

## A MATHEMATICAL DERIVATION OF EQUIVALENT MODEL OF NORMAL MODE HELICAL ANTENNA

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**Abstract**—A mathematical investigation of a normal mode helical antenna (NMHA) is presented to provide an equivalent model. The vector potential of a single-turn NMHA using a developed helix line is first derived. To avoid complexity in the vector potential, a useful relationship between the source point and the helix line is established. Employing this relationship, the integral of the vector potential can be calculated as that of a linear current antenna, and the result leads to an equivalent model that is a combination of the electric dipole and the magnetic dipole, i.e., exactly the same as assumed in previous work. A helix line of several turns can be regarded as a combination of the turns. Thus a general NMHA can be analysed as the sum of the vector potentials of the turns in the helix. To verify the obtained formulas, the calculated radiation characteristics are compared with the results of the commercial simulation, showing good agreement.

### 1. INTRODUCTION

A normal-mode helical antenna (NMHA) has been studied for more than half a century, and the operation principles are well known by a simple equivalent model, proposed by Kraus and Wheeler, based on the assumption that the each turn is the superposition of electric and magnetic dipoles [1, 2]. Based on the equivalent model, radiation characteristics such as the radiation pattern and polarizations were obtained and proved by a number of works. Most analyses used to prove these results are based on numerical methods [3–5]. Although these approaches are widely and successfully used, they do not provide straightforward insights. This paper is an attempt to show a direct calculation of the vector potential of NMHA in order to give more straight-forward for the operation principles.

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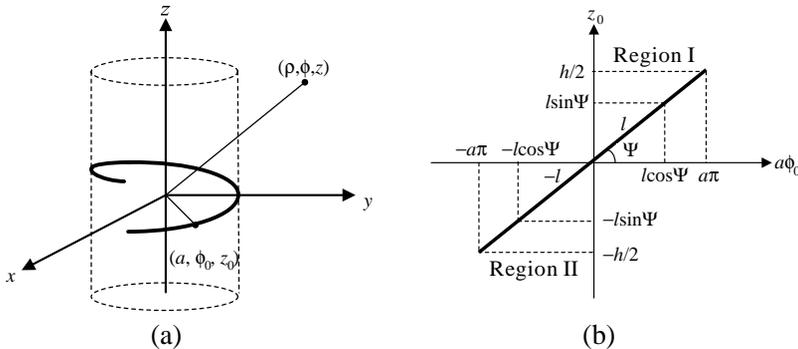
The vector potential, a well-known formulation for an antenna, is employed to analyse a 1-turn NMHA. The major difficulty in using the vector potential to analyse NMHA is that the integrands cannot be separated in general coordinate systems [6]. To overcome this problem, the helix line on the surface of a cylinder-wound helix wire is developed, and the relationship between the source point and the helix line is established. Using this relationship, the integrands in the vector potential can be expressed as functions that depend on only one variable. A simple technique that involves binomial and Maclaurin expansion is also used. Using this procedure, the vector potential can be calculated, and the equivalent model is proposed.

A helix line with  $N$ -turns can be considered to be a combination of the turns. Thus, an NMHA with  $N$ -turns can be analysed using the sum of the vector potentials of the turns. Here the vector potential of the each turn is based on the procedure for a single turn. The radiation resistance and directivity are calculated from the radiation fields. These radiation characteristics were not discussed by the previous papers due to a lack of available formulations. In an electrically small helix with a very thin wire with infinity conductivity, the current distribution can be assumed to be constant and in phase along the wire. Throughout this paper, the time harmonic  $e^{j\omega t}$  is suppressed.

## 2. VECTOR POTENTIAL

### 2.1. Single-turn NMHA

Consider a 1-turn helix wound on a cylinder in the right-hand direction (Fig. 1(a)). The radius of the helix is  $a$ , the height  $h$ , the pitch angle



**Figure 1.** (a) Configuration of 1-turn NM-helix. (b) Developed helix line.

$\psi$ , and the total wire length  $L$ . Unwinding the helix by rolling the cylinder right-handed on a flat surface yields a straight line  $l$  in the  $a\phi' - z'$  plane (Fig. 1(b)). Line  $l$  shows a positive value at region I and a negative value at region II, and has a magnitude equivalent to the length of the wire from the origin to the source point. The line of a 1-turn helix is centered symmetrically about  $(-x)$ -axis; this concept is essentially the same as the conventional developed helix line [4].

The vector potential is a well-known formulation to solve the far-field problem of antennas with a known current distribution. The standard expression for an NMHA in cylindrical coordinates is

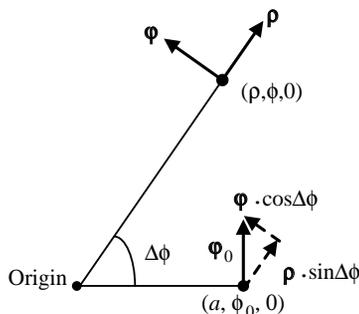
$$\mathbf{A}(\rho, \phi, z) = \int_{-L/2}^{L/2} \mathbf{i}_o \cdot \frac{\mu I_o \exp(-jkR_o)}{4\pi R_o} dl', \quad (1)$$

with

$$R_o = [\rho^2 + z^2 - 2(\rho a \cdot \cos \Delta\phi + z \cdot z') + a^2 + z'^2]^{1/2}, \quad (2)$$

where  $R_o$  is the distance function from the source point  $(a, \phi', z')$  along the helix wire to the field point  $(\rho, \phi, z)$  at a certain arbitrary point in the far-field region,  $\mathbf{i}_o$  the unit vector of the current distribution tangential to the helix wire, and  $\Delta\phi$  the phase difference between the source point and the field point on the  $x$ - $y$  plane. In terms of the integral in (1), the variables of the integrands are the two components  $\phi'$  and  $z'$  of the source point. Define the range of  $\Delta\phi$  to be from  $-\pi$  to  $\pi$ . Because the vector potential will be independent of  $\phi$ , considering only  $\phi = 0$  and  $-\pi \leq \phi' \leq \pi$  is convenient without loss of universality.

The major problem is to calculate the integral in the vector potential. To convert the integrands to functions that depend on only



**Figure 2.** Configuration of unit vectors on  $x$ - $y$  plane.

one variable, we first establish the correspondence between the helix line and the variables in the source point:

$$\phi' = \frac{\cos \psi}{a} \cdot l' = \sigma l', \quad (3a)$$

$$z' = \sin \psi \cdot l', \quad (3b)$$

where  $\psi$  is the pitch angle, and  $\sigma \cos \psi/a$  is used for expression simplicity. Using (3), the distance function  $R_0$  can be written in terms of the helix line  $l$  as

$$\begin{aligned} R_o &= [\rho^2 + z^2 - 2(\rho a \cdot \cos \sigma l' + \sin \psi \cdot z l') + a^2 + \sin^2 \psi \cdot l'^2]^{1/2} \\ &= [r^2 - 2r(a \sin \theta \cos \sigma l' + \cos \theta \sin \psi \cdot l') + a^2 + \sin^2 \psi \cdot l'^2]^{1/2} \end{aligned} \quad (4)$$

and the unit vector of the current distribution (Fig. 2) can be written as

$$\begin{aligned} \mathbf{i}_0 &= \boldsymbol{\rho} \cdot \sin \sigma l' + \boldsymbol{\varphi} \cdot \cos \psi \cos \sigma l' + \mathbf{z} \cdot \sin \psi \\ &= \mathbf{r} \cdot [\sin \theta \cos \psi \sin \sigma l' + \cos \theta \sin \psi] \\ &\quad + \boldsymbol{\theta} \cdot [\cos \theta \cos \psi \sin \sigma l' - \sin \theta \sin \psi] + \boldsymbol{\varphi} \cdot \cos \psi \cos \sigma l', \end{aligned} \quad (5)$$

where  $\boldsymbol{\varphi}_o$  and  $\mathbf{z}_o$  are the unit vector of the source point;  $(\boldsymbol{\rho}, \boldsymbol{\varphi}, \mathbf{z})$  and  $(\mathbf{r}, \theta, \boldsymbol{\varphi})$  are the unit vectors of the field point in cylindrical and spherical coordinates respectively.

In the far-field region, the distance function for amplitude term  $1/R_o$  can be approximated as  $1/r$ . For phase amplitude  $e^{-jkR_o}$ , binomial expansion may be used to obtain a series representation, and the approximate expression is

$$\begin{aligned} R_o &= r[1 - 2r^{-1}(a \sin \theta \cos \sigma l' + \cos \theta \sin \psi \cdot l') + r^{-2}(a^2 + \sin^2 \psi \cdot l'^2)]^{1/2} \\ &\approx r[1 - r^{-1}(a \sin \theta \cos \sigma l' + \cos \theta \sin \psi \cdot l')]. \end{aligned} \quad (6)$$

Substituting (6) into the phase amplitude and using the Maclaurin series expansion yields

$$\exp(-jkR_o) \approx \exp(-jkr) \sum_n \frac{(jka \sin \theta + \cos \theta \sin \psi \cdot jkl')^n}{n!}. \quad (7)$$

Because  $ka \ll 1$  and  $kl \ll kL \ll 1$ ,  $(ka + kl)^n/n!$  decrease rapidly as  $n$  increases, we consider only the first two terms:

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu I_o \exp(-jkr)}{4\pi r} \cdot \int_{-L/2}^{L/2} \mathbf{i}_o \cdot (1 + jka \sin \theta + \cos \theta \sin \psi \cdot jkl) \cdot dl. \quad (8)$$

For simplicity, the vector potential is expressed in terms of the  $r$ ,  $\theta$  and  $\phi$  components, respectively:

$$\begin{aligned}
 A_r &= \frac{\mu I_o \exp(-jkr)}{4\pi r} \int_{-L/2}^{L/2} (\sin \theta \cos \psi \sin \sigma l + \cos \theta \sin \psi) \\
 &\quad \cdot (1 + jka \sin \theta \cos \sigma l + \cos \theta \sin \psi \cdot jkl) \cdot dl \\
 &\approx \frac{\mu \exp(-jkr)}{4\pi r} \cos \theta \cdot I_o L \sin \psi
 \end{aligned} \tag{9a}$$

$$\begin{aligned}
 A_\theta &= \frac{\mu I_o \exp(-jkr)}{4\pi r} \int_{-L/2}^{L/2} (\cos \theta \cos \psi \sin \sigma l - \sin \theta \sin \psi) \\
 &\quad \cdot (1 + jka \sin \theta \cdot \cos(2\pi l/L) + \cos \theta \sin \psi \cdot jkl) \cdot dl \\
 &\approx -\frac{\mu \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o L \sin \psi
 \end{aligned} \tag{9b}$$

$$\begin{aligned}
 A_\phi &= \frac{\mu I_o \exp(-jkr)}{4\pi r} \int_{-L/2}^{L/2} \cos \psi \cos \sigma l \\
 &\quad \cdot (1 + jka \sin \theta \cos \sigma l + \cos \theta \sin \psi \cdot jkl) \cdot dl \\
 &= \frac{jk\mu e^{-jkr}}{4\pi r} \sin \theta \cdot \frac{I_o a L \cos \psi}{2}.
 \end{aligned} \tag{9c}$$

Putting  $\sin \psi = h/L$  and  $\cos \psi = 2\pi ah/L$ , and using  $\mathbf{z} \cdot A_z = \mathbf{r} \cdot A_z \cos \theta - \boldsymbol{\theta} \cdot A_z \sin \theta$ , the vector potential becomes

$$A_z = \frac{\mu \exp(-jkr)}{4\pi r} \cdot I_o h \tag{10a}$$

$$A_\phi = \frac{jk\mu e^{-jkr}}{4\pi r} \sin \theta \cdot I_o S_h, \tag{10b}$$

where  $S_h = \pi a^2$  is the area of the loop for the magnetic dipole. These results lead to an equivalent model that is the combination of electric dipole  $I_o h$  and magnetic dipole  $I_o S_h$ , exactly as assumed in previous work [1, 2].

The electric and magnetic dipoles in NMHAs generate dual modes simultaneously. The radiation fields of excited by the electric dipole are

$$E_\theta = \frac{j\omega\mu \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o h \tag{11a}$$

$$H_\phi = \frac{jk \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o h, \tag{11b}$$

and the radiation fields excited by the magnetic dipole are

$$E_\phi = \frac{\eta k^2 \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o S_h \quad (12a)$$

$$H_\theta = -\frac{k^2 \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o S_h. \quad (12b)$$

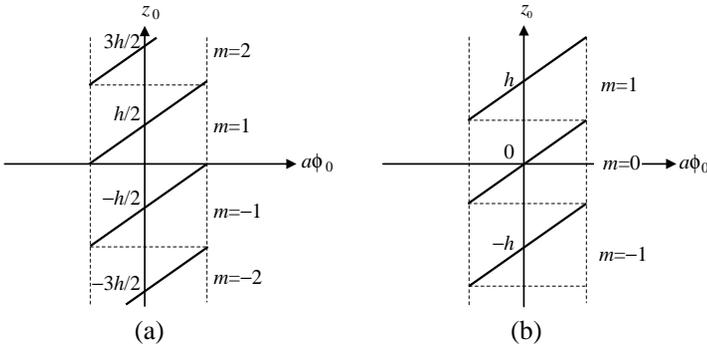
When  $\psi$  is close to zero,  $I_o h = 0$  and the given NMHA degenerates to a small loop. In this case, a 1-turn NMHA predominantly generates  $TE_{01}$  mode excited by the magnetic dipole. If  $\psi$  is close to  $\pi/2$ ,  $I_o S_h = 0$  and the NMHA degenerates to an electrically small dipole. In this case, a 1-turn NMHA radiates  $TM_{01}$  mode excited by the electric dipole.

## 2.2. $N$ -turn NMHA

The wire length  $NL$  and height  $Nh$  of the  $N$ -turn NMHA are assumed to be electrically small and that the current distribution along the helix wire is constant and in phase. If the equivalent form is used, an  $N$ -turn NMHA can be regarded as a combination of the electric dipole  $I_o Nh$  and  $N$ -array of a small loop with spacing  $h$ . However, for explicit formulation, we develop the helix line for  $N$  turns and calculate its vector potential.

The type of helix line for an  $N$ -turn NMHA is divided into odd and even  $N$  (Fig. 3); this characteristic allows simplification of the analysis. For odd  $N = 2N_o + 1$ , the vector potential can be expressed as

$$\mathbf{A} = \sum_{m=1}^{N_e} \mathbf{A}_m + \sum_{m=-N_e}^{-1} \mathbf{A}_m, \quad (13)$$



**Figure 3.** Developed helix line in (a) even  $N$ , (b) odd  $N$ .

and for even  $N = 2N_e$  the vector potential can be written by

$$\mathbf{A} = \sum_{m=-N_o}^{N_o} \mathbf{A}_m, \tag{14}$$

where  $N_e$  and  $N_o$  are any positive integers. The first step to calculate the vector potential of  $N$ -turn NMHA is to establish the relationship of  $m$ th turn.

For odd  $N$ , Equation (3b) is rewritten as

$$z_0 = mh + l \cdot \sin \psi. \tag{15}$$

Substituting Equation (15) into Equation (7), the distance function for the  $m$ th helix line becomes

$$R_m \approx r [1 + r^{-1} (a \sin \theta \cos \sigma l + \cos \theta \sin \psi \cdot l + mh \cos \theta)]. \tag{16}$$

Then the phase amplitude can be extended to

$$\begin{aligned} \exp(-jkR_m) &\approx \exp(-jkr + jkmh \cos \theta + jka \sin \theta + \sin \psi \cos \theta \cdot jkl) \\ &= \exp(-jkr + jkmh \cos \theta) \sum_n \frac{(jka \sin \theta + \cos \theta \sin \psi \cdot jkl)^n}{n!}. \end{aligned} \tag{17}$$

We also consider only the first two terms and substitute (17) into (1) which results in

$$\begin{aligned} \mathbf{A}_m(r, \theta, \phi) &= \frac{\mu I_o \exp(-jkr + jknh \cos \theta)}{4\pi r} \\ &\cdot \int_{-L/2}^{L/2} \mathbf{i}_0 \cdot (1 + jka \sin \theta + \cos \theta \sin \psi \cdot jkl) dl \\ &= \exp(jkmh \cos \theta) \cdot \mathbf{A}_0. \end{aligned} \tag{18}$$

The vector potential of odd  $N$  becomes

$$\begin{aligned} \mathbf{A} &= \sum_{m=-N_e}^{N_e} \mathbf{A}_m = \mathbf{A}_0 + \sum_{m=1}^{N_e} (\mathbf{A}_{-m} + \mathbf{A}_m) \\ &= \mathbf{A}_0 + 2\mathbf{A}_0 \sum_{m=1}^{N_e} \cos(jkmh \cos \theta) \approx \mathbf{A}_0 \cdot (2N_e + 1) = \mathbf{A}_0 \cdot N. \end{aligned} \tag{19}$$

For even  $N$ , Equation (3b) is rewritten as

$$z_0 = h \left( m + \frac{1}{2} \right) + l \cdot \sin \psi. \tag{20}$$

Following the procedure used for odd  $N$ , the vector potential of even  $N$  can be expressed as

$$\begin{aligned} \mathbf{A} &= \sum_{m=1}^{N_e} \mathbf{A}_m + \sum_{m=-N_e}^{-1} \mathbf{A}_m = \mathbf{A}_0 \cdot \sum_{m=1}^{N_e} 2 \cos(jkh(m+1/2) \cos \theta) \\ &\approx \mathbf{A}_0 \cdot 2N_0 = \mathbf{A}_0 \cdot N. \end{aligned} \quad (21)$$

These results demonstrate that the vector potential for an  $N$ -turn NMHA equal  $N$  times of the vector potential for 1-turn NMHA.

The radiation fields excited by the electric dipole of  $N$ -turn NMHA are

$$E_\theta = \frac{j\omega\mu \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o N h \quad (22a)$$

$$H_\phi = \frac{jk \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o N h, \quad (22b)$$

and the radiation fields excited by the magnetic dipole of  $N$ -turn NMHA are

$$E_\phi = \frac{\eta k^2 \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o N S_h \quad (23a)$$

$$H_\theta = -\frac{k^2 \exp(-jkr)}{4\pi r} \sin \theta \cdot I_o N S_h. \quad (23b)$$

### 3. RESULTS AND DISCUSSION

The radiation characteristics are calculated using the obtained radiation fields, to extend the understanding of the NMHA. Let us consider the radiation resistance, normalized radiation pattern, and axial-ratio (AR). Using (22) and (23), the radiation resistance  $R_h^{rad}$  can be expressed as

$$R_h^{rad} = \frac{1}{I_o^2} \operatorname{Re} \int_S [(E_\theta \times H_\phi^*) + (E_\phi \times H_\theta^*)] = 20 [(kNh)^2 + (k^2 NS_h)^2], \quad (24)$$

the normalized radiation pattern in the  $x$ - $y$  and  $y$ - $z$  plane can be calculated as

$$E_n = \frac{E_\theta(\theta, \phi) + E_\phi(\theta, \phi)}{E_\theta(\theta, \phi)_{\max} + E_\phi(\theta, \phi)_{\max}} = \begin{cases} 1, & x\text{-}y \text{ plane} \\ \sin \theta, & y\text{-}z \text{ plane} \end{cases}, \quad (25)$$

and the axial-ratio (AR) can be obtained exactly as given in [2].

$$\mathbf{AR} = \frac{|E_\theta|}{|E_\phi|} = \frac{h}{k \cdot S_h} \quad (26)$$

Now, the obtained radiation characteristics are compared with the results of the commercial simulation FEKO to show validity. Note that the obtained results are only focused on the helix structure as an ideal radiator under the assumption of the uniform current. In practice, the radiation characteristics of the NMHA are critically affected by the fed-configuration, magnitude and phase of the current. To reduce these effects, as shown in Fig. 4, we consider center-fed shorted NMHA on the ground plane with ferrite sheet. The NMHA parameters are the height  $3h = 0.05$  m, radius  $a = 0.02$  m, total turn number  $N = 3$ , wire thickness  $0.0001$  m, and shorted wire length  $d = 0.4$  m. The shorted wires can be regarded as a two-wire transmission line, which does not contribute to the radiation characteristics, and the role of the ferrite sheet is to diminish the ground effect.

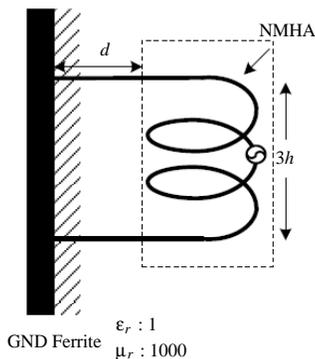


Figure 4. Configuration of NMHA in practice.

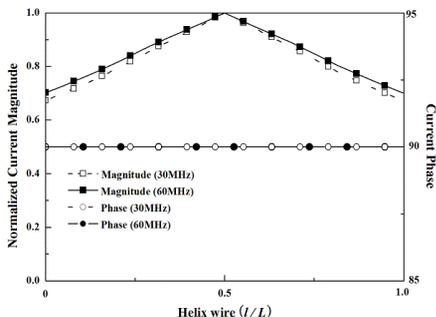


Figure 5. Simulated current in magnitude and phase along the helix wire.

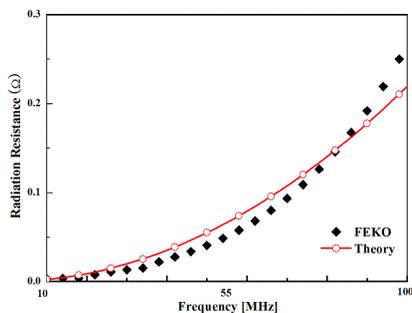
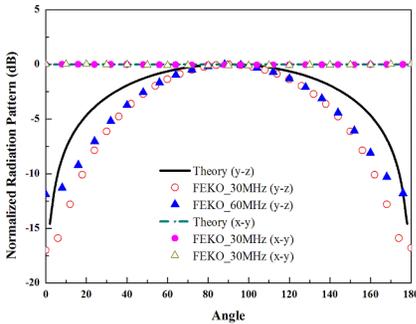
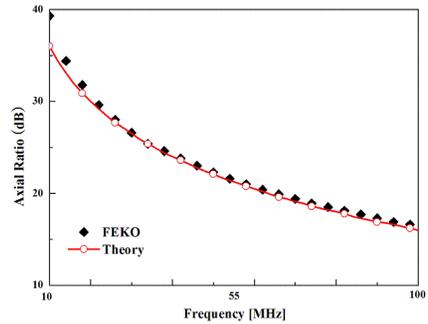


Figure 6. Radiation resistance of NMHA.

The simulated current in magnitude and phase along the helix wire are plotted in Fig. 5, which can be reasonably approximated as the uniform current. In Figs. 6–8, the radiation characteristics are considered. These results are in good agreement with the simulation ones.



**Figure 7.** Normalized radiation pattern of NMHA.



**Figure 8.** Axial ratio of NMHA.

#### 4. CONCLUSION

In this paper, a mathematical investigation of a normal mode helical antenna is presented. Previously, computation of the vector potential of an NMHA was severely limited by the difficulty of separating the integrand in general coordinate systems. To overcome this difficulty, we first developed a straight helix line on the surface plane of the cylinder and established the relationship between the helix line and the source point. Employing this relationship, the integrands become functions that depend only on the helix line and the vector potential of a 1-turn NMHA can be calculated. The equivalent model derived here is a combination of the electric and the magnetic dipoles, and is exactly the same as assumed in previous work. An NMHA with several turns was analyzed using the sum of the vector potentials of  $N$  turns. The results obtained were simplified as the products of the vector potential for a 1-turn helix times the total turn number  $N$ . Using this formulation, the radiation resistance and the directivity were derived. The radiation resistance of an NMHA is a series connection of the radiation resistances of the electric and magnetic dipoles. To verify the proposed equation, the radiation characteristics are compared with the commercial simulation and show a good agreement.

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