply expressed in the form of the Riemann-Silberstein vectors.

Instead of the spherical coordinates $(\mathbf{e}_\vartheta, \mathbf{e}_\varphi)$, the helicity basis vectors $(\mathbf{e}^{+1}, \mathbf{e}^{-1})$ are used. For outgoing waves rotation of one of the Riemann-Silberstein vectors in the helical coordinate system coincides with a circular polarization direction of this vector. A vector component with a consistent direction of rotation has the property of an incident wave. A vector component with an opposite direction of rotation has the property of a reflected wave. The radial component is responsible for a vortex field, tangent to the sphere. The relation between the helical and the spherical unit vectors is

$$\mathbf{e}^{+1} = -\frac{1}{\sqrt{2}}(\mathbf{e}_\vartheta - i\mathbf{e}_\varphi), \quad \mathbf{e}_0 = \mathbf{e}_r, \quad \mathbf{e}^{-1} = \frac{1}{\sqrt{2}}(\mathbf{e}_\vartheta + i\mathbf{e}_\varphi). \tag{26}$$

An arbitrary vector $\mathbf{f}$ is expressed in the helical coordinates as follows

$$\begin{aligned} &\mathbf{f} = f_r\mathbf{e}_r + f_\vartheta\mathbf{e}_\vartheta + f_\varphi\mathbf{e}_\varphi = f_{+1}\mathbf{e}^{+1} + f_0\mathbf{e}^0 + f_{-1}\mathbf{e}^{-1}, \\ &f_{+1} = -\frac{1}{\sqrt{2}}(f_\vartheta + if_\varphi), \quad f_0 = f_r, \quad f_{-1} = \frac{1}{\sqrt{2}}(f_\vartheta - if_\varphi). \end{aligned} \tag{27}$$

Electromagnetic waves satisfying the radiation condition have only one component in the far-field antenna region: $\mathbf{e}^{+1}$ for the right-handed circularly polarized field and $\mathbf{e}^{-1}$ for the left-handed circularly polarized field.

The angular dependence of the Riemann-Silberstein vectors spherical harmonics in the helical coordinates is simply expressed through D-Wigner functions. They were introduced in 1930's quantum mechanics to formalize computations with vector-and-tensor physical objects. These functions have a well-developed mathematical description, as shown in [18].

From the spherical coordinates and components θ and φ we transform the expression to the circular polarization components. The radial component is the same. Curl expression in the helical coordinates system

$$\nabla\times\mathbf{F}_\pm = \begin{cases} i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)F_1 - \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\vartheta} + \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)F_0, \\ \frac{1}{r\sqrt{2}\sin\vartheta}\left[\left(\frac{\partial}{\partial\vartheta}\sin\vartheta - i\frac{\partial}{\partial\varphi}\right)F_1 - \left(\frac{\partial}{\partial\vartheta}\sin\vartheta + i\frac{\partial}{\partial\varphi}\right)F_{-1}\right], \\ \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\vartheta} - \frac{i}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)F_0 - i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)F_{-1}. \end{cases} \tag{28}$$

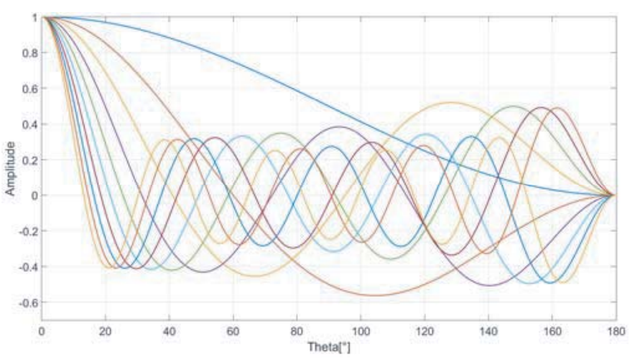


Figure 3. First 10 functions $P^n_{m,l}(\cos\vartheta)$.

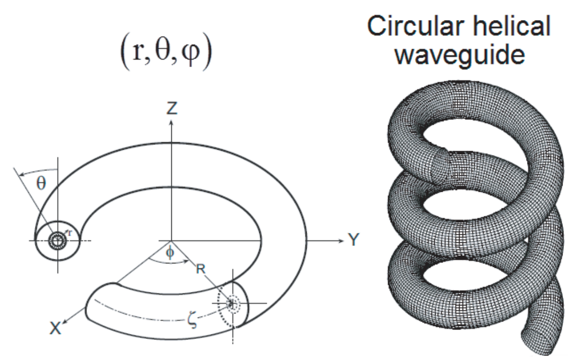


Figure 4. Helical coordinate system.

We solve Maxwell's Equation (3) in the helical coordinates system to derive multipoles field equations for the RS vectors. For the RHCP vector $\mathbf{F}_+$ solution by variation separation

$$\mathbf{F}_+ = \begin{pmatrix} a_1(r)P^n_{m,1}(\cos\vartheta) \\ a_0(r)P^n_{m,0}(\cos\vartheta) \\ a_{-1}(r)P^n_{m,-1}(\cos\vartheta) \end{pmatrix} e^{-im\varphi}. \tag{29}$$

In this expression, the functions $P^n_{m,l}(\cos\vartheta)$ are introduced. Rotation of three-dimensional space can be defined with Euler's angles φ, θ, ψ, where angles φ, θ define spherical coordinates of the new system position, and ψ is the angle of the system rotation relative to the new axis.

For an invariant expression of physical quantities, so-called "matrix elements of irreducible representations of the rotation group" are used. They are expressed through the Euler angles [16]. Functions $P^n_{m,l}(\cos\vartheta)$ are defined in an explicit form. Functions $P^n_{m,l}(\cos\vartheta)$ are defined in an explicit form. There are relations of symmetry, relations with Legendre functions and D-Wigner functions, and also recurrence relations [16–18]

$$\begin{aligned}
&P_{m,l}^{n}(\cos\vartheta)\, e^{-i(m\varphi+l\psi)}, \quad n \geq \max\left(|m|, |l|\right),\\
&P_{m,l}^{n}(\cos\vartheta)\, e^{-i(m\varphi+l\psi)} = (-i)^{m-l}\cdot D_{m,l}^{n}(\vartheta),\\
&\sqrt{1-z^2}\frac{dP_{m,l}^{n}(z)}{dz} \pm \frac{lz-m}{\sqrt{1-z^2}}P_{m,l}^{n}(z) = -i\sqrt{(n\mp l)(n\pm l+1)}P_{m,l\pm 1}^{n}(z).
\end{aligned} \tag{30}$$

Scalar functions correspond to the matrix elements with the index $l = 0$ and describe the radial component of the field. For representation of a vector function the index $l = \pm 1$ is used. "Plus" is for a right circular polarization wave propagating in $\mathbf{r}$ direction, and "minus" is for a right circular polarization wave propagating in $-\mathbf{r}$ direction. In the far-field we need only $l = +1$ for the RHCP wave and $l = -1$ for the LHCP. Therefore, only functions $P_{m,l}^{n}(\cos\vartheta)$ with the indices $l = 0, \pm 1$ are needed.

In a scalar case, the functions describe the radial component of the vector field, and up to a constant factor, correspond to associated Legendre polynomials [18]

$$P_n^m(z) = i^m\sqrt{(n+m)!/(n-m)!}P_{m,0}^{n}(z). \tag{31}$$

The functions $\{e^{-im\varphi}P_{m,l}^{n}(\cos\vartheta)\}$ form an orthogonal basis for components of an arbitrary square-integrable vector function $\mathbf{F}(\vartheta,\varphi) = (F_1, F_0, F_{-1})$ defined on the sphere. F_1 is the component of the right-hand circular polarized field propagating in Z direction, F_{-1} the component of the RHCP field propagating in $-Z$ direction, and F_0 the radial component. The functions can be interpreted as generalized spherical harmonics.

Orthogonality relations for the functions $\{e^{-im\varphi}P_{m,l}^{n}(\cos\vartheta)\}$ are the same as for Legendre polynomials

$$\int_0^{2\pi}\int_0^{\pi} e^{i(m_1-m_2)\varphi}P_{m_1,l}^{n_1}(\cos\vartheta)\overline{P_{m_2,l}^{n_2}(\cos\vartheta)}\sin\vartheta\, d\vartheta d\varphi = 4\pi/(2n_1+1)\delta_{m1,m2}^{n1,n2}. \tag{32}$$

Substituting Eq. (29) into Eq. (3) and using recurrence relations we obtain the equation system with respect to the radial functions [19, 20]

$$\begin{pmatrix}
i\dfrac{d}{dr}+\dfrac{i}{r}-k & \dfrac{-i}{r}\sqrt{\dfrac{n(n+1)}{2}} & 0\\
\dfrac{i}{r}\sqrt{\dfrac{n(n+1)}{2}} & -k & \dfrac{-i}{r}\sqrt{\dfrac{n(n+1)}{2}}\\
0 & \dfrac{i}{r}\sqrt{\dfrac{n(n+1)}{2}} & -i\dfrac{d}{dr}-\dfrac{i}{r}-k
\end{pmatrix}
\begin{pmatrix} a_1(r)\\ a_0(r)\\ a_{-1}(r)\end{pmatrix} = 0. \tag{33}$$

Expression of the functions $a_1(r)$, $a_{-1}(r)$ through $a_0(r)$ leads to the spherical Bessel differential equation

$$\begin{aligned}
&\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\left(k^2-\frac{n(n+1)}{r^2}\right)\right)(ra_0(r)) = 0,\\
&a_0(r) = \sqrt{\frac{n(n+1)}{2}}\,\frac{1}{r}\,z_n(kr),
\end{aligned} \tag{34}$$

where $z_n(kr)$ — Bessel spherical functions. The solution that satisfies the radiation condition is the Hankel function of the second kind.

Finally, the Riemann-Silberstein vector spherical harmonic

$$\mathbf{F}_{\pm} = e^{-im\varphi}\begin{pmatrix}
P_{m,1}^{n}(\cos\vartheta)\left(\dfrac{d}{dr}+\dfrac{1}{r}\mp ik\right)\\
P_{m,0}^{n}(\cos\vartheta)\sqrt{\dfrac{n(n+1)}{2}}\,\frac{1}{r}\\
P_{m,-1}^{n}(\cos\vartheta)\left(\dfrac{d}{dr}+\dfrac{1}{r}\pm ik\right)
\end{pmatrix} z_n(kr). \tag{35}$$

When $n = 1$ the expression (35) represents the fields of three elementary radiators of the RHCP. The index $m = 1$ corresponds to the Huygens element radiating in the axial direction; $m = -1$ to the Huygens element in the opposite direction; $m = 0$ to the electric dipole.

For the far-field representation, we use an asymptotic expression of the spherical Hankel function

$$h_n^{(2)}(kr) \underset{kr\to\infty}{=} i^{n+1}\frac{e^{-ikr}}{kr}\left[1+O\left(\frac{1}{kr}\right)\right]. \tag{36}$$

Thus, in the far-field region there is only one component of each vector

$$\mathbf{F}_+ = 2i^n\frac{e^{-ikr}}{r}\begin{pmatrix} P^n_{m,1}(\cos\vartheta) \\ 0 \\ 0 \end{pmatrix} e^{-im\varphi}, \quad \mathbf{F}_- = 2i^n\frac{e^{-ikr}}{r}\begin{pmatrix} 0 \\ 0 \\ P^n_{m,-1}(\cos\vartheta) \end{pmatrix} e^{-im\varphi}. \tag{37}$$

We have obtained vector spherical expansion for the Riemann-Silberstein vectors field representation. Such expansion uses $P^n_{m,l}(\cos\vartheta)$ functions which are similar to D-Wigner functions for the 3D case. The Wigner D functions give matrix elements of the rotation operator R. These functions can be used as generalized spherical harmonics. Using reverse transformations, from the vectors $\mathbf{F}_+$ and $\mathbf{F}_-$, it is simple to obtain a solution written in traditional vectors $\mathbf{E}$, $\mathbf{H}$, and equations for electric and magnetic multipoles.

6. FIELD EXPANSION IN VECTOR SPHERICAL HARMONICS OF THE RS VECTORS

An arbitrary vector function $\mathbf{f}(r,\vartheta,\varphi) = f_r\mathbf{e}_r + f_\vartheta\mathbf{e}_\vartheta + f_\varphi\mathbf{e}_\varphi = f_{+1}\mathbf{e}^{+1} + f_0\mathbf{e}^0 + f_{-1}\mathbf{e}^{-1}$ on the sphere $r = r_0$ can be represented by a convergent series [17]

$$\mathbf{f}(r_0,\theta,\varphi) = \sum_n \sum_{m=-n}^{n} \sum_{l=-1}^{1} C^n_{m,l}\, P^n_{m,l}(\cos\vartheta)\, e^{-im\varphi} \tag{38}$$

In Eq. (38) $C^n_{m,l}$ are the Fourier transform or the expansion coefficients

$$C^n_{m,l} = \frac{(-1)^{m-l}(2n+1)}{16\pi^2}\int_0^{2\pi}\int_0^{\pi} f_l(r_0,\vartheta,\varphi)e^{im\varphi}P^n_{m,l}(\cos\vartheta)\sin\vartheta\, d\vartheta\, d\varphi. \tag{39}$$

We also consider the radial functions as in the representation of Eq. (35) in order to obtain the vector field at any arbitrary point. The complete field $\mathbf{f}(r,\vartheta,\varphi)$ can be represented as follows

$$\mathbf{f}(r,\theta,\varphi) = \sum_n \sum_{m=-n}^{n} \sum_{l=-1}^{1} C^n_m\, P^n_{m,l}(\cos\vartheta)\, e^{-im\varphi}\, D_l z_n(kr) \tag{40}$$

In Eq. (40) $D_l\, z_n(kr)$ are the corresponding radial functions with its differential operators, defined according to

$$D_{\pm1}z_n(kr) = \left(\frac{d}{dr}+\frac{1}{r}\mp ik\right)z_n(kr), \quad D_0 z_n(kr) = \sqrt{\frac{n(n+1)}{2}}\frac{1}{r}z_n(kr) \tag{41}$$

In order to define the coefficients expression in a general form consider a spherical waveguide shown in Fig. 5.

The general solution is given by

$$\mathbf{F} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C^n_m\, \mathbf{F}^n_m \tag{42}$$

where $\mathbf{F}_N$ is a vector spherical harmonic defined by Eqs. (35), (37), and C_N is an expansion coefficient. We consider the outward-propagating field supposing that we have obtained the vector field components

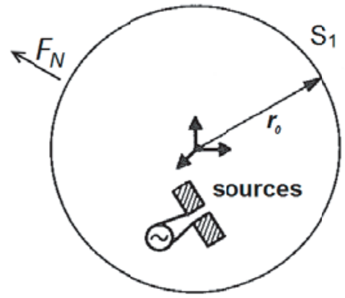


Figure 5. Spherical waveguide.

on the sphere r_0, e.g., a measured near-field radiated from a planar aperture. Lorentz reciprocity in the source-free region outside of the sphere

$$\oint_{S_1} \left(\left[\mathbf{F}, \mathbf{F}_N^*\right], \mathbf{n}\right) ds = \oint_{S_2} \left(\left[\mathbf{F}, \mathbf{F}_N^*\right], \mathbf{n}\right) ds \tag{43}$$

In Eq. (43), S_1 is the sphere which has the radius r_0, and S_2 is the infinite sphere $r \to \infty$. On the left side, the field $\mathbf{F}$ is the field obtained on the sphere S_1. The expanded field on the right side is written according to Eq. (37)

$$\mathbf{F} = 2\frac{e^{-ikr}}{r} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} C_m^n \, i^n \begin{pmatrix} P_{m,1}^n(\cos\vartheta) \\ 0 \\ 0 \end{pmatrix} e^{-im\varphi} \tag{44}$$

Notice some properties of a helical coordinate system. According to Eq. (26)

$$\begin{aligned} \mathbf{A}^* &= A_{+1}^* \left(\mathbf{e}^{+1}\right)^* + A_0^* \left(\mathbf{e}^{0}\right)^* + A_{-1}^* \left(\mathbf{e}^{-1}\right)^* = -A_{-1}^* \mathbf{e}^{+1} + A_0^* \mathbf{e}^0 - A_{+1}^* \mathbf{e}^{-1} \\ [\mathbf{A}, \mathbf{B}^*] &= -i \left(A_0 B_{-1}^* + A_{+1} B_0^*\right) \mathbf{e}^{+1} - i \left(A_{-1} B_{-1}^* - A_{+1} B_{+1}^*\right) \mathbf{e}^0 + i \left(A_{-1} B_0^* + A_0 B_{+1}^*\right) \mathbf{e}^{-1} \end{aligned} \tag{45}$$

Substituting Eqs. (44), (37) and (35) into Eq. (43) and considering Eq. (45) finally the expansion coefficients are

$$\begin{aligned} C_m^n \;=\; & \frac{2n+1}{16\pi} \int_0^{2\pi} \int_0^{\pi} \left(f_{+1}(r_0, \vartheta, \varphi) \left(P_{m,+1}^n(\cos\vartheta)\right)^* \left(a_{+1}(r_0)\right)^*\right. \\ & \left. - f_{-1}(r_0, \vartheta, \varphi) \left(P_{m,-1}^n(\cos\vartheta)\right)^* \left(a_{-1}(r_0)\right)^*\right) e^{im\varphi} r_0^2 \sin\vartheta \, d\vartheta d\varphi. \end{aligned} \tag{46}$$

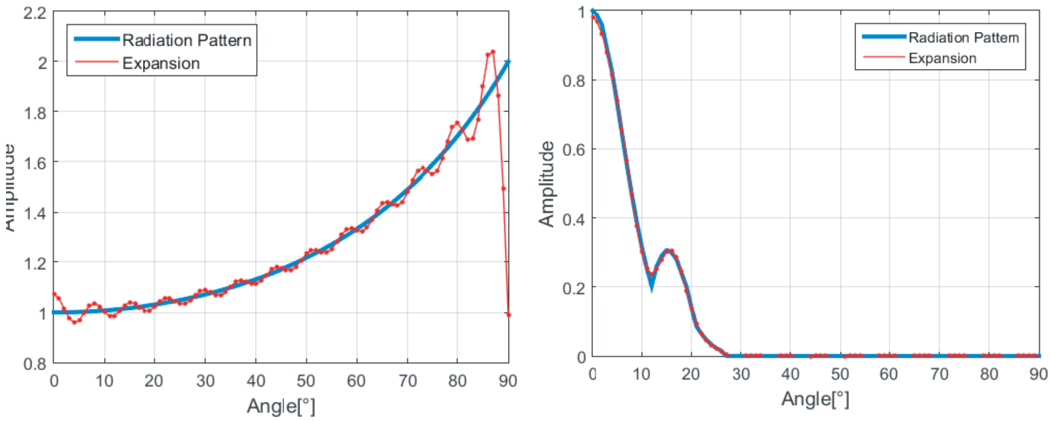
The complete field at any point of space can be found through the tangential field components on the surface.

The simplicity of the electromagnetic field vector expansion and RS vectors application is demonstrated by the example of a plane wave incidence on a reflector parabolic antenna. The reflected field found with the geometrical optics method

$$f(r, \vartheta) = \sec^2\left(\frac{\vartheta}{2}\right) e^{-i\varphi} \frac{e^{-ikr}}{r} \tag{47}$$

We take into consideration only harmonics with the index $m = 1$ assuming that radiation is unidirectional. The expanded field is written in the form of Eq. (40) with the spherical Bessel functions of the first kind. Radiation pattern expansions in $P_{m,l}^n(\cos\vartheta)$ functions are shown in Fig. 6 in two examples. The number of harmonics $N = 50$.

For an invariant representation of the complete field, e.g., near field, the vectors $\mathbf{F}_+$ and $\mathbf{F}_-$ are transformed into the helical coordinates components according to Eq. (27). Then each vector component is expanded into the vector spherical harmonics series on the sphere. Using independence of the vectors



(a) (b)

Figure 6. (a) Amplitude radiation pattern expansion: reflected field from a parabolic antenna irradiated by an incident plane wave; (b) arbitrary feed radiation pattern.

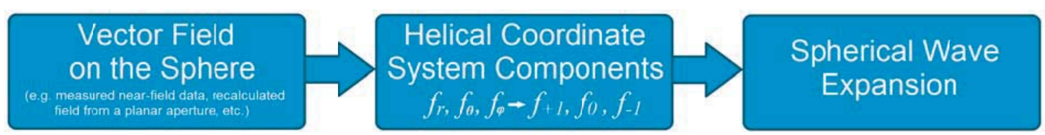


Figure 7. Field expansion steps.

$\mathbf{F}_+$ and $\mathbf{F}_-$ and symmetry relations for the functions $P^n_{m,l}(\cos\vartheta)$, it is possible to reduce the amount of computational work involved.

Thus, the procedure of the spherical wave expansion for the RS vector field representation is the following:

1) If required, calculation of the field spherical components on the sphere (using Stratton-Chu solution, Lorentz reciprocity, etc.);
2) RS vectors components calculation on the sphere r_0 from the field, obtained by measurements or computations;
3) Transformation of the spherical components into the helical coordinate system components;
4) Spherical wave expansion for each vector $\mathbf{F}_+$ and $\mathbf{F}_-$, utilizing representation (40)–(41), (46).

The results can be obtained directly for a circularly polarized field. For example, if the field is right-hand circular polarized, it is represented in the form of Eq. (37) in the far-field region. Therefore, one can obtain the results for the co-polarized far-field directly utilizing the expansion in Eqs. (40)–(41), (46). The number of coefficients calculation is reduced by half compared to TE and TM-wave expansion. The spherical harmonics are represented in an explicit form. In order to obtain the results for the electric and magnetic field vectors, both vectors $\mathbf{F}_+$ and $\mathbf{F}_-$ have to be considered, and the inverse conversion is performed.

Application of the Riemann-Silberstein vectors also gives some advantages in diffraction theory. In particular, boundary conditions can be defined which do not cause excitation of a cross-polarization component for a circularly polarized field.

As a result, the electromagnetic field in terms of the Riemann-Silberstein vectors representation is presented in Fig. 7 calculated according to Equations (40)–(41), (46) for the parabolic reflector antenna. The power distribution in the focus area is found according to Equation (5). The focal distance is $F = 3000\,\text{mm}$, and the radius is $R = 6000\,\text{mm}$. Frequency $f = 3\,\text{GHz}$. The field distribution on the reflector surface is constant. The aperture illumination efficiency is maximal for the antenna with a feed field distribution of the secant function.

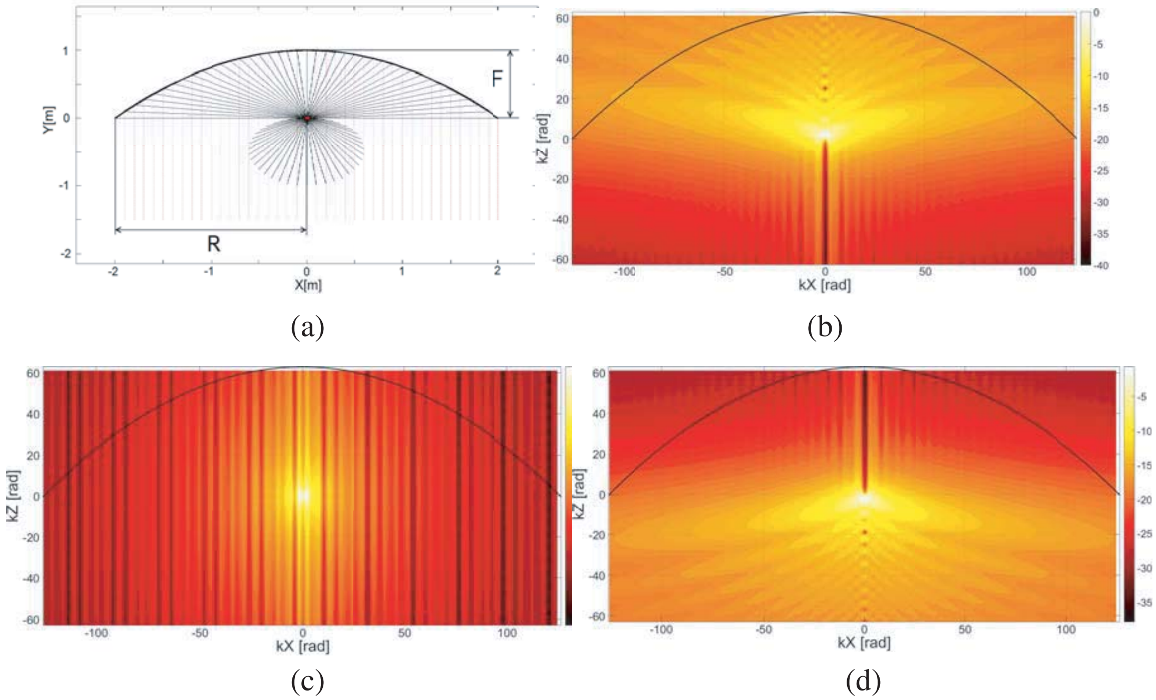


Figure 8. (a) Field distribution [dB] in the reflector antenna with the plane wave normal incidence: incident and reflected beams; (b) field distribution, vector $\mathbf{F}_+$ component F_{+1}; (c) radial component F_0; (d) component F_{-1}.

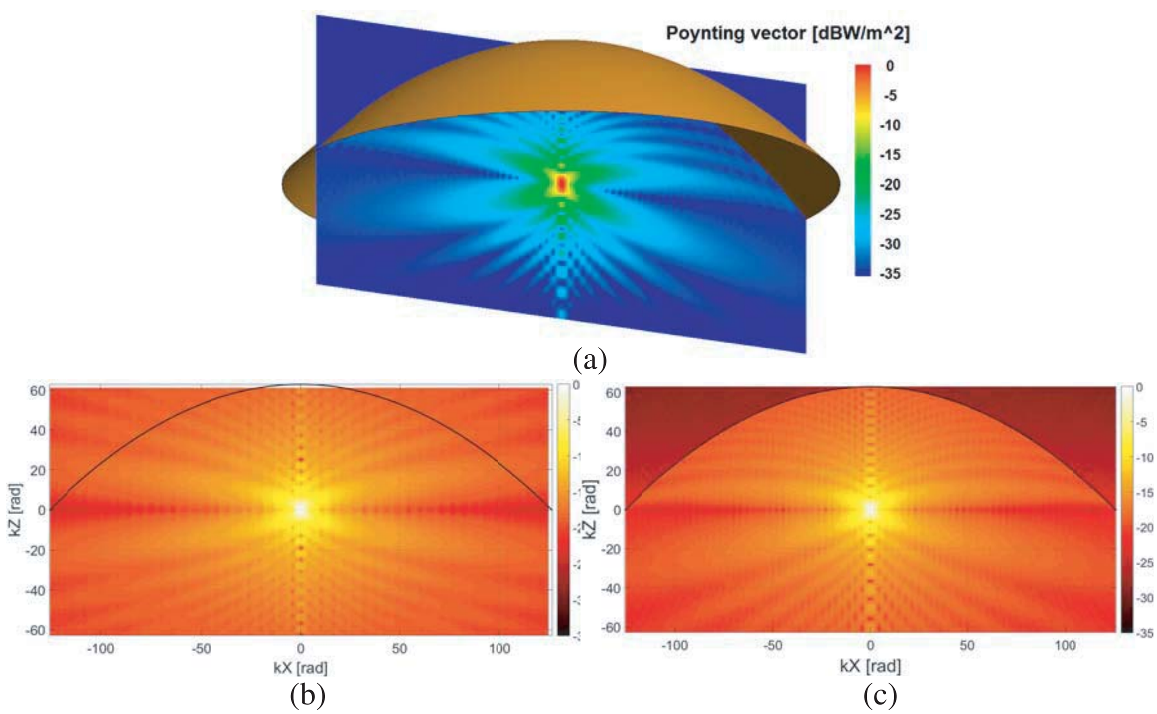


Figure 9. (a) Magnitude of the power distribution [dB] in the reflector antenna with the plane wave normal incidence: FEKO analysis; (b) calculated distribution; (c) simulated distribution in FEKO converted for the comparison.

The comparison between the calculated and simulated fields is given in Fig. 8. The simulation has been performed in FEKO using physical optics method. The difference in the field distributions is caused by the diffraction effects taken into account in FEKO simulation. The simulated field is the interference of the incident plane wave and reflected field.

The functions $\{e^{-im\varphi}P^{n}_{m,l}(\cos\vartheta)\}$ form a complete orthogonal basis of functions on a sphere. Traditional spherical harmonics expansion contains the first constant harmonic, which cannot be a field of any existing radiator. An arbitrary field can be clearly represented in the RS vectors using generalized spherical harmonics.

7. CONCLUSION

The new approach of electromagnetic field representation in terms of the Riemann-Silberstein vectors has been presented here. The benefits of the RS vectors utilization are based on 2 facts:

1) The independence of the vectors;
2) The efficiency of the multipole expansion in helical coordinates.

The modal field expansion into vector spherical harmonics in the helical coordinate system has the same application as traditional TE and TM wave expansions. However, it differs in simplicity and symmetry while having a more physically apparent expression. When an arbitrary radiation pattern is impossible to expand into TE and TM components without numerous integration procedures, expansion in the RS vectors of orthogonal circular polarizations is carried out directly. The number of integral calculations in the expansion is reduced by at least half due to the initial independence of the Riemann-Silberstein vectors for certain applications. Also the expansion does not include a monopole which cannot physically exist. Therefore, we eliminate redundancy in the expansion. As a result, there is a significant reduction in the amount of computational work. The spherical harmonics and vectors are better adapted for invariant mathematical and computer calculations.

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