

TOPOLOGICAL SOLITONS IN 1+2 DIMENSIONS WITH TIME-DEPENDENT COEFFICIENTS

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Abstract—This paper obtains the topological 1-soliton solution of the nonlinear Schrödinger's equation in 1+2 dimensions, with power law nonlinearity and time-dependent coefficients. The solitary wave ansatz is used to obtain the solution. It will also be proved that the power law nonlinearity must reduce to Kerr law nonlinearity for the topological solitons to exist.

1. INTRODUCTION

The dynamics of dark optical solitons, that is also known as topological solitons, in 1+2 dimensions will be studied in this paper. The power law nonlinearity will be considered. The coefficients of the dispersion, nonlinearity and attenuation terms are all time-dependent. It will be seen that these coefficients must be simply Riemann integrable, but could be otherwise arbitrary, for the solitons to exist. It will also be proved that for the solitons to exist, the power law nonlinearity must reduce to Kerr law nonlinearity.

The governing equation will be the nonlinear Schrödinger's equation (NLSE) with power law nonlinearity. This NLSE in 1+2 dimensions with power law nonlinearity and having time-dependent coefficients will not be integrable by the classical method of Inverse Scattering Transform (IST) since the Painleve test of integrability will fail in this case. However, there are various modern methods of integrability that obtain the solution of many nonlinear evolution equations when the IST approach fails [1–15]. Some of these commonly studied techniques are G'/G method, exponential function method, Adomian decomposition method, F -expansion method, sub-ODE method and Lie symmetry approach [5], just to name a few. However, one has to be extremely careful in using these techniques, simply because it could lead to incorrect results [8, 9]. In this paper, there will be one such method that will be used to carry out the integration. This is the solitary wave ansatz method.

2. MATHEMATICAL ANALYSIS

The dimensionless form of the NLSE in 1+2 dimensions with power law nonlinearity and time-dependent coefficients is given by [2, 12]

$$iq_t + a(t)(q_{xx} + q_{yy}) + b(t)|q|^{2m}q = i\alpha(t)q \quad (1)$$

In (1), the first term represents the evolution term, while the second and third terms together in parentheses represent dispersion terms in x and y directions respectively, and the coefficient of the dispersion terms is $a(t)$, while $b(t)$ is the coefficient of the nonlinear term and the parameter m dictates the power law nonlinearity. Finally, on the right hand side, $\alpha(t)$ represents the coefficient of the linear attenuation term. If the parameter $m = 1$, the power law nonlinearity collapses to the case of Kerr law nonlinearity. In this paper the focus will be on obtaining the 1-soliton solution to (1) using the solitary wave ansatz method.

In order to solve (1), it is first assumed that the solution is given in the following phase-amplitude format [2, 12]

$$q(x, y, t) = Pe^{i\phi} \tag{2}$$

where $P = P(x, y, t)$ is the amplitude portion while $\phi = \phi(x, y, t)$ is the phase portion of the soliton. It is assumed that

$$\phi(x, y, t) = -\kappa_1x - \kappa_2y + \omega t + \theta \tag{3}$$

where κ_1 and κ_2 are the frequencies of the soliton in the x - and y -directions respectively while ω is the wave number of the soliton and θ is the phase constant. Since the dispersion, nonlinearity and attenuation terms have time-dependent coefficients, it is therefore assumed that these soliton parameters κ_1 , κ_2 , ω and θ are all time-dependent. Therefore, from (2) and (3)

$$iq_t = \left\{ i \frac{\partial P}{\partial t} - \left(\omega + t \frac{d\omega}{dt} - x \frac{d\kappa_1}{dt} - y \frac{d\kappa_2}{dt} + \frac{d\theta}{dt} \right) P \right\} e^{i\phi} \tag{4}$$

$$q_{xx} = \left(\frac{\partial^2 P}{\partial x^2} - 2i\kappa_1 \frac{\partial P}{\partial x} - \kappa_1^2 P \right) e^{i\phi} \tag{5}$$

$$q_{yy} = \left(\frac{\partial^2 P}{\partial y^2} - 2i\kappa_2 \frac{\partial P}{\partial y} - \kappa_2^2 P \right) e^{i\phi} \tag{6}$$

Substituting (2)–(6) into (1) and decomposing into real and imaginary parts, respectively, yields

$$P \left(\omega + t \frac{d\omega}{dt} - x \frac{d\kappa_1}{dt} - y \frac{d\kappa_2}{dt} + \frac{d\theta}{dt} \right) - a(t) \left\{ \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - (\kappa_1^2 + \kappa_2^2) P \right\} - b(t)P^{2m+1} = 0 \tag{7}$$

and

$$\frac{\partial P}{\partial t} - 2a(t) \left(\kappa_1 \frac{\partial P}{\partial x} + \kappa_2 \frac{\partial P}{\partial y} \right) = \alpha(t)P \tag{8}$$

For dark solitons or topological solitons, the assumption is [1, 8]

$$P(x, y, t) = A \tanh^p \tau \tag{9}$$

where

$$\tau = B_1x + B_2y - vt \tag{10}$$

where the exponent p is unknown at this stage and this will be determined in course of derivation of the exact soliton solution to (1). The typical boundary conditions for this ansatz is $q \rightarrow \pm A$ as $|x|, |y| \rightarrow \pm\infty$. For dark solitons, the parameters A, B_1 and B_2 in (9) and (10) are free parameters, while v is the velocity of the soliton. Again, since the coefficients of dispersion, nonlinearity and attenuation terms are all time-dependent, consequently, the soliton parameters A, B_1, B_2 and v are all time-dependent, in general. So, Equations (7) and (8) respectively reduce to

$$\begin{aligned} & A \tanh^p \tau \left(\omega + t \frac{d\omega}{dt} - x \frac{d\kappa_1}{dt} - y \frac{d\kappa_2}{dt} + \frac{d\theta}{dt} \right) \\ & - a(t)pA (B_1^2 + B_2^2) \left\{ (p-1) \tanh^{p-2} \tau - 2p \tanh^p \tau + (p+1) \tanh^{p+2} \tau \right\} \\ & + a(t) (\kappa_1^2 + \kappa_2^2) A \tanh^p \tau - b(t)A^{2m+1} \tanh^{(2m+1)p} \tau = 0 \end{aligned} \quad (11)$$

and

$$\begin{aligned} & \frac{dA}{dt} \tanh^p \tau + pA (\tanh^{p-1} \tau - \tanh^{p+1} \tau) \left(x \frac{dB_1}{dt} + y \frac{dB_2}{dt} - v - t \frac{dv}{dt} \right) \\ & - 2a(t)pA (\kappa_1 B_1 + \kappa_2 B_2) (\tanh^{p-1} \tau - \tanh^{p+1} \tau) = \alpha(t)A \tanh^p \tau \end{aligned} \quad (12)$$

From (11), equating the exponents $(2m+1)p$ and $p+2$ yields

$$(2m+1)p = p+2 \quad (13)$$

that gives

$$p = \frac{1}{m} \quad (14)$$

In (11), the functions $\tanh^{p+j} \tau$, for $j = -2, 0$ and 2 are linearly independent functions. Thus, setting the coefficients of $\tanh^{p-2} \tau$ to zero gives

$$p = 1 \quad (15)$$

so that from (14),

$$m = 1 \quad (16)$$

This shows that dark solitons exist for Kerr law nonlinearity only. Now, in (11), setting the coefficients of $\tanh^p \tau$ and $\tanh^{p+2} \tau$ to zero respectively yields

$$\omega = -a(t) \left\{ (\kappa_1^2 + \kappa_2^2) + 2(B_1^2 + B_2^2) \right\} \quad (17)$$

and

$$B_1^2 + B_2^2 = -\frac{b(t)A^2}{2a(t)} \tag{18}$$

Equation (11) also implies that ω , κ_1 , κ_2 and θ are all constants. These follow from the fact that besides these two linearly independent functions, the other linearly independent functions are $t \tanh^p \tau$, $x \tanh^p \tau$ and $y \tanh^p \tau$. Also, θ being a phase constant, is always a constant. Again, from (12), equating the coefficients of $\tanh^p \tau$ yields

$$\frac{dA}{dt} = \alpha(t)A \tag{19}$$

so that

$$A(t) = A_0 e^{\int \alpha(t) dt} \tag{20}$$

provided the Riemann integral of $\alpha(t)$ exists. In (20), A_0 is the initial value of the free parameter A . Similarly, from this Equation (12), setting the coefficients of the other linearly independent functions to zero, one obtains B_1 and B_2 are also constants. Also, from (17), since κ_1 , κ_2 and ω are constants,

$$a(t) = \text{constant} \tag{21}$$

Therefore, from (12),

$$v(t) = 2a(\kappa_1 B_1 + \kappa_2 B_2) \tag{22}$$

Finally, from (18) and (20), it is possible to write

$$b(t)e^{2\int \alpha(t) dt} = c \tag{23}$$

where c is a constant and also the restriction

$$ab(t) < 0 \tag{24}$$

must be valid, as seen from (18), for the topological solitons to exist. Thus, the topological or dark soliton solution to (1) is given by

$$q(x, y, t) = A \tanh(B_1 x + B_2 y - vt) e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{25}$$

with six degrees of freedom. The relation between the free parameters A , B_1 and B_2 is given in (18) and the velocity of the soliton is seen in (22). The variation of A with time is given by (20). These impose the restrictions (23) and (24) for dark solitons to exist.

3. CONCLUSION

In this paper, topological optical solitons were studied in 1+2 dimensions in presence of time-dependent dispersion, nonlinearity and linear attenuation. The considered law of nonlinearity is the power law. It was, however, proved that the dark solitons exists only when the power law nonlinearity reduces to the Kerr law nonlinearity. It has also been observed that the time-dependent coefficient of the linear attenuation term must be simply Riemann integrable for these topological solitons to exist. The other simple constrains for these time-dependent coefficients have also been identified.

It needs to be noted that the study of solitons appears not only in Nonlinear Optics, but also in the area of Hydrodynamics. Some nonlinear evolution equations that are of the type in (1) are used to study Fluid Dynamics, but the non-existence of solution sometimes complicates this sort of result [3, 4, 11]. The NLSE in 1+1 dimensions also appears in the context of Plasma Physics, Mathematical Biology and others [1, 14, 15]. The NLSE given by (1) is the elliptic form while the hyperbolic form of NLSE where the coefficients of $a(t)$ have opposite signs appear in the context of Fluid Dynamics [1, 13].

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REFERENCES

1. Ablowitz, M. J. and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, PA, 1981.
2. Biswas, A., "Topological 1-soliton solution of the nonlinear Schrödinger's equation with Kerr law nonlinearity in 1+2 dimensions," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 14, No. 7, 2845–2847, 2009.
3. Escudero, C., "Blow-up of the hyperbolic Burger's equation," *Journal of Statistical Physics*, Vol. 127, No. 2, 327–338, 2007.

4. Jordan, P. M., "Growth and decay of shock and acceleration waves in a traffic flow model with relaxation," *Physical D*, Vol. 207, No. 3–4, 220–229, 2005.
5. Khalique, C. M. and A. Biswas, "Optical solitons with parabolic and dual-power law nonlinearity via lie symmetry analysis," *Journal of Electromagnetic Waves and Applications*, Vol. 23, No. 7, 963–973, 2009.
6. Konar, S., S. Jana, and S. Shwetanshumala, "Incoherently coupled screening photovoltaic spatial solitons in biased photovoltaic photorefractive crystals," *Optics Communications*, Vol. 273, No. 2, 324–333, 2007.
7. Konar, S., S. Jana, and M. Mishra, "Induced focussing and all optical switching in cubic-quintic nonlinear media," *Optics Communications*, Vol. 255, No. 1–3, 114–129, 2005.
8. Kudryashov, N. A. and N. B. Loguinova, "Be careful with exp-function method," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 14, No. 5, 1881–1890, 2009.
9. Kudryashov, N. A., "Seven common errors in finding exact solutions of nonlinear differential equations," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 14, No. 9–10, 3507–3529, 2009.
10. Lott, D. A., A. Henriquez, B. J. M. Sturdevant, and A. Biswas, "A numerical study of optical soliton-like structures resulting from the nonlinear Schrödinger's equation with square-root law nonlinearity," *Applied Mathematics and Computation*, Vol. 207, No. 2, 319–326, 2009.
11. Makarenko, A. S., M. N. Moskalkov, and S. P. Levkov, "On blow-up solutions in turbulence," *Physics Letters A*, Vol. 235, No. 4, 391–397, 1997.
12. Sturdevant, B. J. M., D. A. Lott, and A. Biswas, "Dynamics of topological optical solitons with time-dependent dispersion, nonlinearity and attenuation," *Communicat. in Nonlinear Science and Numerical Simulation*, Vol. 14, No. 8, 3305–3308, 2009.
13. Sulem, C. and P. Sulem, *The Nonlinear Schrödinger's Equation*, Springer Verlag, New York, NY, 1999.
14. Wazwaz, A. M., "Reliable analysis for nonlinear Schrödinger equations with a cubic nonlinearity and a power law nonlinearity," *Mathem. and Comp. Mod.*, Vol. 43, No. 1–2, 178–184, 2006.
15. Wazwaz, A. M., "A study on linear and nonlinear Schrödinger equations by the variational iteration method," *Chaos, Solitons & Fractals*, Vol. 37, No. 4, 1136–1142, 2008.