IMPROVEMENT IN THE ANALYTICAL CALCULATION OF THE MAGNETIC FIELD PRODUCED BY PERMANENT MAGNET RINGS

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Abstract—In this paper, we present an improved Coulombian-based analytical calculation of magnetic fields created by permanent-magnetic rings. The 3 dimensional (3D) components of two types of magnetized rings (axially and radially) were analytically evaluated. The obtained components of the magnetic field are expressed over complete elliptical integrals of the first and second kind, as well as by Heuman’s Lambda function. These expressions permit fast and accurate calculations of the magnetic field at any point of interest, for both regular and singular cases. The presented method gives an improvement of already known expressions for calculating the magnetic fields of the aforementioned magnetized rings, and we consider that these improved analytical expressions are more extendable to numerical applications.

1. INTRODUCTION

Permanent magnet rings are used in many technical applications, such as magnetic separators, magnetic holding devices, magnetic torque drives, magnetic bearing devices, generators, alternators, loudspeakers, and actuators. In these applications, the magnetic fields created by these magnets need to be accurately calculated to optimize devices used therein. Until now, most analytical and numerical approaches have
evaluated two major kinds of applications, such as parallelepipedic and cylindrical magnets, [1–21]. Even though parallelepipedic magnets are easy to produce, are easily magnetized, and create a magnetic field that is easily calculated, we wish to evaluate cylindrical magnets with radial and axial polarizations. The analytical solutions of the magnetic field, based on the Coulombian approach, created by axially and radially magnetized permanent magnetic rings have been evaluated in [1]. The proposed expressions obtained over the complete elliptical integrals of the first, second, and third kind, as evaluated by Mathematica, produce tedious expressions which are complicated for the numerical treatment. In addition, Mathematica-based solutions contain some magnetic field components with imaginary parts nearly equal to zero (Hz), making these components more difficult to plot with Mathematica, especially in the treatment of singular cases that can be solved analytically. Herein, for the previous reason, we propose purely analytical solutions for these components of the magnetic field for the aforementioned types of magnets. The obtained 3D analytical expressions ameliorate already calculated magnetic fields for permanent magnetic rings, and are more suitable for numerical applications. All results are expressed over the complete elliptical integrals of the first, and second kind, as well by Heuman’s Lambda function.

2. BASIC EXPRESSIONS

The solutions for the magnetic fields, produced by permanent magnet rings (See Figs. 1 and 2) can be obtained by using the Coulombian method [1]. Rings with axial (Fig. 1) and radial (Fig. 2) magnetic polarizations were investigated in this study.

The corresponding magnetic fields are:

a) Axial magnetic polarization

\[
\vec{H}^+(r, z) = \frac{\sigma^*}{4\pi\mu_0} \int_{\theta=0}^{\theta=2\pi} \int_{r_1=r_{in}}^{r_1=r_{out}} \frac{P_{1+M}}{|P_{1+M}|^3} r_1 d r_1 d \theta
\]

\[
\vec{H}^-(r, z) = \frac{-\sigma^*}{4\pi\mu_0} \int_{\theta=0}^{\theta=2\pi} \int_{r_1=r_{in}}^{r_1=r_{out}} \frac{P_{1-M}}{|P_{1-M}|^3} r_1 d r_1 d \theta
\]

\[
\vec{H}(r, z) = \vec{H}^+(r, z) + \vec{H}^-(r, z)
\]
Figure 1. Given geometry: A ring with $z$ axis of symmetry and inner and outer radius $r_{in}$ and $r_{out}$, respectively. The magnetic polarization is axial [1].

Figure 2. Given geometry: A ring with $z$ axis of symmetry and inner and outer radius $r_{in}$ and $r_{out}$, respectively. The magnetic polarization is radial, [1].

where

\begin{align*}
P_{1+}\vec{M} &= (r - r_1 \cos \theta)\vec{i}_r - r_1 \sin \theta \vec{i}_\theta + (z - h)\vec{i}_k \\
P_{1-}\vec{M} &= (r - r_1 \cos \theta)\vec{i}_r - r_1 \sin \theta \vec{i}_\theta + (z + h)\vec{i}_k
\end{align*}
b) Radial magnetic polarization

\[ \vec{H}^+(r, z) = \frac{\sigma^*}{4\pi\mu_0} \int_{\theta=0}^{\theta=2\pi} \left( \frac{P_{1+}\vec{M}}{P_{1+}} \right) \sum_{n=1}^{\theta} \frac{1}{k_n^2} \sqrt{r_n} \frac{E(k_n^+)}{r} \left[ E(k_n^+) - (1 - \frac{k_n^+}{2})K(k_n^+) \right] \]

\[ \vec{H}^-(r, z) = -\frac{\sigma^*}{4\pi\mu_0} \int_{\theta=0}^{\theta=2\pi} \left( \frac{P_{1-}\vec{M}}{P_{1-}} \right) \sum_{n=1}^{\theta} \frac{1}{k_n^2} \sqrt{r_n} \frac{E(k_n^-)}{r} \left[ E(k_n^-) - (1 - \frac{k_n^-}{2})K(k_n^-) \right] \]

\[ \vec{H}(r, z) = \vec{H}^+(r, z) + \vec{H}^-(r, z) \]

where

\[ P_{1+}\vec{M} = (r - r_{in} \cos \theta)\vec{i}_r - r_{in} \sin \theta \vec{i}_\theta + (z - z_1)\vec{i}_k \]

\[ P_{1-}\vec{M} = (r - r_{out} \cos \theta)\vec{i}_r - r_{out} \sin \theta \vec{i}_\theta + (z - z_1)\vec{i}_k \]

\[ \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \quad \text{— magnetic permeability of vacuum.} \quad \sigma^* = 1 \text{T} \quad \text{— surface magnetic pole density,} \quad \vec{i}_r, \vec{i}_\theta, \vec{i}_z \quad \text{— unit vectors in the cylindrical coordinate system.} \]

### 3. CALCULATION METHOD

a) Axial magnetic polarization

By substituting the variable \( \theta = \pi - 2\beta \) in (1), integrating twice, and respecting angle variable symmetry, we obtained the 3D components of the magnetic field at the point \( M(r, 0, z) \). In this case all integrals will be transformed on the interval of integration \( 0 \leq \beta \leq \pi/2 \) so that all results will be expressed over complete elliptical integrals.

\[ H_r(r, z) = H_r^+(r, z) + H_r^-(r, z) \]

\[ H_\theta(r, z) = H_\theta^+(r, z) + H_\theta^-(r, z) = 0 \]

\[ H_z(r, z) = H_z^+(r, z) + H_z^-(r, z) \]

where

\[ H_r^+(r, z) = \frac{\sigma^*}{\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} \frac{1}{k_n^2} \sqrt{r_n} \frac{E(k_n^+)}{r} \left[ E(k_n^+) - (1 - \frac{k_n^+}{2})K(k_n^+) \right] \]

\[ H_r^-(r, z) = -\frac{\sigma^*}{\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} \frac{1}{k_n^2} \sqrt{r_n} \frac{E(k_n^-)}{r} \left[ E(k_n^-) - (1 - \frac{k_n^-}{2})K(k_n^-) \right] \]
Thus, we obtained the 3D components of the magnetic field expressed over the complete elliptical integrals of the first, second, and third kind [22,23]. Ravaud et al. [1] obtained the same expressions for the component \( H_r \) and relatively tedious expressions for the component \( H_z \) wherein, the incomplete integral of the third kind appears. They used Mathematica to evaluate this component and obtained its real and the imaginary components of \( H_z \). Obviously this component has to be purely real. Even though the imaginary part is a consequence of ‘numerical noise’ and nearly equals zero they obtained satisfactory results for this component. In our calculation we obtained the complete elliptical integrals of the third kind. These expressions are easy to program and can be used to evaluate the magnetic field anywhere except in the singular case. In the singular case, it is possible to replace the complete elliptic integral of the third kind with the elliptic integral of the first kind and with Heuman’s Lambda function [22,23].

After some mathematical modifications the component \( H_z(r, z) \) which is suitable for either regular or singular cases appears in the
following form:

\[ H^+_z(r, z) = \frac{\sigma^*}{2\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} \left\{ \frac{k_1^+(z-h)\sqrt{r^2 + (z-h)^2}}{\sqrt{r^2 + (z-h)^2 + r}} K(k_1^+) \\
+ \frac{\pi}{2} \text{sign}(z-h)\text{sign}(\sqrt{r^2 + (z-h)^2} - r_n)[1 - \Lambda_0(\theta^+_1, k_1^+)] \\
+ \frac{\pi}{2} \text{sign}(z-h)[1 - \Lambda_0(\theta^+_2, k_1^+)] \right\} \]

\[ H^-_z(r, z) = -\frac{\sigma^*}{2\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} \left\{ \frac{k_1^-\sqrt{r^2} + z^2}{\sqrt{r^2 + z^2 + r}} K(k_1^-) \\
+ \frac{\pi}{2} \text{sign}(z)\text{sign}(\sqrt{r^2 + z^2} - r_n)[1 - \Lambda_0(\theta^-_1, k_1^-)] \\
+ \frac{\pi}{2} \text{sign}(z)[1 - \Lambda_0(\theta^-_2, k_1^-)] \right\} \]

\[ \theta^+_1 = \arcsin \sqrt{\frac{1 - h^+_1}{1 - k_1^{+2}}} , \quad \theta^+_2 = \arcsin \frac{|z-h|}{\sqrt{r^2 + (z-h)^2 + r}} \]

\[ \theta^-_1 = \arcsin \sqrt{\frac{1 - h^-_1}{1 - k_1^{-2}}} , \quad \theta^-_2 = \arcsin \frac{|z|}{\sqrt{r^2 + z^2 + r}} \]

a) Singularity treatment

The component \( H^+_r \) has two singularities (See Fig. 3) for \( k_1^{+2} = 1 \) (\( z = h, \ r = r_{in}, \ r = r_{out} \)) for which \( H^+_r \) values are \(-\infty\) and \(+\infty\).

The component \( H^-_z \) has two removable singularities (See Fig. 5) for \( k_1^{+2} = 1 \) (\( z = h, \ r = r_{in}, \ r = r_{out} \)). Using the L’Hopital rule we obtained \( H^-_z \to 0 \) for \( k_1^{+2} \to 1 \).

\( K(k) \) — complete elliptic integral of the first kind [22, 23].

\( E(k) \) — complete elliptic integral of the second kind [22, 23].

\( \Pi(h, k) \) — complete elliptic integral of the third kind [22, 23].

\( \Lambda(\epsilon, k) \) — Heuman’s Lambda function [22, 23].

b) Radial magnetic polarization

By substituting the variable \( \theta \) with \( \theta = \pi - 2\beta \) in (2), integrating twice, and respecting angle variable symmetry we obtained the 3D components of the magnetic field at the point \( M(r, 0, z) \):

\[ H_r(r, z) = H^+_r(r, z) + H^-_r(r, z) \]

\[ H_\theta(r, z) = H^+_\theta(r, z) + H^-_\theta(r, z) = 0 \]

\[ H_z(r, z) = H^+_z(r, z) + H^-_z(r, z) \]
where

\[ H^+_r(r, z) = -\frac{\sigma^*}{4\pi\mu_0} \sum_{n=1}^{2} \frac{(-1)^{n-1} t_n k_n^+}{r} \sqrt{\frac{r_m}{r}} \left[ K(k_n^+) + \frac{r - r_{in}}{r + r_{in}} \Pi(h^+, k_n^+) \right] \]

\[ H^-_r(r, z) = \frac{\sigma^*}{4\pi\mu_0} \sum_{n=1}^{2} \frac{(-1)^{n-1} t_n k_n^-}{r} \sqrt{\frac{r_{out}}{r}} \left[ K(k_n^-) + \frac{r - r_{out}}{r + r_{out}} \Pi(h^-, k_n^-) \right] \]

\[ H^+_\theta(r, z) = H^-_\theta(r, z) = 0 \]

\[ H^+_z(r, z) = \frac{\sigma^*}{2\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} k_n^+ \sqrt{\frac{r_{in}}{r}} \left[ 2t_n k_n^+ + \frac{\pi}{r} \sqrt{\frac{r_{in}}{r}} \text{sign}(r - r_{in})\text{sign}(t_n) \right] \left[ 1 - \Lambda_0(\varepsilon_n^+, k_n^+) \right] \]

\[ H^-_z(r, z) = -\frac{\sigma^*}{2\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} k_n^- \sqrt{\frac{r_{out}}{r}} \left[ 2t_n k_n^- + \frac{\pi}{r} \sqrt{\frac{r_{out}}{r}} \text{sign}(r - r_{out})\text{sign}(t_n) \right] \left[ 1 - \Lambda_0(\varepsilon_n^-, k_n^-) \right] \]

Thus, we obtained the 3D components of the magnetic field expressed over the complete elliptical integrals of the first, second, and third kind [22, 23]. Ravaud et al. [1] obtained the same expressions for the component \( H_z \) and relatively tedious expressions for the component \( H_r \) wherein, the incomplete integrals of the first and third kind appear. Above obtained expressions can be easily used to calculate the magnetic field anywhere except in the singular case. In the singular case, it is possible to replace the elliptic integral of the third kind with the elliptic integral of the first kind and Heumann Lambda function [22, 23]. After some mathematical modifications, the component \( H_r(r, z) \) suitable for either regular or singular cases appears in the following form:

\[ H^+_r(r, z) = -\frac{\sigma^*}{4\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} t_n k_n^+ \sqrt{\frac{r_{in}}{r}} \left[ 2t_n k_n^+ K(k_n^+) \right] \]

\[ +\frac{\pi}{r} \sqrt{\frac{r_{in}}{r}} \text{sign}(r - r_{in})\text{sign}(t_n) \left[ 1 - \Lambda_0(\varepsilon_n^+, k_n^+) \right] \]

\[ H^-_r(r, z) = \frac{\sigma^*}{4\pi\mu_0} \sum_{n=1}^{2} (-1)^{n-1} t_n k_n^- \sqrt{\frac{r_{out}}{r}} \left[ 2t_n k_n^- K(k_n^-) \right] \]

\[ +\frac{\pi}{r} \sqrt{\frac{r_{out}}{r}} \text{sign}(r - r_{out})\text{sign}(t_n) \left[ 1 - \Lambda_0(\varepsilon_n^-, k_n^-) \right] \]
where

\[ \varepsilon_n^+ = \arcsin \sqrt{\frac{1 - h^+}{1 - k_n^{-2}}}, \quad \varepsilon_n^- = \arcsin \sqrt{\frac{1 - h^-}{1 - k_n^{-2}}} \]

b1) Singularity treatment

The component \( H_r^+ \) has one removable singularity (See Figs. 7) for \( k_n^+ = 1 \) (\( z = h, \ r = r_{in} \)). Using the L’Hopital rule we obtained \( H_r^+ \rightarrow 0 \) for \( k_n^+ ightarrow 1 \).

The component \( H_z^+ \) has one singularity (See Figs. 9) for \( k_n^+ = 1 \) (\( z = h, \ r = r_{in} \)) for which \( H_z^+ \) value is \(+\infty\).

4. EXAMPLES

a) Axial magnetic polarization

In Figs. 3 and 4 we show the radial and the axial components of the magnetic field (axial magnetic polarization) as a function of the radial distance of the observation point for a given altitude with \( z = 3 \text{ mm} \) (component \( H_r \)) and \( z = 4 \text{ mm} \) (component \( H_z \)) for a magnet with dimensions \( r_{in} = 25 \text{ mm}, \ r_{out} = 28 \text{ mm}, \ h = 3 \text{ mm} \) and \( \sigma^* = 1 \text{ T} \) [1].

We only considered the upper face of the permanent magnet ring to simplify the analytical calculation [1]. If we compare our results to those obtained in [1], we observed the same form for the components.

\[ \text{Figure 3.} \quad \text{The radial component } H_r \text{ is a function of the radial distance: the observation height is } z = 3 \text{ mm}, \ r_{in} = 25 \text{ mm}, \ r_{out} = 28 \text{ mm}. \]
Figure 4. The axial component $H_z$ is a function of the radial distance: the observation height is $z = 4$ mm, $r_{in} = 25$ mm, $r_{out} = 28$ mm.

Figure 5. The radial component $H_r$ is a function of the radial distance: the observation height is $z = 1.5$ mm, $r_{in} = 25$ mm, $r_{out} = 28$ mm [1].
of $H_r$ (Fig. 3). From the components of $H_z$ (Fig. 4) we can see that there is a difference in the right side where the magnetic field does not exist for $r > 0.025$ m [1]. This component of the magnetic field is difficult to plot with Mathematica as mentioned in [1]. We find that our approach, in which we replaced the elliptic integral of third kind by Heumans Lambda function, can considerably improve the magnetic field calculation near singular points where the elliptic integral of third kind is tedious for numerical evaluation [20].

b) Radial magnetic polarization

In Figs. 5 and 6 we show the radial and the axial components of the magnetic field as a function of the radial distance of the observation point for a given altitude with $z = 1.5$ mm (component $H_r$) and $z = 3$ mm (component $H_z$) for given magnet with dimensions $r_{in} = 25$ mm, $r_{out} = 28$ mm, $h = 3$ mm and $\sigma^* = 1$ T [1]. We only consider the ring inner face of the permanent magnet ring to simplify the analytical calculation [1]. If we compare our results to those obtained in [1] we observe the same form for all of the components of either $H_r$ or $H_z$.

Thus, we can confirm that obtained expressions (3) and (4) perfectly correspond to those given in [1] but they are more suitable for the calculation of magnetic fields especially in singular cases. The comparative calculation was made using MATLAB programming.
5. CONCLUSION

In this paper, we presented an improved analytical approach, based on a previously developed analytical method, to calculate the magnetic field of axially- and radially-polarized magnetic rings. The proposed analytical expressions are easily extended to the numerical calculation of the magnetic field at any regular or singular point in space.

REFERENCES


