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ANALYSIS OF NONLINEAR OSCILLATORS WITH u^n FORCE BY HE'S ENERGY BALANCE METHOD

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Abstract—In this letter, an application of energy balance method is applied to solve the nonlinear oscillators with u^n force. Comparison is made between the modification of harmonic balance method and energy balance method. The results reveal that the energy balance method is very effective and simple. Energy balance method is very effective and convenient and quite accurate to both linear and nonlinear physics and engineering problems.

1. INTRODUCTION

This paper considers the following general nonlinear oscillators [2]:

$$u'' + \omega_0^2 u + \varepsilon f(u) = 0 \tag{1}$$

With the initial condition

$$u(0) = A, \quad u'(0) = 0 \tag{2}$$

where f is a nonlinear function of u'', u', u, in this preliminary report, we limit ourselves to the simplest case, i.e., f depends upon only the function of u.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Recently, considerable attention has been directed towards the analytical solutions for nonlinear equations without possible small parameters. The traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have appeared in open literature, for example,

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d-perturbation method [3, 4], variational iteration method (VIM) [5–10], homotopy perturbation method [11–18], bookkeeping parameter Perturbation method [19], just to name a few, a review on some recently developed nonlinear analytical methods can be found in detail in Refs. [20–24].

In energy balance method, a variational principle for the nonlinear oscillation is established, then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones. Some examples reveal that even the lowest order approximations are of high accuracy.

2. BASIC IDEA

First we consider the Duffing equation [2]:

$$u'' + u + \varepsilon u^3 = 0,\ u(0) = A,\ u'(0) = 0 \tag{3}$$

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 \right\} d\tau \tag{4}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 = \frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4 \tag{5}$$

Or:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 - \frac{1}{2}A^2 - \frac{1}{4}\varepsilon A^4 = 0 \tag{6}$$

In Eqs. (5) and (6) the kinetic energy (E) and potential energy (T) can be respectively expressed as $u'^2/2$, $u^2/2 + \varepsilon u^4/4$ throughout the oscillation, it holds that $H = E + T$ constant.

We use the following trial function to determine the angular frequency ω.

$$u = A\cos\omega t \tag{7}$$

Substituting (7) into (6), we obtain the following residual equation:

$$R(t) = \omega^2 \sin^2\omega t + \cos^2\omega t + \frac{1}{2}\varepsilon A^2 \cos^4\omega t - 1 - \frac{1}{2}\varepsilon A^2 \tag{8}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (7) is only an approximation to

the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2} \tag{9}$$

We can apply various other techniques, for examples, least square method, Galerkin method, to identify the constant ω.

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\varepsilon A^2}} \tag{10}$$

The approximate period obtained by the traditional perturbation method reads (Nayfeh, 1985).

$$T_{pert} = 2\pi\left(1 - \frac{3}{8}\varepsilon A^2\right) \tag{11}$$

So our theory, in case $\varepsilon \ll 1$, gives exactly the same result with those obtained by perturbation method.

What is rather surprising about the remarkable range of validity of (10) is that the actual asymptotic period as $\varepsilon \to \infty$ is also of high accuracy.

$$\lim_{\varepsilon\to\infty} \frac{T_{ex}}{T} = \frac{2\sqrt{3/4}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = 0.9294 \tag{12}$$

The lowest order approximation given by (10) is actually within 7.6% of the exact frequency regardless of the magnitude of εA^2.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly [2].

3. APPLICATIONS

We consider the following nonlinear oscillator [1]:

$$u'' + f(u) = 0 \tag{13}$$

We will study the properties of the periodic solutions to certain nonlinear oscillators by applying He's energy balance method for which the elastic restoring forces are non-polynomial functions of the displacement. In particular, this term is chosen to be [2]:

$$f(u) = ku^n,\ n \in R \tag{14}$$

when k designates the stiffness of system.

With the boundary condition of:

$$u(0) = A,\ u'(0) = 0 \tag{15}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + k\frac{u^{n+1}}{n+1} - k\frac{A^{n+1}}{n+1} = 0 \tag{16}$$

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2\omega t + k\frac{(A\cos\omega t)^{n+1}}{n+1} - k\frac{A^{n+1}}{n+1} = 0 \tag{17}$$

we obtain:

$$\omega = \sqrt{2k\frac{A^{n-1}}{n+1}\left\{\frac{1-\cos^{n+1}\omega t}{\sin^2\omega t}\right\}} \tag{18}$$

Its period can be written in the form:

Table 1. Comparison of energy balance frequency with harmonic balance frequency [1]. ($k = 1$) example 1.

A	Energy balance frequency	harmonic balance frequency
0.1	3.2879	3.5191
0.2	2.3249	2.4884
0.3	1.8983	2.0318
0.4	1.6440	1.7595
0.5	1.4704	1.5738
0.6	1.3423	1.4367
0.7	1.2427	1.3301
0.8	1.1625	1.2442
0.9	1.0960	1.1730
1.0	1.0397	1.1128
5.0	0.4650	0.4977
10.0	0.3288	0.3519

$$T = \frac{2\pi}{\sqrt{2k\dfrac{A^{n-1}}{n+1}\left\{\dfrac{1-\cos^{n+1}\omega t}{\sin^2\omega t}\right\}}} \tag{19}$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \sqrt{4k\frac{A^{n-1}}{n+1}\left\{1-\left(\frac{\sqrt{2}}{2}\right)^{n+1}\right\}} \tag{20}$$

In order to compare, we write harmonic balance solution [1]:

$$\omega = \sqrt{2k\frac{A^{n-1}}{\sqrt{\pi}}\left\{\frac{\Gamma\left(\dfrac{1}{2}(n+2)\right)}{\Gamma\left(\dfrac{1}{2}(n+3)\right)}\right\}} \tag{21}$$

Γ is gamma function.

Table 2. Comparison of energy balance frequency with harmonic balance frequency [1]. ($k = 1$) example 2.

A	Energy balance frequency	harmonic balance frequency
0.1	1.6096	1.6498
0.2	1.4012	1.4363
0.3	1.2921	1.3244
0.4	1.2198	1.2503
0.5	1.1666	1.1958
0.6	1.1248	1.1529
0.7	1.0907	1.1179
0.8	1.0619	1.0885
0.9	1.0372	1.0631
1.0	1.0156	1.0410
5.0	0.7361	0.7545
10.0	0.6408	0.6568

Table 3. Comparison of energy balance frequency with harmonic balance frequency [1]. ($k = 1$) example 3.

A	Energy balance frequency	harmonic balance frequency
0.1	1.0000	1.0000
0.2	1.0000	1.0000
0.3	1.0000	1.0000
0.4	1.0000	1.0000
0.5	1.0000	1.0000
0.6	1.0000	1.0000
0.7	1.0000	1.0000
0.8	1.0000	1.0000
0.9	1.0000	1.0000
1.0	1.0000	1.0000
5.0	1.0000	1.0000
10.0	1.0000	1.0000

3.1. Example 1

We consider a one-dimensional, nonlinear oscillator governed by:

$$u'' + ku^n = 0,\ n = 0.5 \tag{22}$$

With the boundary condition of:

$$u(0) = A,\ u'(0) = 0 \tag{23}$$

3.2. Example 2

We consider a one-dimensional, nonlinear oscillator governed by:

$$u'' + ku^n = 0,\ n = 0.8 \tag{24}$$

With the boundary condition of:

$$u(0) = A,\ u'(0) = 0 \tag{25}$$

3.3. Example 3

For special case $n = 1$, oscillator is linear and:

$$u'' + ku^n = 0,\ n = 1.0 \tag{26}$$

With the boundary condition of:

$$u(0) = A,\ u'(0) = 0 \tag{27}$$

3.4. Example 4

We consider a one-dimensional, nonlinear oscillator governed by:

$$u'' + ku^n = 0,\ n = 1.4 \tag{28}$$

With the boundary condition of:

$$u(0) = A,\ u'(0) = 0 \tag{29}$$

Table 4. Comparison of energy balance frequency with harmonic balance frequency [1]. ($k = 1$) example 4.

A	Energy balance frequency	harmonic balance frequency
0.1	0.3862	0.3705
0.2	0.5096	0.4888
0.3	0.5994	0.5749
0.4	0.6725	0.6450
0.5	0.7352	0.7052
0.6	0.7909	0.7586
0.7	0.8412	0.8068
0.8	0.8873	0.8511
0.9	0.9301	0.8922
1.0	0.9702	0.9306
5.0	1.8468	1.7715
10.0	2.4369	2.3375

3.5. Example 5

We consider a one-dimensional, nonlinear oscillator governed by:

$$u'' + ku^n = 0,\ n = 2.0 \tag{30}$$

With the boundary condition of:

$$u(0) = A,\ u'(0) = 0 \tag{31}$$

Table 5. Comparison of energy balance frequency with harmonic balance frequency [1]. ($k = 1$) example 5.

A	Energy balance frequency	harmonic balance frequency
0.1	0.0928	0.0849
0.2	0.1857	0.1698
0.3	0.2785	0.2546
0.4	0.3714	0.3395
0.5	0.4642	0.4244
0.6	0.5570	0.5093
0.7	0.6099	0.5942
0.8	0.7427	0.6791
0.9	0.8356	0.7639
1.0	0.9284	0.8488
5.0	4.6420	4.2441
10.0	9.2840	8.8483

4. CONCLUSIONS

In this work, we use an application of energy balance method for solving the nonlinear oscillators with u^n force. The solution obtained by energy balance method is valid for not only weakly nonlinear equations, but also strong ones. Moreover, we showed that the obtained solutions are valid for the whole domain. The examples show that even the lowest order approximations obtained by the present theory are actually of high accuracy.

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