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A SYNTHESIS OF UNEQUALLY SPACED ANTENNA ARRAYS USING LEGENDRE FUNCTIONS

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Abstract—In the wireless communication systems, the objective of using array antennas is to extract the desired signal while filtering out the unwanted interferences. New methods are developed for the optimization and synthesis, in terms of the directivity and the side lobe level. For the optimization of the array pattern, it is proposed to adjust both the excitation and the spacing of the antenna elements. The determination of the optimal excitation and spacing is shown to be a polynomial problem; this is described in terms of a unified mathematical approach to nonlinear optimization of multidimensional array geometries. The approach utilizes a class of limiting properties of Legendre functions that are dictated by the array geometry addressed. The objective of this paper is to describe a unified mathematical approach to nonlinear optimization of multidimensional array geometries.

1. INTRODUCTION

During the last years, wireless technology has grown at a formidable rate. This has resulted in an increase in the number of subscribers and a better quality of the system. The most practical solution to this problem is to use spatial processing [1]. Spatial processing is the central idea of the smart antennas technology. A system of smart antennas

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is constituted by array antennas and the processing related to each element of antenna. With this processing it is possible to identify the space location of each user and to establish a constant space filtering.

In a system of array antennas, an open problem is the synthesis of the radiation pattern with characteristics of high directivity, minimized side lobes, and adaptability to the radio channel. Within the synthesis of the radiation pattern, the determination of the amplitude excitation of the antenna elements that form an array, as well as, the structure of the array, that improves the properties of radiation of the array, is one of the issues of greater interest within the technology of smart antennas in wireless communications. Side lobe level reduction has been widely studied [1, 2]. Many of these techniques try to find the right elements excitation to diminish side lobe level, other techniques proposed in [3] and [4] were developed to reduce the side lobe level using non-uniform spacing. More recently, genetic algorithms (GA) and differential evolution (DE) illustrated in [5] and [6], respectively, were used to solve the problem of the synthesis of the radiation pattern. In this work we considered the use of unified mathematical approaches, based on using Legendre functions [7, 8], in order to optimize the synthesis of the radiation pattern in terms of small amplitude of side lobes and maximum gain. The mathematical model has an additional advantage, of being a generic model so that can be applied to different types of antenna arrays geometries.

The emphasis of this work is to develop a model of array antennas, seeking to optimize the directivity, the side lobe level and the computation time of the algorithm improving the response time.

The paper is organized as follows: Section 2 presents the problem formulation to describe the linear and the planar array factor in terms of Legendre polynomials. Section 3 presents the numerical results for the directivity, side lobe level and the computation time. Finally, Section 4 states the conclusion of this work.

NOMENCLATURE

$k = 2\pi/\lambda$: free-space wave number.
d_n: element position.
I_n: element excitation.
ε_n: position perturbation of the element.
Δu: sampling interval.
M: sampled points.
α_p, β_n: transformation vectors.
$P_{m-1/2}(\cos\alpha)$: Legendre function of fractional order.
φ: progressive phase.
θ_d: direction of the main lobe.

2. PROBLEM FORMULATION

2.1. Linear Array Synthesis Using Legendre Functions

The linear antenna array considered in this work is a symmetric linear array of $2N + 1$ elements as shown in Fig. 1.

The characteristics of the array factor and the field of the array can be controlled varying the separation and excitation between elements. The array factor is given as follows [1]

$$AF = E(u) = \sum_{n=0}^{N} \varepsilon_n I_n \cos(kd_n u) \tag{1}$$

where $u = \cos(\theta) + \varphi$, in the limit $0 \leq \theta \leq \pi$ radians. In order to establish the array factor in terms of Legendre polynomials is considered a desired array pattern defined as [7]:

$$E_d(u), \text{ in the interval } -1 \leq u \leq 1. \tag{2}$$

According to the Figure 1, the array factor is symmetric, i.e., $E(-u) = E(u)$ [4], therefore, we consider the synthesis problem in the interval $0 \leq u \leq 1$. Its response is uniformly sampled at M points ($M \gg 1$) in the interval $0 \leq u \leq 1$ to obtain

$$\begin{aligned} E(u_m) &= \sum_{n=0}^{N} \varepsilon_n I_n \cos(m\beta_n - \varphi_n); \\ m &= 0,\ 1,\ 2,\ \ldots,\ M-1 \end{aligned} \tag{3}$$

where $\Delta u = 1/(M-1)$, $u_m = m\Delta u$, $\varphi_n = kd_n \cos(\theta_d)$ and $\beta_n = kd_n\Delta u$. The following step is to apply the Legendre transformation

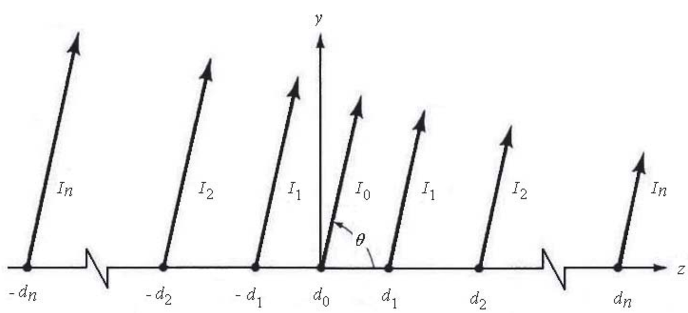

Figure 1. Geometry of $2N + 1$ element no periodic symmetric linear array.

$F(\alpha_p)$ to the array factor. The transformation is implemented to get a triangular set of equations and the final expression is:

$$\begin{aligned} F(\alpha_p) &= \sum_{m=0}^{M-1} \varepsilon_m E_d(u_m) P_{m-1/2}(\cos\alpha_p); \\ p &= 0,\, 1,\, 2,\, 3,\, \ldots,\, N \end{aligned} \tag{4}$$

where $\varepsilon_m = 1$, $m = 0$; $\varepsilon_m = 2$, $m > 0$; the Legendre transformation of the desired array pattern $E_d(u)$ is motivated by the following limiting relation for the Legendre polynomial of fractional order [9]:

$$\begin{aligned} f(\alpha,\, \beta) &= \sum_{m=0}^{\infty} \varepsilon_m P_{m-1/2}(\cos\alpha)\cos(m\beta) \\ &= \left[2/(\cos\beta - \cos\alpha)\right]^{1/2}, \quad 0 \le \beta < \alpha \\ &= 0, \qquad\qquad\qquad\qquad\quad \alpha < \beta < \pi \end{aligned} \tag{5}$$

Utilizing (4) and (5), we obtain the following triangular system of equations:

$$F(\alpha_p) = \sum_{n=0}^{p} I_n f(\alpha_p,\, \beta_n) \tag{6}$$

From (6), this system is invertible to obtain the value of first element current and the p-th element current.

$$\begin{aligned} I_0 &= F(\alpha_0)/f(\alpha_0,\, \beta_0) \\ I_p &= \frac{F(\alpha_p) - \sum\limits_{n=0}^{p-1} I_n f(\alpha_p,\, \beta_n);}{f(\alpha_p,\, \beta_p)}; \\ p &= 1,\, 2,\, 3,\, \ldots,\, N \end{aligned} \tag{7}$$

with the above parameter (I), it is possible to synthesize the radiation pattern of linear antenna arrays, in a fast and simple manner.

2.2. Planar Array Synthesis Using Legendre Functions

Consider a symmetric planar array of $N_1 \times N_2$ elements as show in Fig. 2.

A desired 2-D array pattern is defined as follows [8]:

$$E_d(u,\, v), \text{ in the interval } -1 \le u \le 1;\; -1 \le v \le 1 \tag{8}$$

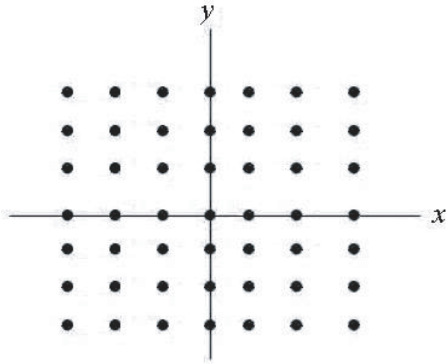

Figure 2. Geometry of $N_1 \times N_2$ element planar array.

where $u = \sin(\theta)\cos(\phi)$, $v = \sin(\theta)\sin(\phi)$ in the limits $0 \le \theta \le \pi$; $0 \le \phi \le 2\pi$ radians. The desired pattern is sampled uniformly $M_1 \times M_2$ points $(M_1, M_2 \gg 1)$ in the interval $0 \le u \le 1$, $0 \le v \le 1$ to yield

$$E(u_{m1}, v_{m2}) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} \varepsilon_{n1n2} I_{n1n2} \cos(m_1\beta_{n1})\cos(m_2\beta_{n2});$$
$$m_1 = 0, 1, 2, \ldots, M_1-1;\ m_2 = 0, 1, 2, \ldots, M_2-1 \quad (9)$$

where $\beta_{n1} = kd_{n1}\Delta u$; $\beta_{n2} = kd_{n2}\Delta v$, the sampling interval $\Delta u = 1/(M_1-1)$; $\Delta v = 1/(M_2-1)$, $In_1 In_2$ is the current distribution of the array and (d_{n1}, d_{n2}) represent the (x, y) position of the element in the array. Since the array factor is symmetric in each quadrant of the $u-v$ space, i.e., $E(u, v) = E(u, -v) = E(-u, v) = E(-u, -v)$ [4], the synthesis problem is considered in one quadrant only, i.e., $0 \le u \le 1$, $0 \le v \le 1$. The following step is to apply the Legendre transformation $F(\alpha_{p1}, \alpha_{p2})$ to the planar array factor, defined as follows:

$$F(\alpha_{p1}, \alpha_{p2}) = \sum_{m_1=0}^{M_1-1} \varepsilon_{m1} \sum_{m_2=0}^{M_2-1} \varepsilon_{m2} E(u_{m1}, v_{m2})$$
$$\times P_{m1-\frac{1}{2}}(\cos\alpha_{p1}) P_{m2-\frac{1}{2}}(\cos\alpha_{p2});$$
$$p_1 = 0, 1, 2, N_1-1, \quad p_2 = 0, 1, 2, \ldots, N_2-1 \quad (10)$$

where $\varepsilon_m = 1$, $m = 0$; $\varepsilon_m = 2$, $m > 0$; this transformation of the planar array factor $E(u, v)$ is motivated by the limiting relation for the Legendre polynomial of fractional order presented in (5). The application of (10) and (5) yields the following triangular system of equations:

$$F(\alpha_{p1},\, \alpha_{p2}) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} I_{n1n2} f(\alpha_{p1},\, \beta_{n2}) f(\alpha_{p2},\, \beta_{n1}) \tag{11}$$

From (11), we can determine the first p-th and q-th element current.

$$I_{00} = \frac{F(\alpha_0,\, \alpha_0)}{f(\alpha_0,\, \beta_0) f(\alpha_0,\, \beta_0)} \tag{12}$$

$$I_{pq} f(\alpha_q,\, \beta_q) f(\alpha_p,\, \beta_p) = F(\alpha_p,\, \alpha_q) - \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} I_{ij} f(\alpha_p,\, \beta_i) f(\alpha_q,\, \beta_j) \tag{13}$$

The simulation results for uniform and non-uniform spacing in a planar array are given in the next Section. The processor used during the experiment was an Intel Core$^{\mathrm{TM}}$2 Duo T5600 @1.83 GHz. (1 GB RAM) with platform MATLAB 7.8 for Windows Vista Service pack 1, 32-bit Math Works

3. SIMULATION AND RESULTS

The following section presents the results of simulation for the linear and planar geometries described in the previous sections.

3.1. Linear Array

The purpose of this section is to evaluate the performance of a linear array, which has been synthesized by Legendre functions. First, for comparison purposes, the number of the elements of the linear array was set as 17 and the main lobe was steered to 90°, following the results presented in [7]. Fig. 3 illustrates the results comparatives for uniform and non-uniform array patterns in a linear array. From the results, we observe a side lobe level (SLL) improvement by ~ 6 dB for the non-uniform array (-20.13 dB) with respect to the uniform array (-13.89 dB).

On the other hand, since the focus of the technique is the minimization of SLL, half power beam width (HPBW) presents a slight increase of 3% ($\sim 0.18°$) in reference to a non-uniform linear array. With these improvements we reaffirm the results reported in [5], that for a non-uniform array we obtain better performance on SLL and HPBW.

However, in order to report an extended evaluation of the performance of the arrays, Fig. 4 illustrates the SLL when the main lobe has been steered in a range of $-60° \leq \theta_0 \leq 60°$ using the non-uniform spacing. With the same procedure, the HPBW was evaluated and shown in Fig. 5. This analytical technique has a characteristic that it can change the SLL by modifying its space-broadening factor (Δ), this Δ factor was chosen from [7]. Both cases present the performance

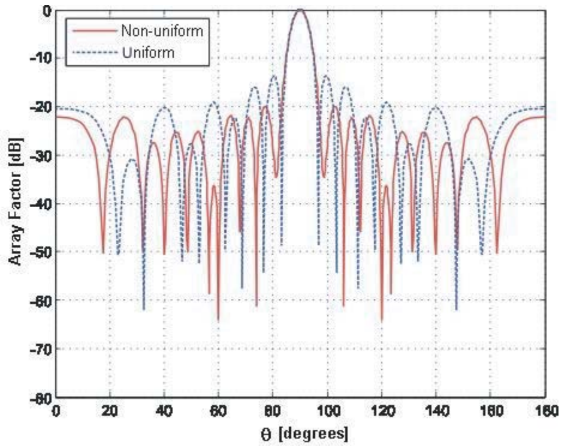

Figure 3. Array pattern for the Legendre functions synthesis with uniform (dashed line) and non-uniform spacing (solid line) of a 17 element array.

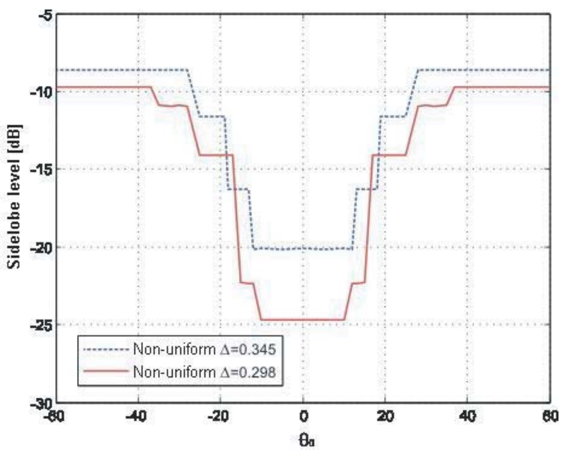

Figure 4. Sidelobe level (SLL) when the main lobe is steered in the range $-60° \leq \theta_0 \leq 60°$ with non-uniform spacing in a 17 element array for different space broadening factors Δ.

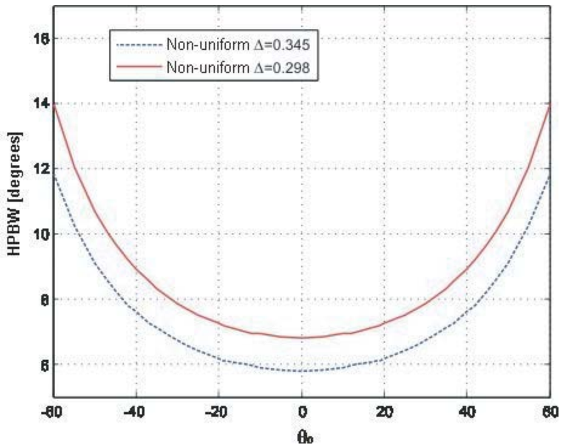

Figure 5. Half power beam width (HPBW) when the main lobe is steered in the range $-60^\circ \leq \theta_0 \leq 60^\circ$ with non-uniform spacing in a 17 element array for different space broadening factors Δ.

of the Legendre functions, but with different space broadening factor (Δ). In the range of $|\theta_0| \leq 12.29^\circ$ has a reduction in the SLL of 1.13 dB and an increment of 0.16° in HPBW for a Δ factor of 0.345.

But if now the angular region is evaluated on $|\theta_0| \leq 15.8^\circ$, 5.7 dB reduction in SLL and an increment of a maximum of 0.29° in HPBW are illustrated, for a Δ factor of 0.298. If these results are compared with differential evolution (DE) [6], we have a disadvantage, because the range of steering diminishes 47.33% (28.40°). But the principal advantage using Legendre functions is that we obtain a narrower beam width.

3.2. Planar Array

For comparison purposes, a 31-element symmetric linear array is selected, therefore the number of variables to calculate is 15, which is a typical number of variables used to test optimization algorithms [5, 6]. As was proposed in the Section 2, the Legendre functions were applied to synthesize the position of 15 elements of a symmetric linear array. Fig. 6 illustrates the array factor obtained to use the Legendre functions with uniform spacing ($\lambda/2$). In this case, it is achieved a SLL of -16 dB in the planar array. The unique parameter that has a better result is in the SLL. However, we need to use non-uniform spacing between elements of antenna, for obtaining a better performance in the array factor.

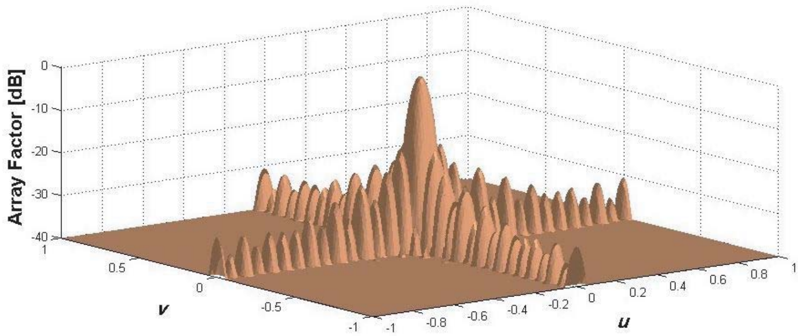

Figure 6. Array factor for the 31×31 uniform element spacing planar array.

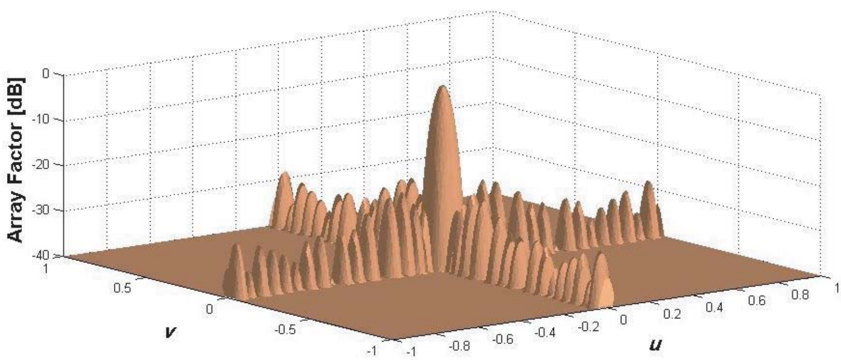

Figure 7. Array factor for the 31×31 non-uniform element spacing.

The array factor obtained for the 31 element planar array with non-uniform spacing is shown in Fig. 7. The SLL achieved with non-uniform spacing is -22 dB. In addition to, we obtained a HPBW of $3.18° \times 3.18°$ and 35.05 dB of directivity. With these results we achieved a reduction of 27.27% (6 dB) in the SLL. In the sense of the HPBW, we obtained a decrease of 8.36% (0.29°), and the directivity presents an increase of 2.14% (0.75 dB) with respect to Legendre functions with uniform spacing.

3.3. Computation Time

In order to know the computation time for a linear array, we used an average time of 1000 repetitions, varying the number of antenna elements from 5 to 37 elements. The processor used during the

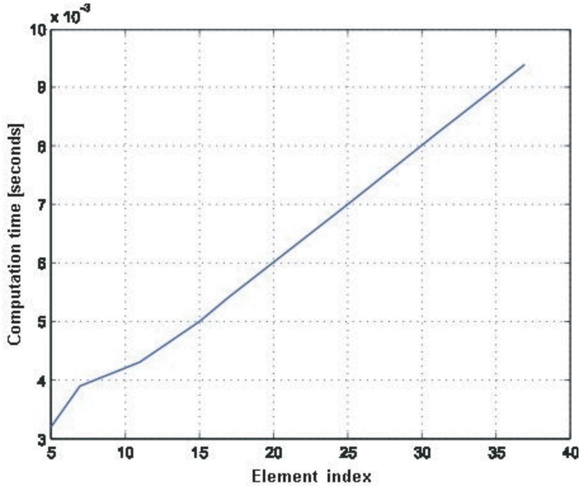

Figure 8. Computation time in a linear array applying Legendre functions.

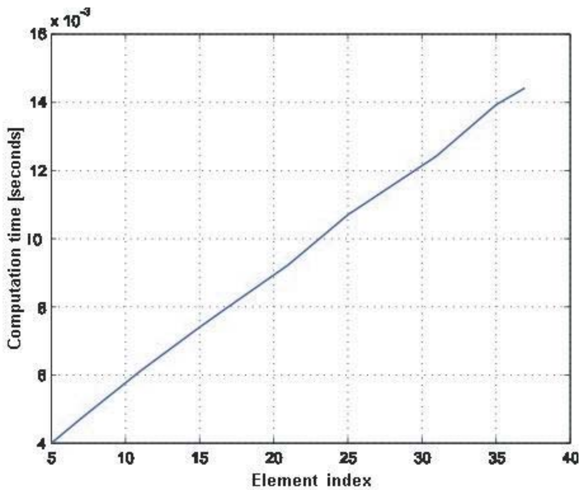

Figure 9. Computation time in a planar array applying Legendre functions.

experiment was an Intel Core$^{\mathrm{TM}}$2 Duo T5600 @1.83 GHz. (1 GB RAM) with platform MATLAB 7.8 for Windows Vista Service pack 1, 32-bit Math Works.

Fig. 8 illustrates the computation time for a linear array, where this time is very fast using Legendre functions in comparison with heuristic techniques [5, 6]. Applying Legendre functions we obtained a maximum value of 9.5 milliseconds using 37 elements of antenna.

Figure 9 illustrates the results for a planar array, using Legendre functions; the computation time of the algorithm presents a maximum value of 14.5 milliseconds. Likewise that a linear array we achieved an almost linearly behavior, using planar arrays.

Results for both cases show that the computation time from this analytical technique are faster in comparison with other techniques as differential evolution [6]. The evaluation of this parameter is important because we found that the principal advantage of using Legendre functions is its computation time.

4. CONCLUSIONS

According to the results obtained in the simulation, the application of Legendre functions in non-uniform arrays (linear and planar) obtain a considerable improvement in the side lobe level (SLL) and also in the half power beam (HPBW), obtaining with this a good directivity in the radiation pattern. The disadvantage of this technique is the range of steering, because it diminishes $\sim 28.4^\circ$ in comparison with evolution differential for a linear array, but this analytical technique presents two advantages in comparison with DE, one is that, it is obtained a narrower beam width without affecting the SLL and other is the computation time.

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APPENDIX A.

A.1. Legendre Functions

In order to establish the array factor in terms of Legendre polynomials we must initiate with the differential equation of second order [9].

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0; \quad n = 0, 1, 2, \ldots \tag{A1}$$

The solutions are called Legendre functions. If n is zero or positive integer, these functions denominate Legendre polynomials. A compact expression of $P_n(x)$ is given by the following expression [9]:

$$P_n(x) = \frac{1}{2^n n!} D^n \left[(x^2 - 1)^n \right] \tag{A2}$$

where the operator symbol D^n denotes n-th derivative. The Legendre polynomials have a property of using a recurrence relation, where the equation defines a recurrent sequence, i.e.:

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x); \quad n = 1, 2, 3, \ldots \tag{A3}$$

A.2. Legendre Functions of Fractional Order

In this paper we use a Legendre function of fractional order $P_{m-1/2}$, and to obtain this function a recurrence relation is required. This relation have Legendre polynomials of fractional order, the recurrence relation is given for the values of $m \geq 2$ and $n \geq 3$

$$(n_1 + 0.5)P_n(x) = 2n_1 x P_{n-1}(x) - (n_1 - 0.5)P_{n-2}(x) \tag{A4}$$

For the value of $m = 0,\ n = 1$, the function $P_{m-1/2}$, will be equal to $P_{-1/2}$, therefore its value is [9]:

$$P_n(x) = P_{m-1/2}(x) = \frac{2}{\pi} K \left[\left(\frac{1-x}{2} \right)^{1/2} \right] \tag{A5}$$

According to the explicit expressions of Legendre [11], $x = \cos(\theta)$ and using identities trigonometric the Legendre polynomial in elliptical integrals of first order is:

$$P_{-1/2}(\cos\theta) = \frac{2}{\pi} K \left[\text{sen} \left(\frac{\theta}{2} \right) \right] \tag{A6}$$

where $K\left[\text{sen}\left(\frac{\theta}{2}\right)\right] = \int_0^{\pi/2} \left[1 - \left(\text{sen}\frac{\theta}{2}\right)\sin^2\theta\right]^{-1/2} d\theta$.

In the same way for the value of $m = 1$ and $n = 2$, the function $P_{m-1/2}$, will be equal to $P_{1/2}$, therefore its value in elliptical integrals of first and second order ($K[\text{sen}(\theta/2)]$ and $E[\text{sen}(\theta/2)]$) is:

$$P_{1/2}(\cos\theta) = \frac{2}{\pi} \left\{ 2E \left[\text{sen} \left(\frac{\theta}{2} \right) \right] - K \left[\text{sen} \left(\frac{\theta}{2} \right) \right] \right\} \tag{A7}$$

where $E\left[\text{sen}\left(\frac{\theta}{2}\right)\right] = \int_0^{\pi/2} \left[1 - \left(\text{sen}\frac{\theta}{2}\right)\sin^2\theta\right]^{-1/2} d\theta$

$K\left[\text{sen}\left(\frac{\theta}{2}\right)\right] = \int_0^{\pi/2} \left[1 - \left(\text{sen}\frac{\theta}{2}\right)\sin^2\theta\right]^{-1/2} d\theta$

Now with these two Legendre polynomials and using the recurrence relation will be simpler to obtain the following values of the polynomials, for $m \geq 2$ and $n \geq 3$ from the equation (A4) we obtained

$$P_n(x) = P_{m-1/2}(x) = \frac{2n_1 x P_{n-1}(x) - (n_1 - 0.5)P_{n-2}(x)}{(n_1 + 0.5)} \quad \text{(A8)}$$

$$P_n(\cos\theta) = \frac{2n_1(\cos\theta)P_{n-1}(\cos\theta) - (n_1 - 0.5)P_{n-2}(\cos\theta)}{(n_1 + 0.5)} \quad \text{(A9)}$$

where $n_1 = n - 2$.

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