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On the Mutual Inductance between Non-Coaxial Coplanar Circular Loops

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Abstract—A simple and efficient explicit solution is derived for the mutual inductance of two non-coaxial coplanar circular loops, which is valid in the quasi-static as well as non-quasi-static frequency ranges. The solution is obtained by rigorously evaluating the Sommerfeld Integral describing the inductance, starting from expanding the integrand into a power series of the loop radius. As a result, a sum of simpler integrals is obtained, and term-by-term analytical integration is straightforwardly performed. The inductance is finally expressed as a series of spherical Hankel functions, with algebraic coefficients depending on the electrical size of the loops. Conducted numerical tests lead to conclude that, accuracy being equal, the proposed expression offers advantages in terms of time cost over conventional numerical integration techniques.

1. INTRODUCTION

Evaluation of the inductive coupling between current-carrying loops of wire is required in many fields of scientific interest, including wireless power transfer, magnetic resonance imaging, radio direction finding, electromagnetic sounding for exploration of terrestrial areas, electromagnetic induction heating [1–34]. The widespread interest in solving this problem responds to the need for strengthening or nullifying the magnetic coupling between any pair of coils that constitute the considered coil system, depending on the application. For instance, enhancement of magnetic coupling between a transmitting coil and a receiving coil is required in wireless power transfer systems, every time that the transmission efficiency of the inductive link must be optimized [1–3, 30]. On the other hand, reduction of magnetic coupling effects is desired in applications like magnetic resonance imaging (MRI), where multiple receiving coils are located in close proximity to each other to assure compact coverage of an area and obtain high signal-to-noise ratio images [4, 5]. It is easily understood how, in such a situation, inductive crosstalk among neighboring receivers must be minimized [4, 5].

In the last decades, different analytical expressions have been proposed that describe the mutual inductance between two circular loops [7, 26, 35, 36]. Yet, these solutions either consists of integral expressions that require intensive and time-demanding numerical evaluation [4, 5, 26], or are valid in the quasi-static frequency range only [7, 35, 36] and cannot be used when the effects of the displacement currents are not negligible. This may be the case, for instance, of applications where the operating frequency exceeds a few tens of MHz, like magnetic resonance imaging [1, 2] and shortwave inductive diathermy for therapeutic heating of tissues [30, 32–34, 37]. Here, the overall size of the whole two-coil system may not be sufficiently small for electromagnetic retardation to have negligible impact on the field distribution, and the quasi-static field assumption fails.

The purpose of the present paper is to determine a simple and efficient series-form expression for the mutual inductance of two identical coplanar loops, which allows to accurately calculate the

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inductance and, at the same time, to overcome the limitations implied by the previously published solutions to the same problem. In particular, the expression must be valid in both the quasi-static and non-quasi-static frequency regions, as long as the assumption of uniform current in the source loop is reasonable. This is true approximately up to the frequency at which the circumference of the loop is equal to one third of the free-space wavelength [38]. The proposed expression is obtained by turning the Sommerfeld-type integral describing the mutual inductance into a sum of simpler integrals, and this results from expanding the integrand into a power series of the loop radius. Then, term-by-term analytical integration is performed, and the mutual inductance is finally described by a power series of the electrical size of the loops, with the dependence on the distance between the loops expressed by spherical Hankel functions. The derived expression is valid as far as the thin-wire hypothesis, underlying the present theoretical development, holds. This implies that the wire radius must be much smaller than the loop radius. Numerical simulations are performed to show the advantages of the derived formula, in terms of accuracy and time cost, over the standard numerical techniques conventionally used to calculate self and mutual inductances of coils.

2. THEORY

The problem under study is sketched in Fig. 1. Two thin-wire air-cored co-planar circular loops have radius a, and are separated by the distance ρ. The time-harmonic integral representation for the total flux linkage per unit current between the coils is well known and given by [18, 26]

$$M = \pi\mu_0 a^2 \int_0^\infty \frac{1}{u_0} \left[J_1(\lambda a)\right]^2 \, J_0(\lambda\rho)\, \lambda d\lambda, \tag{1}$$

where $J_n(\cdot)$ is the nth-order Bessel function, and

$$u_0 = \sqrt{\lambda^2 - k_0^2}, \qquad k_0^2 = \omega^2\mu_0\epsilon_0, \tag{2}$$

being ϵ_0 and μ_0 the free-space dielectric permittivity and magnetic permeability, respectively. The scope of this work is to derive an explicit analytical expression for M. To do this, we first use the Maclaurin series expansion of the quantity $[J_1(\lambda a)]^2$, that is [39]

$$\left[J_1\left(\lambda a\right)\right]^2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-1)!!}{(2n)!!(n-1)!(n+1)!} \left(\lambda a\right)^{2n}, \tag{3}$$

and rewrite Eq. (1) as

$$M = \pi\mu_0 a^2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-1)!! a^{2n}}{(2n)!!(n-1)!(n+1)!} \int_0^\infty \frac{1}{u_0} J_0(\lambda\rho)\lambda^{2n+1} d\lambda. \tag{4}$$

Next, since it holds

$$\lambda^2 J_0(\lambda\rho) = -\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right) J_0(\lambda\rho) = -\nabla_t^2 J_0(\lambda\rho), \tag{5}$$

one obtains the identity

$$\lambda^{2n} J_0(\lambda\rho) = (-1)^n \left(\nabla_t^2\right)^n J_0(\lambda\rho), \tag{6}$$

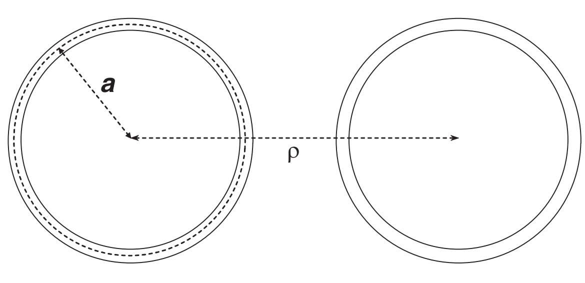

Figure 1. Sketch of two co-planar circular loops.

which makes it possible to turn Eq. (4) into

$$M = -\pi\mu_0 a^2 \sum_{n=1}^{\infty} \frac{(2n-1)!!\, a^{2n}}{(2n)!!(n-1)!(n+1)!} \left(\nabla_t^2\right)^n \int_0^{\infty} \frac{1}{u_0} J_0(\lambda\rho)\,\lambda d\lambda, \tag{7}$$

where the semi-infinite integral on the right-hand side is recognized to be the well-known tabulated Sommerfeld Integral [40–43]

$$\int_0^{\infty} \frac{1}{u_0} J_0(\lambda\rho)\lambda d\lambda = \frac{e^{-jk_0\rho}}{\rho} = -\,jk_0 h_0^{(2)}\left(k_0\rho\right), \tag{8}$$

being $h_0^{(2)}(\cdot)$ the zeroth-order spherical Hankel function of the second kind. Thus, after substituting Eq. (8) into Eq. (7), so as to obtain

$$M = jk_0\pi\mu_0 a^2 \sum_{n=1}^{\infty} \frac{(2n-1)!!\, a^{2n}}{(2n)!!(n-1)!(n+1)!} \left(\nabla_t^2\right)^n h_0^{(2)}\left(k_0\rho\right), \tag{9}$$

and performing the differentiations, it yields

$$M = j\pi\mu_0 a \sum_{n=1}^{\infty} b_n (k_0 a)^{2n+1} \sum_{m=0}^{n} c_{m,n} \frac{h_m^{(2)}\left(k_0\rho\right)}{(k_0\rho)^m}, \tag{10}$$

with

$$b_n = \frac{(2n-1)!!}{2^n(n-1)!(n+1)!}, \tag{11}$$

and

$$c_{m,n} = (-1)^{m+n} \frac{(2m-1)!!}{(m)!(n-m)!}. \tag{12}$$

Expression (10) is valid under the assumption that the current in the source loop is uniformly distributed, which underlies the derivation of Eq. (1) [18]. Since a nearly uniform current in a loop antenna can be obtained as long as $k_0 a<0.3$ [38], it turns out that Eq. (10), seen as a power series of $k_0 a$, gives the magnetic field radiated by the primary loop in the radial direction as the superposition of spherical waves, with decreasing amplitude as the order n of the wave function increases.

3. NUMERICAL RESULTS

To test the proposed approach, expression (10) is first used to calculate the amplitude-frequency spectrum of the mutual inductance between two loops, 2 cm in radius, separated by the radial distance ρ=6 cm. Different values for the index N at which Eq. (10) is truncated are considered, and the results of the computation, shown in Fig. 2, are compared with the data arising from numerical integration of the complete integral representation in Eq. (1). Numerical integration is performed by using a Gauss-Kronrod G7-K15 scheme, resulting from the combination of a 7-point Gauss rule with a 15-point Kronrod rule. The plotted curves point out how the outcomes from Eq. (10) with N=5 are in excellent agreement with G7-K15 data all over the considered frequency range. Moreover, a variation of frequency does not affect convergence of Eq. (10). It should be also noted that, as frequency is decreased, the trend of $|M|$ becomes nearly horizontal and approaches its quasi-static limit, where the fields exhibit a predominantly static behavior. Here, the inductance may be computed by assuming that the displacement currents are negligible (quasi-static field approximation). As an example, for a=2 cm and ρ=6 cm the inductance extraction software Fast-Henry [44], based on the magnetoquasistatic condition, provides $M\cong$0.989 nH, a value that is in agreement with the low-frequency limit in Fig. 2. On the other hand, the outcomes from the quasi-static approximation do not depend on frequency and, starting from about 50 MHz, they cannot reproduce the effective trend of $|M|$ any longer. This aspect is further clarified by Fig. 3, which depicts ρ-profiles of the amplitude of M, calculated by using the proposed approach, Fast-Henry computational package, and Gauss-Kronrod integration of Eq. (1). The operating frequency is 100 MHz, while the radius of the loops is taken to be a=5 cm. As is seen, the curves arising from retaining only

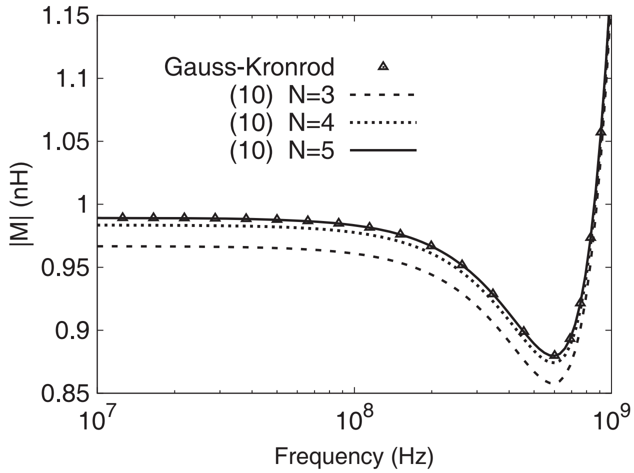

Figure 2. Mutual inductance between two loops with radius a=2 cm, computed against frequency.

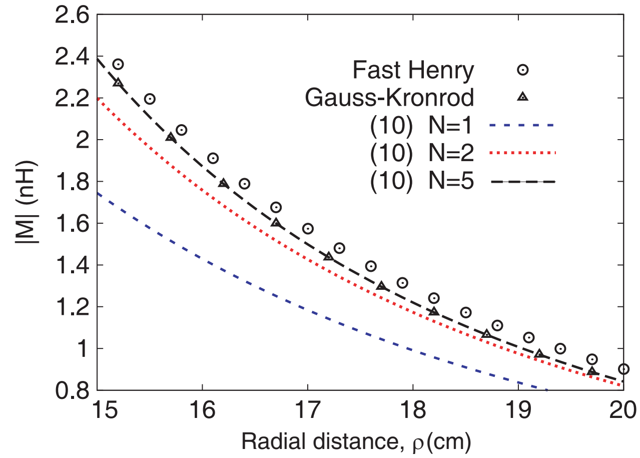

Figure 3. Mutual inductance between two loops with radius a=5 cm, computed against the radial distance ρ.

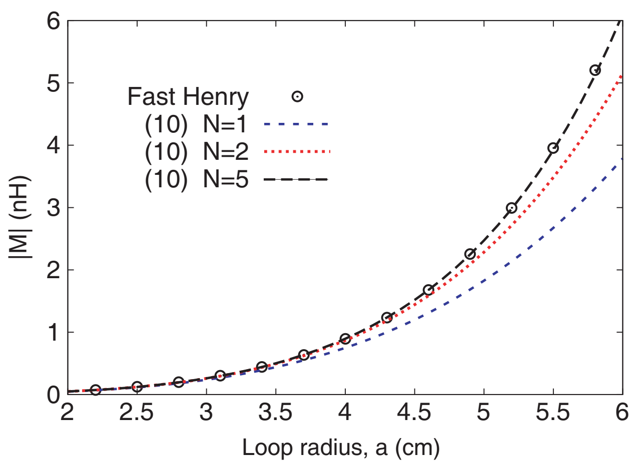

Figure 4. Mutual inductance between two loops separated by the distance ρ=15 cm, computed against the loop radius a.

5 terms in Eq. (10) and the data from numerical integration are still overlapping, regardless of the value of the radial distance ρ. Instead, Fast-Henry program overestimates the mutual inductance, and this happens because at the frequency of 100 MHz the two-loop system has entered the non-quasi-static frequency region. In fact, if D is the diagonal of the rectangular bounding box that encloses the two loops, its minimum value is $D_{\min}=\sqrt{(\rho_{\min}+2a)^2+(2a)^2}=\sqrt{(15+10)^2+10^2}\cong 27\,\mathrm{cm}$, while $k_0=2\pi/3\cong 2.094\,\mathrm{m}^{-1}$. As a consequence, $k_0 D_{\min}\cong 0.57>0.1$, which means that, for all the considered values of ρ, the overall size of the whole two-loop system is not sufficiently small for electromagnetic retardation to have negligible impact on the field distribution. On the other hand, it suffices to decrease frequency by one order of magnitude (that is down to 10 MHz) to have $k_0 D_{\min}\cong 0.057<0.1$ and, hence, to make the quasi-static field assumption valid again. This point is illustrated by Fig. 4, which depicts profiles of the amplitude of M against the loop radius a, calculated at the operating frequency 10 MHz by using the proposed method and Fast-Henry program. The radial distance between the loops is taken to be ρ=15 cm. As can be observed, Fast-Henry data now agree perfectly with the curve originating from Eq. (10) with N=5. This is expected, since the largest diagonal of the rectangular bounding box that encloses the two loops is $D_{\max}=\sqrt{(\rho+2a_{\max})^2+(2a_{\max})^2}=\sqrt{(15+12)^2+12^2}\cong 29.55\,\mathrm{cm}$, while

$k_0{=}2\pi/30{\cong}0.2094\,\mathrm{m}^{-1}$. As a consequence, $k_0 D_{\max}{\cong}0.062{<}0.1$, and the quasi-static field assumption holds for all the considered values of the loop radius a.

Accuracy being equal, use of Eq. (10) instead of Gauss-Kronrod scheme permits to achieve significant time savings. This is confirmed by Table 1, which shows the average CPU time taken by the two methods to compute the ρ-profiles of M depicted in Fig. 3. Table 1 also illustrates the speed-up offered by the new method with respect to Fast-Henry tool, that is the ratio of the time taken by the computational package to that required by Eq. (10). As seen, use of the proposed series expression with $N{=}10$ instead of Fast-Henry program allows both to improve the accuracy of the result of the computation and to significantly reduce the time cost.

Table 1. CPU time comparisons for the computation of M.

Approach	average CPU time [s]	Speed-Up
Fast-Henry	1.69	-
Numerical Integration	1.36×10^{2}	1.24×10^{-2}
(10) with $N = 5$	6.79×10^{-4}	2.49×10^{3}
(10) with $N = 7$	9.25×10^{-4}	1.83×10^{3}
(10) with $N = 10$	3.37×10^{-3}	5.01×10^{2}

4. CONCLUSIONS

The scope of this work has been to present a simple and efficient explicit formula for the mutual inductance between two coplanar single-turn coils, which is valid in the quasi-static as well as non-quasi-static frequency ranges. The formula has been obtained by expanding the integrand of the integral representation for the inductance into a power series of the loop radius. This has made it possible to convert the original integral representation into a sum of simpler integrals, and then to perform term-by-term analytical integration. As a result, the inductance is expressed by a series of spherical Hankel functions, with algebraic coefficients depending on the electrical size of the loops. Numerical simulations have been carried out to show the advantages of the proposed formula in terms of accuracy and time cost.

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