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A NEW INTEGRAL EQUATION FORMULATION FOR SCATTERING OF ELECTROMAGNETIC WAVES BY 2D CONDUCTING STRUCTURES, USING CYLINDRICAL HARMONICS

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Abstract—Using cylindrical harmonics and Fourier series, a new integral equation formulation is derived for perfectly conducting 2D scattering problems. This new integral equation is based on the fact that, all of the electric and magnetic field components are zero inside a perfect electric conductor. The incident and scattered fields are expressed in the cylindrical coordinate system with respect to a common origin inside the scatterer, using the addition theorem for Bessel and Hankel functions. The resulting electric or magnetic field is set equal to zero for all the points inside the largest cylinder that is contained in and tangent to the surface of the scatterer. As a result the field point variables are eliminated from the integral equation and only the source points are present in this formulation. Therefore the size of the problem is reduced considerably. A dramatic improvement in the computation speed is seen compared to the classical method of moments. TE and TM scattering problems are considered and the integral equation formulation is derived and solved for both cases.

1. INTRODUCTION

In the past decades we have witnessed great breakthroughs in developing fast algorithms, improving the speed of the method of moments (MoM). Iterative solvers [1, 2] reduce the number of computations from $O\left(N^3\right)$ in direct solvers to $O\left(N^2\right)$, in which N is the number of unknowns. The multilevel matrix decomposition algorithm [3] (MLMDA) simplifies the matrix vector multiplication,

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by decomposing the MoM matrix into large blocks describing the interactions between distant parts of the scatterer. The introduction of the fast multipole method [4–6] (FMM) and its multilevel counterpart [7–10] has reduced the number of computations to $O\left(N\log\left(N\right)\right)$. Parallel [11–14] and hybrid [15–17] approaches help break the computations between different processors and different methods. Although these algorithms address the problem of speed in an efficient way , still a fine discretization of the scatterer in the order of 10 to 20 elements per wavelength is required to obtain a good accuracy.

Although lots of papers have been published on the ways to speed up the computations, less focus has been on the nature of the integral equation itself and the ways to reduce the number of elements. In this paper a method is proposed to simplify the integral equation for the problem of scattering by 2D conducting structures. Using cylindrical harmonics an integral equation is derived in which the contributions of the field point variables are eliminated and only the source points are present in the formulation. In this way the size of the problem is reduced dramatically. Fourier series are used to reduce the number of unknowns as well. The size of the final matrix is much smaller compared to the classical MoM. Therefore the computational difficulties encountered when handling large matrices, are avoided.

2. THEORY

The algorithm proposed in this paper, is based on the fact that all of the electric and magnetic field components are zero inside a PEC structure. The incident and scattered fields are expressed in terms of cylindrical harmonics with respect to a common origin, using the addition theorem for Hankel functions. Then the total field is set equal to zero inside the largest cylinder that is contained inside the structure. As a result a new integral equation is derived in which there is no contribution of the field variables. This equation is only valid inside the largest cylinder contained inside the structure. But, since there are no free currents or charges inside the structure and therefore no discontinuity in the electric and magnetic fields, the solution to this equation would be the answer to the problem. The TE and TM modes are treated differently. First consider the TM mode.

2.1. TM Mode

Consider the program of Fig. 1, where a plane wave is incident on a PEC structure. Assume the boundary of the PEC structure is

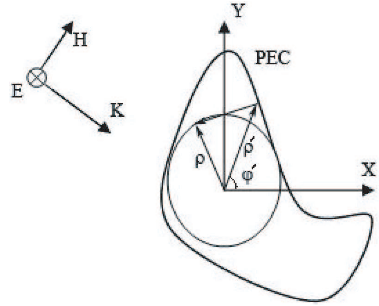

Figure 1. Scattering by an arbitrary PEC — TM mode.

expressed by $\rho' = f(\varphi')$ in the polar coordinate system. For the TM mode the currents are directed toward the z direction i.e., $J = I(\varphi')\,\mathbf{a}_z$. The incident and scattered fields can be expressed in the cylindrical coordinate system by (1)–(3), in which φ_i is the incident angle and $\frac{\omega\mu}{4} H_0^{(2)}(\beta|\rho - \rho'|)$ is the 2D Green function. Equation (2) is the Fourier series expansion of (1).

$$\mathbf{E}^i = \mathbf{a}_z E_z^i = \mathbf{a}_z E_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{a}_z E_0 e^{-j\beta\rho\cos(\varphi-\varphi_i)} \tag{1}$$

$$\mathbf{E}^i = \mathbf{a}_z E_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta\rho)\, e^{jn(\varphi-\varphi_i)} \tag{2}$$

$$\mathbf{E}^s = \mathbf{a}_z \frac{\omega\mu}{4} \int_C I(\varphi')\, H_0^{(2)}(\beta|\boldsymbol{\rho} - \boldsymbol{\rho}'|)\, dl' \tag{3}$$

Using the addition theorem for Hankel functions, the 2D Green function can be expanded in the following form [18]. This series is convergent only for the values of ρ satisfying the condition $\rho \le \rho'$.

$$H_0^{(2)}(\beta|\boldsymbol{\rho} - \boldsymbol{\rho}'|) = \sum_{n=-\infty}^{\infty} J_n(\beta\rho)\, H_n^{(2)}(\beta\rho')\, e^{jn(\varphi-\varphi')} \qquad \rho \le \rho' \tag{4}$$

Using (4) the total electric field for all the points (ρ, φ) inside the cylinder shown in Fig. 1 is expressed by:

$$\mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s = \sum_{n=-\infty}^{\infty} \Bigg(E_0 j^{-n} e^{-jn\varphi_i} - \frac{\omega\mu}{4} \int_0^{2\pi} I(\varphi')\, H_n^{(2)}(\beta\rho')\, e^{-jn\varphi'} \sqrt{\rho'^2 + \left(\frac{d\rho'}{d\varphi'}\right)^2}\, d\varphi' \Bigg) J_n(\beta\rho)\, e^{jn\varphi} \tag{5}$$

This series is valid and convergent for all the points inside the largest cylinder contained in the PEC structure (see Fig. 1). The total electric field is zero inside this region for all the values of ρ and φ. This condition is satisfied if and only if the integral equation:

$$E_0 j^{-n} e^{-jn\varphi_i} - \frac{\omega\mu}{4} \int_0^{2\pi} I\left(\varphi'\right) H_n^{(2)}\left(\beta\rho'\right) e^{-jn\varphi'} \sqrt{\rho'^2 + \left(\frac{d\rho'}{d\varphi'}\right)^2} d\varphi' = 0 \qquad n = -\infty, \ldots, \infty \quad (6)$$

is satisfied, in which the field points contributions are totally eliminated. Equation (6) makes the total electric field zero inside the cylinder shown in Fig. 1. But, since there are no free currents inside the structure and therefore no discontinuity in the electric field, the solution to this equation is the only solution of the problem.

Equation (6) should be solved for all values of n. The presence of the term $e^{-jn\varphi'}$ inside the integral helps solving this equation very effectively using Fourier series. The current $I\left(\varphi'\right)$ is periodic with period 2π and therefore it has a Fourier series of the form:

$$I\left(\varphi'\right) = \sum_{m=-\infty}^{\infty} c_m e^{jm\varphi'} \quad (7)$$

All the other terms inside the integral are expressed by their Fourier series for each value of n.

$$\sqrt{\rho'^2 + \left(\frac{d\rho'}{d\varphi'}\right)^2} H_n^{(2)}\left(\beta\rho'\right) = \sum_{k=-\infty}^{\infty} d_{n,k} e^{jk\varphi'} \quad n = -\infty, \ldots, \infty \quad (8)$$

Substituting (7) and (8) into (6) results in:

$$\int_0^{2\pi} \left[\left(\sum_{m=-\infty}^{\infty} c_m e^{jm\varphi'}\right)\left(\sum_{k=-\infty}^{\infty} d_{n,k} e^{jk\varphi'}\right) e^{-jn\varphi'}\right] d\varphi' = \frac{4E_0}{\omega\mu} j^{-n} e^{-jn\varphi_i} \qquad n = -\infty \cdots \infty \quad (9)$$

and after taking the integration:

$$2\pi \sum_{m=-\infty}^{\infty} c_m d_{n,n-m} = \frac{4E_0}{\omega\mu} j^{-n} e^{-jn\varphi_i} \quad n = -\infty, \ldots, \infty \quad (10)$$

Depending on the behavior of the function $\rho' = f\left(\varphi'\right)$ and the size of the problem only a few number of terms in (10) are enough for solving the problem with a good accuracy. Doing so would result in a system of $2N+1$ equations with $2N+1$ unknowns $(c_{-N}, \ldots, c_N)$:

$$2\pi \sum_{m=-N}^{N} c_m d_{n,n-m} = \frac{4E_0}{\omega\mu} j^{-n} e^{-jn\varphi_i} \quad n = -N, \ldots, N \tag{11}$$

By using this technique the size of the problem is reduced dramatically. The only bottleneck to this method is finding the Fourier coefficients in (8) in a fast and effective way. Using DFT and FFT these coefficients can be computed very fast (see Appendix A), resulting in a very efficient algorithm for finding the currents on a 2D scatterer with a very good accuracy.

2.2. TE Mode

In this case the incident and scattered fields are expressed by [18]:

$$\begin{aligned}\mathbf{H}^i &= H_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \mathbf{a_z} = H_0 e^{-j\beta\rho\cos(\varphi-\varphi_i)} \mathbf{a_z} \\ &= H_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta\rho)\, e^{jn(\varphi-\varphi_i)} \mathbf{a_z}\end{aligned} \tag{12}$$

$$\mathbf{H}^s = \int_C \frac{j\beta}{4} H_1^{(2)}\left(\beta\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)\left(-I_x\left(\varphi'\right)\sin(\psi) + I_y\left(\varphi'\right)\cos(\psi)\right) dl' \mathbf{a_z} \tag{13}$$

The parameter ψ is the polar angle of the vector $\boldsymbol{\rho} - \boldsymbol{\rho}'$, φ_i is the incident angle and $I_x(\varphi')$, $I_y(\varphi')$ are the x and y components of the current on the boundary of the PEC. Using the addition theorem for Hankel functions [19], the following equations can be derived:

$$\begin{aligned}&H_1\left(\beta\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)\cos(\psi) \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(\beta\rho)\left(H_{n-1}\left(\beta\rho'\right)e^{j\varphi'} - H_{n+1}\left(\beta\rho'\right)e^{-j\varphi'}\right)e^{-jn\varphi'}e^{jn\varphi}\end{aligned} \tag{14}$$

$$\begin{aligned}&H_1\left(\beta\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)\sin(\psi) \\ &= \frac{1}{2j}\sum_{n=-\infty}^{\infty} J_n(\beta\rho)\left(H_{n-1}\left(\beta\rho'\right)e^{j\varphi'} + H_{n+1}\left(\beta\rho'\right)e^{-j\varphi'}\right)e^{-jn\varphi'}e^{jn\varphi}\end{aligned} \tag{15}$$

These equations are convergent for all the values of $\rho \le \rho'$. After substituting (14) and (15) in (13) the total magnetic field inside the cylinder shown in Fig. 1 can be expressed in the form:

$$\begin{aligned}\mathbf{H}^t = \sum_{n=-\infty}^{\infty}\Bigg[\frac{j\beta}{4}\int_{C'} -I_x\frac{1}{2j}\left(H_{n-1}\left(\beta\rho'\right)e^{j\varphi'}+H_{n+1}\left(\beta\rho'\right)e^{-j\varphi'}\right)e^{-jn\varphi'}\\ +I_y\frac{1}{2}\left(H_{n-1}\left(\beta\rho'\right)e^{j\varphi'}-H_{n+1}\left(\beta\rho'\right)e^{-j\varphi'}\right)e^{-jn\varphi'}dl'\\ +H_0 j^{-n}e^{-jn\varphi_i}\Bigg]J_n\left(\beta\rho\right)e^{jn\varphi}\end{aligned} \tag{16}$$

Therefore the total magnetic field inside this cylinder is zero if and only if:

$$\int_0^{2\pi}\left(I_xF_n^x+I_yF_n^y\right)e^{-jn\varphi'}d\varphi' = \frac{4j}{\beta}H_0 j^{-n}e^{-jn\varphi_i} \tag{17}$$

$$F_n^x = -\tfrac{1}{2j}\left(H_{n-1}\left(\beta\rho'\right)e^{j\varphi'}+H_{n+1}\left(\beta\rho'\right)e^{-j\varphi'}\right)\sqrt{\rho'^2+\left(\tfrac{d\rho'}{d\varphi'}\right)^2} \tag{18}$$

$$F_n^y = \tfrac{1}{2}\left(H_{n-1}\left(\beta\rho'\right)e^{j\varphi'}-H_{n+1}\left(\beta\rho'\right)e^{-j\varphi'}\right)\sqrt{\rho'^2+\left(\tfrac{d\rho'}{d\varphi'}\right)^2} \tag{19}$$

Using the unit tangent to the surface of the PEC at a point (ρ', φ'),

$$\begin{aligned}\mathbf{a}_c &= \frac{\frac{\partial\rho'}{\partial\varphi'}\mathbf{a}'_\rho+\rho'\mathbf{a}'_\varphi}{\sqrt{\rho'^2+\left(\frac{\partial\rho'}{\partial\varphi'}\right)^2}} = \frac{\left(\frac{\partial\rho'}{\partial\varphi'}\cos\left(\varphi'\right)-\rho'\sin\left(\varphi'\right)\right)}{\sqrt{\rho'^2+\left(\frac{\partial\rho'}{\partial\varphi'}\right)^2}}\mathbf{a_x}\\ &\quad+\frac{\left(\frac{\partial\rho'}{\partial\varphi'}\sin\left(\varphi'\right)+\rho'\cos\left(\varphi'\right)\right)}{\sqrt{\rho'^2+\left(\frac{\partial\rho'}{\partial\varphi'}\right)^2}}\mathbf{a_y}\end{aligned} \tag{20}$$

the currents $I_x(\varphi')$, $I_y(\varphi')$ can be expressed in terms of the complex current $I_c(\varphi')$ on the surface of the PEC at a point (ρ', φ') by:

$$I_x = I_c\left(\frac{\partial\rho'}{\partial\varphi'}\cos\left(\varphi'\right)-\rho'\sin\left(\varphi'\right)\right)/\sqrt{\rho'^2+\left(\frac{\partial\rho'}{\partial\varphi'}\right)^2} \tag{21}$$

$$I_y = I_c\left(\frac{\partial\rho'}{\partial\varphi'}\sin\left(\varphi'\right)+\rho'\cos\left(\varphi'\right)\right)/\sqrt{\rho'^2+\left(\frac{\partial\rho'}{\partial\varphi'}\right)^2} \tag{22}$$

Substituting (21) and (22) in (17) leads to:

$$\int_0^{2\pi} I_c\left(\varphi'\right) G_n\left(\varphi'\right) e^{-jn\varphi'} d\varphi' = \frac{4j}{\beta} H_0 j^{-n} e^{-jn\varphi_i} \tag{23}$$

$$G_n = \tfrac{1}{2} H_{n-1}\left(\beta\rho'\right)\left(\rho' + j\frac{\partial\rho'}{\partial\varphi'}\right) + \tfrac{1}{2} H_{n+1}\left(\beta\rho'\right)\left(-\rho' + j\frac{\partial\rho'}{\partial\varphi'}\right) \tag{24}$$

Which can be solved very effectively using the method discussed in the previous section. Assuming $I_c = \sum_{m=-\infty}^{\infty} c_m e^{jm\varphi'}$ and $G_n = \sum_{m=-\infty}^{\infty} d_{n,m} e^{jm\varphi'}$ the final results are:

$$2\pi \sum_{m=-N}^{N} c_m d_{n,n-m} = \frac{4j}{\beta} H_0 j^{-n} e^{-jn\varphi_i} = \frac{4j}{\beta\eta} E_0 j^{-n} e^{-jn\varphi_i} \qquad n = -N, \ldots, N \tag{25}$$

3. NUMERICAL EXAMPLE

In this part the accuracy and the speed of the algorithm proposed in this paper is examined. First consider the problem of TM_z scattering by an elliptical cylinder shown in Fig. 2, with dimensions $a = 4\lambda$, $b = 8\lambda$. The incident angle is assumed to be $\varphi_i = -45°$. The currents on the surface of the scatterer are calculated using the method introduced in this paper, using 40 harmonics. The result is compared to the exact solution obtained using MoM in Fig. 2. About 400 elements are required in the MoM to reach this level of accuracy. Using the method proposed here, only 45 harmonics are required to solve the problem with the same accuracy.

To compare the error introduced in this method for different number of harmonics, the normalized error is calculated using the relation:

$$error = 2\pi \frac{\int_0^{2\pi} \left|I\left(\varphi\right) - I_{exact}\left(\varphi\right)\right|^2 d\varphi}{\int_0^{2\pi} \left|I_{exact}\left(\varphi\right)\right|^2 d\varphi} \tag{26}$$

and is plotted in Fig. 3 for the problem of Fig. 2.

The number of harmonics to be chosen depends on the size of the scatterer and the required error. For very small errors this number is much less compared to the MoM. Also the final system of linear Equations (11), (25) has a better behavior compared to the MoM,

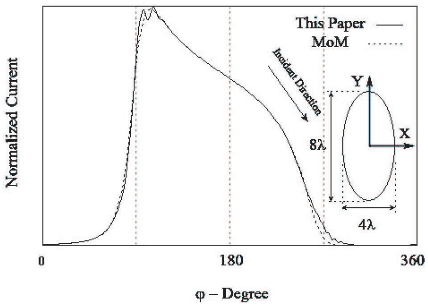

Figure 2. TM$_z$ scattering by an elliptical cylinder. Normalized currents on the surface of the scatterer are calculated using 40 harmonics and compared to the MoM with 400 elements.

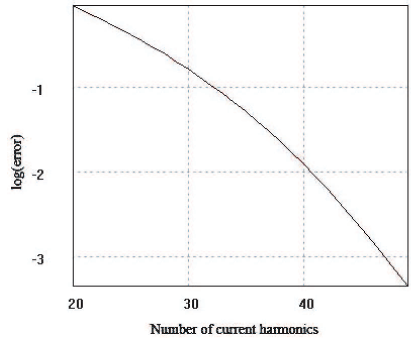

Figure 3. The error with respect to the number of harmonics for the problem of Fig. 2.

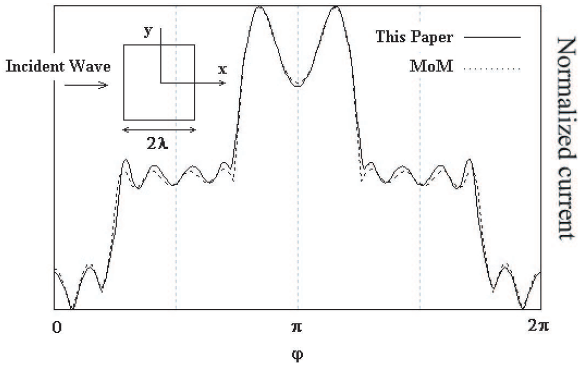

Figure 4. TE scattering by a square metallic structure, compared to the MoM.

when conventional methods such is Gauss elimination is used to solve it, and is less prone to divergence due to round off and truncation errors.

For non smooth structures and objects with edges, the derivative $\frac{\partial \rho'}{\partial \varphi'}$ appearing in the formulation is discontinuous. In this case the average of the left and right derivatives at the edge can be used. Using just the left derivative or just the right derivative instead, would also lead to the same results. This is shown in Fig. 4 for the problem of TE scattering by a metallic cylinder with square cross section. The incident angle is assumed to be $\varphi_i = 0$. The current on the surface of the scatterer is derived using 20 harmonics and compared to the MoM with 120 elements.

The method proposed here provides a good way for reducing the number of variables for scattering by large structures. This method is more appropriate for structures with a good aspect ratio. For these structures only the first few terms of the Fourier series is sufficient to solve the problem with a good accuracy. However for objects with a poor aspect ratio the number of harmonics needs to be increased. This method provides another advantage for medium and large problems with a good aspect ratio. In these cases a small number of harmonics are sufficient for a good accuracy and therefore the problem can be solved using direct matrix solvers. For example for an object with a maximum size of 20λ and with a good aspect ratio, less than 100 harmonics are required. In this case using direct matrix solvers, the whole procedure takes less than a few seconds. However for the classic MoM, the number of elements would be in the order of 1000 and iterative matrix solvers should be used. In this case using MoM takes much longer, at least in the order of several minutes.

4. CONCLUSIONS

A new integral equation was derived for 2 dimensional scattering problems, in which the field point variables were eliminated, using cylindrical harmonics. Since only the source point variables are present in this formulation it can be solved faster. A method was proposed to solve the integral equation using the Fourier series, resulting in the reduction of the number of unknowns as well. The system of equations derived in this paper is smaller compared to MoM and shows a better behavior with conventional matrix solvers.

APPENDIX A.

The Fourier series coefficients of a periodic function $f(x)$ of period 2π are normally calculated by (A2). For most functions this integration is taken numerically. Due to the oscillatory nature of the term e^{-jnx}, the integration process may be computationally very inefficient, even with efficient integration algorithms such as Gauss-Quadrature. Therefore a lot of samples would be required to perform the integration with a satisfactory accuracy.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{jnx} \tag{A1}$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-jnx} dx \tag{A2}$$

However, If it is assumed that the function $f(x)$ can be approximated with the desired accuracy by its first N Fourier harmonics in (A3), these coefficients can be calculated very effectively using Discreet Fourier Transform (DFT).

$$f(x) = \sum_{n=-N}^{N} c_n e^{jnx} \tag{A3}$$

Each function $f(x)$ defined by (A3) is a member of the vector space V_{2N+1} spanned by the set of vectors:

$$V_{2N+1} = span\left\{e^{-jnx},\ n = 0, \pm 1, \ldots, \pm N\right\} \tag{A4}$$

First we introduce the Dirichlet Kernel $\delta_N(x)$ defined by:

$$\delta_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{jnx} = \frac{1}{2\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right) x\right)}{\sin\left(\frac{x}{2}\right)} \tag{A5}$$

The set of the functions

$$g_k(x) = \frac{2\pi}{2N+1} \delta_N(x - x_k) \quad k = 0, \ldots, 2N, \quad x_k = \frac{2\pi}{2N+1} k \tag{A6}$$

form an orthogonal basis, and therefore they span the space V_{2N+1} as well. i.e., every function in this vector space can be expressed uniquely by a series of the form:

$$f(x) = \sum_{k=0}^{2N} a_k g_k(x) \tag{A7}$$

These functions have an interesting property which makes them very appropriate for computational purposes.

$$g_k(x_l) = \delta_{kl} \tag{A8}$$

$$\delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases} \tag{A9}$$

δ_{kl} is the Dirac delta function. Using (A8) and (A7), a_k satisfies:

$$f(x_k) = a_k \tag{A10}$$

Therefore each function $f(x)$ in V_{2N+1} can be expressed uniquely by only $2N+1$ samples using the Dirichlet Kernel. i.e.,

$$f(x) = \frac{2\pi}{2N+1}\sum_{k=0}^{2N} f(x_k)\,\delta_N(x - x_k) \tag{A11}$$

Substituting the Dirichlet Kernel by its Fourier series leads to:

$$\begin{aligned} f(x) &= \frac{1}{2N+1}\sum_{k=0}^{2N} f(x_k) \sum_{n=-N}^{N} e^{jn(x-x_k)} \\ &= \frac{1}{2N+1}\sum_{n=-N}^{N}\left(\sum_{k=0}^{2N} f(x_k)\,e^{-jnx_k}\right) e^{jnx} = \sum_{n=-N}^{N} c_n e^{jnx} \end{aligned} \tag{A12}$$

$$\begin{aligned} c_n &= \frac{1}{2N+1}\sum_{k=0}^{2N} f(x_k)\,e^{-jnx_k} \\ &= \frac{1}{2N+1}\sum_{k=0}^{2N} f(x_k)\,w_{2N+1}^{nk} \quad n = -N\cdots N \end{aligned} \tag{A13}$$

$$w_{2N+1} = e^{-j\frac{2\pi}{2N+1}} \tag{A14}$$

Equation (A13) is the DFT of the periodic function $f(x)$ sampled at the points $x_k = \frac{2\pi}{2N+1}k$, $k = 0, \ldots, 2N$.

When $2N+1 = PQ$, where P and Q are positive integers, the computations can be speed up even more using the Fast Fourier Transform (FFT) [20].

Equation (A13) provides a computationally effective way for calculating the Fourier coefficients of functions using only $2N+1$ samples, avoiding numerical integration.

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