ELECTROMAGNETIC WAVE SCATTERING BY A THIN LAYER IN WHICH MANY SMALL PARTICLES ARE EMBEDDED

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Abstract—Scattering of electromagnetic (EM) waves by many small particles (bodies), embedded in a thin layer, is studied. Physical properties of the particles are described by their boundary impedances. The thin layer of depth of the order \( O(a) \), with many embedded small particles of characteristic size \( a \), is described by a boundary condition on the surface of the layer. The limiting interface boundary condition is obtained for the effective EM field in the limiting medium, in the limit \( a \to 0 \), where the number \( M(a) \) of the particles tends to infinity at a suitable rate.

1. INTRODUCTION

It is known (see, e.g., [2, 9]) that the light propagation through diffraction gratings may exhibit strong resonances at certain frequencies, which is useful in applications. In this paper, we study electromagnetic (EM) wave scattering by many small impedance particles \( D_m, 1 \leq m \leq M, M = M(a) \), embedded in a thin layer of the depth \( h(a) \sim a \), where \( a \) is the characteristic dimension of a small particle. The shape of the particles may be fairly arbitrary, not necessarily spherical.

We assume that

\[
\lim_{a \to 0} a/d(a) = 0, \quad \lim_{a \to 0} d(a) = 0, \quad ka \ll 1,
\]

where \( k \) is the wavenumber, and \( d(a) \) is the distance between neighboring particles.

The thin layer is located on a smooth surface \( S \). The permittivity \( \varepsilon_0 \), conductivity \( \sigma_0 \geq 0 \), \( \varepsilon' = \varepsilon_0 + i\sigma_0/\omega \) and permeability \( \mu_0 \) of the space are assumed constants, \( k^2 = \omega^2 \varepsilon' \mu_0 \), \( \omega \) is the frequency.

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For example, one may assume that $S$ is the plane $x_3 = 0$, but our arguments are valid for an arbitrary smooth $S$. The $M$ particles on $S$ are distributed according to the following law: in any open subset $\Delta$ of $S$ there are $\mathcal{N}(\Delta)$ particles, where

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(s)ds[1 + o(1)], \quad a \to 0,$$

(2)

$N(s) \geq 0$ is a continuous function, vanishing outside of a finite domain $\Omega \subset S$ in which small particles (bodies) $D_m$ are distributed, $\kappa \in (0, 1)$ is a number, and the boundary impedances of the small particles are defined by the formula

$$\zeta_m = \frac{h(s_m)}{a^{\kappa}}, \quad s_m \in D_m,$$

(3)

where $s_m \in S$ is a point inside $m$–th particle $D_m$, $\text{Re}h(s) \geq 0$, and $h(s)$ is a continuous function vanishing outside $\Omega$.

We can choose $\kappa$ and $h(s)$ as we wish.

Denote by $[E, H] = E \times H$ the cross product of two vectors, and by $(E, H) = E \cdot H$ the dot product of two vectors.

The impedance boundary condition on the surface $S_m$ of the $m$–th particle $D_m$ is $E^t = \zeta_m[H^t, N]$, where $E^t(H^t)$ is the tangential component of $E(H)$ on $S_m$, and $N$ is the unit normal to $S_m$, pointing out of $D_m$. We define $E^t = [N, [E, N]] = E - N(E, N)$. This corresponds to the geometrical meaning of the tangential component of $E$, and differs from the definition $E^t = [N, E]$ that is used sometimes.

In this paper, we use the methodology, developed in [4–7] and some results from [7].

2. EM WAVE SCATTERING BY MANY SMALL PARTICLES

EM wave scattering problem consists of finding vectors $E$ and $H$ satisfying the Maxwell equations:

$$\nabla \times E = i\omega \mu_0 H, \quad \nabla \times H = -i\omega \epsilon_0 E \quad \text{in} \ D := \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m,$$

(4)

the impedance boundary conditions:

$$[N, [E, N]] = \zeta_m[H, N] \quad \text{on} \ S_m, \quad 1 \leq m \leq M,$$

(5)

and the radiation conditions:

$$E = E_0 + v_E, \quad H = H_0 + v_H,$$

(6)

where $\zeta_m$ is the impedance, defined in (3), $N$ is the unit normal to $S_m$ pointing out of $D_m$, $E_0$, $H_0$ are the incident fields satisfying
Equation (1) in all of $\mathbb{R}^3$. One often assumes that the incident wave is a plane wave, i.e., $E_0 = \mathcal{E} e^{ik\alpha \cdot x}$, $\mathcal{E}$ is a constant vector, $\alpha \in S^2$ is a unit vector, $S^2$ is the unit sphere in $\mathbb{R}^3$, $\alpha \cdot \mathcal{E} = 0$, $v_E$ and $v_H$ satisfy the radiation condition: $r (\frac{\partial v}{\partial r} - ikv) = o(1)$ as $r := |x| \to \infty$, $k = \omega \sqrt{\epsilon_0 \mu_0}$.

By impedance $\zeta_m$ we assume in this paper a constant, $\text{Re} \ zeta_m \geq 0$, or a matrix function $2 \times 2$ acting on the tangential to $S_m$ vector fields, such that $\text{Re}(\zeta_m E^t, E^t) \geq 0 \ \forall E^t \in T_m$, (7)

where $T_m$ is the set of all tangential to $S_m$ continuous vector fields such that $\text{Div} E^t = 0$, where $\text{Div}$ is the surface divergence, and $E^t$ is the tangential component of $E$. Smallness of $D_m$ means that $ka \ll 1$, where $a = 0.5 \text{max}_{1 \leq m \leq M} \text{diam} D_m$.

**Lemma 1.** Problem (4)–(7) has at most one solution.

Lemma 1 is proved in [7].

Let us note that problem (4)–(7) is equivalent to the problems (6)–(9), where

$$\nabla \times \nabla \times E = k^2 E \ \text{in} \ D, \quad H = \frac{\nabla \times E}{i \omega \mu_0},$$

$$[N, [E, N]] = \frac{\zeta_m}{i \omega \mu_0} [\nabla \times E, N] \text{on} \ S_m, \ 1 \leq m \leq M.$$ (9)

This is the impedance boundary condition (see, e.g., [1 p. 301]) The expression $[N, [E, N]] = E - (E, N)N$ is the tangential component of the field $E$ on the surface $S_m$, $N$ is the unit normal to $S_m$ pointing out of $D_m$.

Thus, we have reduced our problem to finding one vector $E(x)$. If $E(x)$ is found, then $H = \frac{\nabla \times E}{i \omega \mu_0}$.

Let us look for $E$ of the form

$$E = E_0 + \sum_{m=1}^{M} \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt, \quad g(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},$$ (10)

where $t \in S_m$, $dt$ is an element of the area of $S_m$, and $\sigma_m(t) \in T_m$. This $E$ for any continuous $\sigma_m(t)$ solves Equation (8) in $D$ because $E_0$ solves (8) and

$$\nabla \times \nabla \times \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt = \nabla \nabla \cdot \int_{S_m} g(x, t) \sigma_m(t) dt$$

$$- \nabla^2 \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt$$

$$= k^2 \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt, \ x \in D.$$ (11)
Here we have used the known identity \( \text{div} \, \text{curl} E = 0 \), valid for any smooth vector field \( E \), and the known formula
\[
-\nabla^2 g(x, y) = k^2 g(x, y) + \delta(x - y).
\] (12)

The integral \( \int_{S_m} g(x, t) \sigma_m(t) dt \) satisfies the radiation condition. Thus, formula (10) gives \( E(x) \) that solves problem (8), (9), and satisfies the radiation condition, if \( \sigma_m(t) \) are chosen so that boundary conditions (9) are satisfied.

Define the effective field \( E_e(x) = E_{e}^{m}(x) = E_e^{(m)}(x, a) \), acting on the \( m \)-th body \( D_m \):
\[
E_e(x) := E(x) - \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt := E_e^{(m)}(x),
\] (13)

where we assume that \( x \) is in a neighborhood of \( S_m \). However, \( E_e(x) \) is defined for all \( x \in \mathbb{R}^3 \).

Away from \( S \), the field \( E_e(x, a) \) tends to a limit \( E(x) = E_e(x) \) as \( a \to 0 \), and \( E_e(x) \) is a twice continuously differentiable function away from \( S \), see [7]. To derive an integral equation for \( \sigma_m = \sigma_m(t) \), substitute
\[
E = E_e + \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt
\]
into boundary condition (9), use the known formula (see, e.g., [3])
\[
[N, \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt] = \int_{S_m} [N_s, [\nabla_s g(x, t), \sigma_m(t)]] dt \pm \frac{\sigma_m(t)}{2},
\] (14)

where the \(- (+)\) signs denote the limiting values of the left-hand side of (14) as \( x \to s \) from \( D \) (\( D_m \)), and get
\[
\sigma_m(t) = A_m \sigma_m + f_m, \quad 1 \leq m \leq M.
\] (15)

Here \( A_m \) is a linear Fredholm-type integral operator, and \( f_m \) is a continuously differentiable function.

Let us specify \( A_m \) and \( f_m \). One has
\[
f_m = 2[N_s, f_e(s)], \quad f_e(s) := [N_s, [E_e(s), N_s]] - \frac{\zeta_m}{i \omega \mu_0} [\nabla \times E_e, N_s].
\] (16)

Condition (9) and formula (14) yield
\[
f_e(s) + \frac{1}{2} [\sigma_m(s), N_s] + \left[ \int_{S_m} [N_s, [\nabla_s g(s, t), \sigma_m(t)]] dt, N_s \right] - \frac{\zeta_m}{i \omega \mu_0} \left[ \nabla \times \nabla \times \int_{S_m} g(x, t) \sigma_m(t) dt, N_s \right] \bigg|_{x \to s} = 0
\] (17)
Using the known formula $\nabla \times \nabla = \text{grad div} - \nabla^2$, the relation
\[
\nabla_x \nabla_x \int_{S_m} g(x, t) \sigma_m(t) dt = \nabla_x \int_{S_m} (-\nabla_t g(x, t), \sigma_m(t)) dt
\]
\[
= \nabla_x \int_{S_m} g(x, t) \text{Div} \sigma_m(t) dt = 0, \tag{18}
\]
where $\text{Div}$ is the surface divergence, and a consequence of formula (12)
\[
-\nabla^2_x \int_{S_m} g(x, t) \sigma_m(t) dt = k^2 \int_{S_m} g(x, t) \sigma_m(t) dt, \quad x \not\in S, \tag{19}
\]
one gets from (17) the following equation
\[
[N_s, \sigma_m(s)] + 2f_e(s) + 2B\sigma_m = 0. \tag{20}
\]
Here
\[
B\sigma_m := \left[ \int_{S_m} [N_s, \nabla_s g(s, t), \sigma_m(t)] dt, N_s \right] \tag{21}
\]
Take cross product of $N_s$ with the left-hand side of (20) and use the formulas $N_s \cdot \sigma_m(s) = 0$, $f_m := f_m(s) := 2[N_s, f_e(s)]$, and
\[
[N_s, [N_s, \sigma_m(s)]] = -\sigma_m(s), \tag{22}
\]
to get from (20) Equation (15):
\[
\sigma_m(s) = 2[N_s, f_e(s)] + 2[N_s, B\sigma_m] := A_m\sigma_m + f_m, \tag{23}
\]
where $A_m\sigma_m = 2[N_s, B\sigma_m]$. The operator $A_m$ is linear and compact in the space $C(S_m)$, so that Equation (23) is of Fredholm type. Therefore, Equation (23) is solvable for any $f_m \in T_m$ if the homogeneous version of (23) has only the trivial solution $\sigma_m = 0$. In this case the solution $\sigma_m$ to Equation (23) is of the order of the right-hand side $f_m$, that is, $O(a^{-\kappa})$ as $a \to 0$, see formula (16). Moreover, it follows from Equation (23) that the main term of the asymptotics of $\sigma_m$ as $a \to 0$ does not depend on $s \in S_m$.

**Lemma 2.** Assume that $\sigma_m \in T_m$, $\sigma_m \in C(S_m)$, and $\sigma_m(s) = A_m\sigma_m$. Then $\sigma_m = 0$.

Lemma 2 is proved in [7].

Let us write (10) as
\[
E(x) = E_0(x) + \sum_{m=1}^M [\nabla_x g(x, x_m), Q_m] + \sum_{m=1}^M \nabla \times \int_{S_m} (g(x, t) - g(x, x_m)) \sigma_m(t) dt, \tag{24}
\]
where
\[ Q_m := \int_{S_m} \sigma_m(t) dt. \] (25)

Since \( \sigma_m = O(a^{-\kappa}) \), one has \( Q_m = O(a^{2-\kappa}) \). We want to prove that the second sum in (24) is negligible compared with the first sum. One has
\[ j'_1 := ||\nabla_x g(x, x_m), Q_m|| \leq O \left( \max \left\{ \frac{1}{d^2}, \frac{k}{d} \right\} \right) O(a^{2-\kappa}), \] (26)

\[ j'_2 := |\nabla \times \int_{S_m} (g(x,t) - g(x,x_m)) \sigma_m(t) dt| \leq a O \left( \max \left\{ \frac{1}{d^3}, \frac{k^2}{d} \right\} \right) O(a^{2-\kappa}), \] (27)

and
\[ \frac{j'_2}{j'_1} = O \left( \max \left\{ \frac{a}{d}, k \right\} \right) \rightarrow 0, \quad \frac{a}{d} = o(1), \quad a \rightarrow 0. \] (28)

Thus, one may neglect the second sum in (26), and write
\[ E(x) = E_0(x) + \sum_{m=1}^{M} [\nabla_x g(x, x_m), Q_m] \] (29)

with an error that tends to zero as \( a \rightarrow 0 \). Let us estimate \( Q_m \) asymptotically, as \( a \rightarrow 0 \). Integrate Equation (23) over \( S_m \) to get
\[ Q_m = 2 \int_{S_m} [N_s, f^e(s)] ds + 2 \int_{S_m} [N_s, B \sigma_m] ds. \] (30)

It follows from (16) that
\[ [N_s, f^e] = [N_s, E^e] - \frac{\zeta_m}{i \omega \mu_0} [N_s, [\nabla \times E^e, N_s]]. \] (31)

If \( E^e \) tends to a finite limit as \( a \rightarrow 0 \), then formula (31) implies
\[ [N_s, f^e] = O(\zeta_m) = O \left( \frac{1}{a^{\kappa}} \right), \quad a \rightarrow 0. \] (32)

By Lemma 2 the operator \( (I - A_m)^{-1} \) is bounded, so \( \sigma_m = O \left( \frac{1}{a^{\kappa}} \right) \), and
\[ Q_m = O \left( a^{2-\kappa} \right), \quad a \rightarrow 0, \] (33)

because integration over \( S_m \) adds factor \( O(a^2) \). As \( a \rightarrow 0 \), the sum (29) converges to the integral (see [8], Lemma 1)
\[ E = E_0 + \nabla \times \int_S g(x, s) N(s) Q(s) ds, \] (34)
where $N(s)$ is the function from (2), and $Q(s)$ is the function such that

$$Q_m = Q(x_m)a^{2-\kappa}. \quad (35)$$

The function $Q(y)$ can be expressed in terms of $E$:

$$Q(y) = -4\pi i \omega \epsilon_0 h(s)(\nabla \times E)(s), \quad (36)$$

see [7].

Here the factor $4\pi$ appears if $D_m$ are balls. Otherwise a factor $c_m$, depending on the shape of $S_m$, should be used in place of $4\pi$. The factor $c_m$ is defined by the formula

$$\int_{S_m} \nabla \times E_e(s)ds = c_m a^2 \nabla \times E_e(x_m),$$

where $x_m \in S$ is a point in $D_m$.

Thus, Equation (36) takes the form

$$E(x) = E_0(x) - 4\pi i \omega \epsilon_0 \nabla \times \int_{S} g(x,s) \nabla \times E(s)h(s)N(s)ds. \quad (37)$$

It follows from Equation (37) that the limiting field $E(x)$ satisfies Equation (8) away from $S$, and a transmission boundary condition on $S$:

$$[N_s, E_-(s) - E_+(s)] = -4\pi i \omega' h(s)N(s). \quad (38)$$

3. CONCLUSIONS

It is proved that a distribution of many small impedance particles in a thin layer on a smooth surface $S$ can be described by a transmission boundary condition (38). This condition shows that the equivalent surface currents on $S$ are calculated analytically in the limit $a \to 0$ in terms of boundary impedance function $h$ and the distribution density function $N(s)$. Therefore, these currents can be controlled.

REFERENCES


