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SCATTERING BY AN IMPEDANCE HALF-PLANE: COMPARISON OF THE SOLUTIONS OF RAMAN/ KRISHNAN AND MALIUZHINETS/SENIOR

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Abstract—There are three approaches for the solution of the diffraction problem of plane waves by an impedance half-plane in the literature. The diffracted field expressions, obtained by the related methods, are compared numerically. The examination of the scattered field shows that the most reliable solution is the field representation of Raman and Krishnan. Since the diffracted fields of Senior and Maliuzhinets do not compensate the discontinuities of the geometrical optics waves at the transition regions.

1. INTRODUCTION

The diffraction problem of waves by a conducting half-plane is a canonical problem of the diffraction theory. The first rigorous solution of this problem was put forward by Sommerfeld [1]. He defined a spectrum integral of plane waves and chose the amplitude function according to an extended space that has a period of 4π. The evaluation of the complex integral yielded to the uniform expressions of the scattered fields in terms of the Fresnel functions. The method of Sommerfeld was applied to more general case of the conducting wedge by Carslaw [2]. The impedance surfaces are more realistic in the modeling of the scatterers since an object does not generally reflect all of the incoming energy. The first solution of the diffraction problem of waves by a non-perfectly reflecting half-screen was obtained by Raman and Krishnan [3]. They applied the uniform solution of Sommerfeld to the problem by using the reflection coefficient from a whole impedance

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plane. Later Senior studied the problem with the method of Wiener-Hopf factorization [4]. Maliuzhinets studied the scattering problem of plane waves by a wedge with different impedance surfaces by using the extended method of Carslaw [5]. In fact the two approaches were equivalent since they depended on the solution of double integral equations. The resulting field expressions for the diffracted waves were obtained by the asymptotic evaluation of the complex integrals and expressed in terms of meromorphic functions, named as the Maliuzhinets functions. The solutions of Senior and Maliuzhinets are too cumbersome for the practical engineering applications, and simpler expressions are searched for the diffracted waves [6].

The aim of this paper is to compare the solution of Raman and Krishnan with Senior-Maliuzhinets. Our recent studies show that the field expressions of Raman and Krishnan are more rigorous than that of Senior-Maliuzhinets [7]. Also the diffracted field expressions of Raman and Krishnan are more suitable for practical application. The function of the diffracted fields is to compensate the discontinuities of the geometrical optics (GO) waves at the transition regions. We will compare the mentioned scattered fields at these regions numerically. The behavior of the scattered waves shows the reliability of the solutions.

A time factor of $\exp(jwt)$ will be suppressed throughout the paper. w is the angular frequency.

2. THE DEFINITION OF THE PROBLEM

A half-plane with two equal face impedances is illuminated by a plane wave of

$$u_i = u_0 \exp\left[jk\rho\cos\left(\phi - \phi_0\right)\right] \tag{1}$$

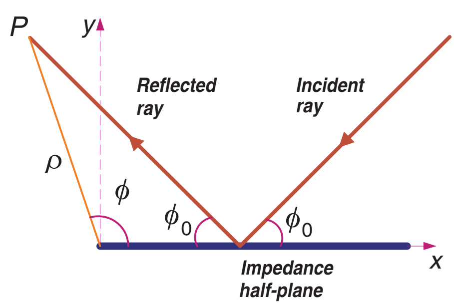

Figure 1. Geometry of the impedance half-plane.

for k is the wave-number. u_0 is the complex amplitude of the wave. The half-plane is located at $S = \{(x, y, z); x \in (0, \infty), y = 0, z \in (-\infty, \infty)\}$. The geometry of the problem is given in Fig. 1.

The impedance boundary condition can be defined as

$$u|_S = \frac{1}{jk\sin\theta} \frac{\partial u}{\partial n}\bigg|_S \tag{2}$$

where $\vec{n}$ is the unit normal vector of the surface. $\sin\theta$ is equal to Z_0/Z for Z_0 and Z are the impedances of the free space and surface. u is the total scattered field. The scattered field by a whole impedance surface can be given by the equation of

$$u = u_0 \exp\left[jk\rho\cos(\phi - \phi_0)\right] + u_0 \frac{\sin\phi_0 - \sin\theta}{\sin\phi_0 + \sin\theta} \exp\left[jk\rho\cos(\phi + \phi_0)\right]. \tag{3}$$

The impedance surface reduces to the soft and hard surfaces for $\sin\theta \to \infty$ and $\sin\theta = 0$, respectively.

3. THE SOLUTION OF RAMAN AND KRISHNAN

Raman and Krishnan derived a basic solution of the diffraction problem of plane waves by an impedance half-plane by using the expression, obtained by Sommerfeld [3]. The scattered waves by a soft or hard whole plane can be given by the representation of

$$u = u_0 \exp\left[jk\rho\cos(\phi - \phi_0)\right] \mp u_0 \exp\left[jk\rho\cos(\phi + \phi_0)\right] \tag{4}$$

where the signs of $+$ and $-$ are valid for hard and soft planes, respectively. The total scattered waves can be written as

$$u = u_0 \exp\left[jk\rho\cos(\phi - \phi_0)\right] F[\xi_-] \mp u_0 \exp\left[jk\rho\cos(\phi + \phi_0)\right] F[\xi_+] \tag{5}$$

when a soft or hard half-plane is taken into account. $\xi_\mp$ is the detour parameter and given by the expression of

$$\xi_\mp = -\sqrt{2k\rho}\cos\frac{\phi \mp \phi_0}{2}. \tag{6}$$

$F[x]$ is the Fresnel function, which can be defined as

$$F[x] = \frac{\exp(j\pi/4)}{\sqrt{\pi}} \int_x^\infty \exp\left(-jt^2\right) dt. \tag{7}$$

Raman and Krishnan applied the same logic to the diffraction problem of plane waves by an impedance half-plane, heuristically. Multiply the fields, obtained for a whole plane, with the Fresnel function in the case of the half-plane. Thus their solution can be given by

$$\begin{aligned} u_s^{RK} &= u_0 \exp\left[jk\rho\cos\left(\phi-\phi_0\right)\right] F\left[\xi_-\right] \\ &\quad + u_0 \frac{\sin\phi_0 - \sin\theta}{\sin\phi_0 + \sin\theta} \exp\left[jk\rho\cos\left(\phi+\phi_0\right)\right] F\left[\xi_+\right]. \end{aligned} \tag{8}$$

The GO and diffracted waves can also be derived from Equation (8) by using the relation of

$$F[x] = U(-x) + sign(x)\, F[|x|] \tag{9}$$

where $U(x)$ is the unit step function, which is equal to one for $x > 0$ and zero otherwise [9]. $sign(x)$ is the signum function, which is equal to 1 for $x > 0$ and -1 for $x < 0$. The total GO field has the expression of

$$\begin{aligned} u_{\mathrm{GO}} &= u_0 \exp\left[jk\rho\cos\left(\phi-\phi_0\right)\right] U\left[-\xi_-\right] \\ &\quad + u_0 \frac{\sin\phi_0 - \sin\theta}{\sin\phi_0 + \sin\theta} \exp\left[jk\rho\cos\left(\phi+\phi_0\right)\right] U\left[-\xi_+\right] \end{aligned} \tag{10}$$

which is valid for all of the solutions. The diffracted waves can be written as

$$u_d^{RK} = u_{di}^{RK} + u_{dr}^{RK} \tag{11}$$

where u_{di}^{RK} and u_{dr}^{RK} can be defined as

$$u_{di}^{RK} = u_0 \exp\left[jk\rho\cos\left(\phi-\phi_0\right)\right] sign\left(\xi_-\right) F\left[|\xi_-|\right] \tag{12}$$

and

$$u_{dr}^{RK} = u_0 \frac{\sin\phi_0 - \sin\theta}{\sin\phi_0 + \sin\theta} \exp\left[jk\rho\cos\left(\phi+\phi_0\right)\right] sign\left(\xi_+\right) F\left[|\xi_+|\right] \tag{13}$$

respectively. In fact the solution of Raman and Krishnan is not rigorous since it is derived heuristically by using the reflection coefficient of a whole impedance plane, given by Eq. (3).

Figure 2 shows the variation of the total scattered, diffracted and GO waves with respect to the observation angle. The observation distance (ρ) is equal to $10/k$. The angle of incidence (ϕ_0) is 60°. $\sin\theta$ has the value of four. It can be seen from the figure that the reflection and shadow boundaries occur at $\phi = 120°$ and $\phi = 240°$. The maximum value of the amplitude decreased to 1.6 from 2 for $\phi < 120°$. This behavior is the result of the surface impedance. The total diffracted field is not equal to zero on the surfaces of the half-plane ($\phi = 0°$ and 360°). For this reason, the field behaves appropriate for the impedance boundary conditions. The diffracted wave compensates the discontinuities, occurring in the GO field, at the transition regions.

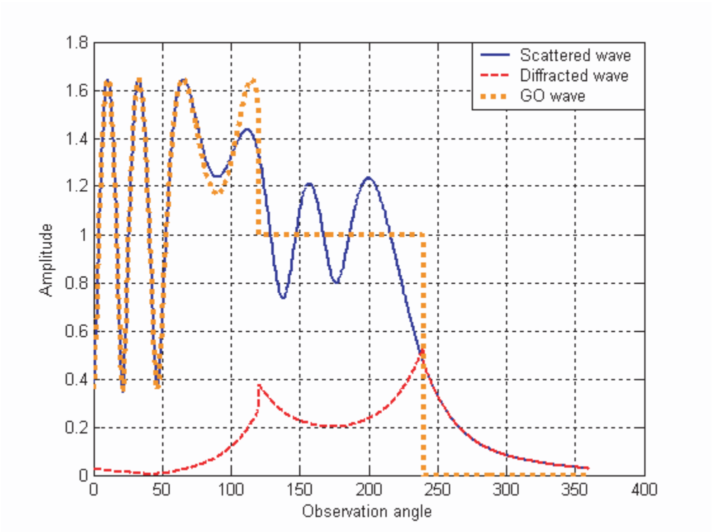

Figure 2. Total scattered waves.

4. THE SOLUTIONS OF SENIOR AND MALIUZHINETS

In [5], Maliuzhinets derived a solution for the diffraction problem of waves by a wedge with different face impedances. The nature of the solution is non-uniform since the amplitude approaches to infinity at the transition regions. A uniform version of the scattered waves by a half-pane with two equal face impedances was given by Senior and Volakis [8], based on the solution of Senior [4]. In this section, we will obtain a uniform representation of the Maliuzhitens' solution.

4.1. The Solution of Senior

The non-uniform expression of the diffracted fields, obtained by Senior [8], can be written as

$$\begin{aligned} u_d^S &= \frac{\exp\left(-j\pi/4\right)}{\sqrt{2\pi}} \frac{1-2\chi\cos\frac{\phi_0}{2}\cos\frac{\phi}{2}}{\cos\phi+\cos\phi_0} K_+\left(\chi,\,\cos\phi\right) \\ & \quad K_+\left(\chi,\,\cos\phi_0\right)\frac{\exp\left(-jk\rho\right)}{\sqrt{k\rho}} \end{aligned} \tag{14}$$

where χ is equal to $1/\sin\theta$. The term of $K_+(\chi,\,\cos u)$ comes from the Wiener-Hopf factorization and can be defined as

$$K_+(\chi,\,\cos u) = \frac{4}{\sqrt{\chi}} f(u,\,\chi) \left\{ \frac{\psi_\pi\left(\frac{3\pi}{2} - u - \theta\right)\psi_\pi\left(\frac{\pi}{2} - u + \theta\right)}{\left[\psi_\pi(\pi/2)\right]^2} \right\}^2 \qquad (15)$$

for the function of $f(u,\,\chi)$ is

$$f(u,\,\chi) = \frac{\sin\frac{u}{2}}{\left\{1+\sqrt{2}\cos\left[\left(\frac{3\pi}{2}-u-\theta\right)/2\right]\right\}\left\{1+\sqrt{2}\cos\left[\left(\frac{\pi}{2}-u+\theta\right)/2\right]\right\}}. \qquad (16)$$

$\psi_\pi(\alpha)$ is the Maliuzhinets function for a half-plane and can be expressed as

$$\psi_\pi(\alpha) = \exp\left[-\frac{1}{8\pi}\int_0^\alpha \frac{\pi\sin t - 2\sqrt{2}\pi\sin(t/2) + 2t}{\cos t}dt\right]. \qquad (17)$$

Equation (14) can be rewritten as

$$u_d^S = \frac{1}{\sin\phi_0}\left(1 - 2\chi\cos\frac{\phi_0}{2}\cos\frac{\phi}{2}\right)(u_{dr} - u_{di})\,K_+(\chi,\,\cos\phi)\,K_+(\chi,\,\cos\phi_0) \qquad (18)$$

where u_{dr} and u_{di} are equal to

$$u_{dr} = \frac{\exp(-j\pi/4)}{2\sqrt{2\pi}}\frac{\sin\frac{\phi+\phi_0}{2}}{\cos\frac{\phi+\phi_0}{2}}\frac{\exp(-jk\rho)}{\sqrt{k\rho}} \qquad (19)$$

and

$$u_{di} = \frac{\exp(-j\pi/4)}{2\sqrt{2\pi}}\frac{\sin\frac{\phi-\phi_0}{2}}{\cos\frac{\phi-\phi_0}{2}}\frac{\exp(-jk\rho)}{\sqrt{k\rho}} \qquad (20)$$

respectively. The uniform fields can be obtained by using the relation of

$$sign(x)\,F[|x|] \approx \frac{\exp(-j\pi/4)}{2\sqrt{\pi}}\frac{\exp\left(-jx^2\right)}{x} \qquad (21)$$

according to the uniform theory of diffraction [9]. As a result the uniform fields can be given by the equations of

$$u_{dr} = -\exp\left[jk\rho\cos(\phi+\phi_0)\right]\sin\frac{\phi+\phi_0}{2}sign(\xi_+)\,F[|\xi_+|] \qquad (22)$$

and

$$u_{di} = -\exp\left[jk\rho\cos(\phi-\phi_0)\right]\sin\frac{\phi-\phi_0}{2}sign(\xi_-)\,F\left[|\xi_-|\right]. \tag{23}$$

4.2. The Solution of Maliuzhinets

The diffracted field, obtained by Maliuzhinets [5], can be written as

$$\begin{aligned} u_d^M \;=\; & -\frac{\exp(-j\pi/4)}{2\sqrt{2\pi}}\frac{\sin(\phi_0/2)}{\Psi(\pi-\phi_0)}\left[\frac{\Psi(-\phi)}{\sin(\phi/2)+\cos(\phi_0/2)}\right. \\ & \left.+\frac{\Psi(2\pi-\phi)}{\sin(\phi/2)-\cos(\phi_0/2)}\right]\frac{\exp(-jk\rho)}{\sqrt{k\rho}} \end{aligned} \tag{24}$$

for the geometry, in Fig. 1. $\Psi(x)$ can be defined as

$$\Psi(x)=\psi_\pi\left(x+\frac{3\pi}{2}-\theta\right)\psi_\pi\left(x+\frac{\pi}{2}+\theta\right)\psi_\pi\left(x-\frac{\pi}{2}-\theta\right)\psi_\pi\left(x-\frac{3\pi}{2}+\theta\right). \tag{25}$$

Equation (24) can be arranged as

$$u_d^M = -\frac{\exp(-j\pi/4)}{2\sqrt{2\pi}}\frac{\sin(\phi_0/2)}{\Psi(\pi-\phi_0)}g(\phi,\,\phi_0)\,\frac{\exp(-jk\rho)}{\sqrt{k\rho}} \tag{26}$$

for $g(\phi,\,\phi_0)$ is equal to

$$g(\phi,\,\phi_0) = \frac{\begin{array}{c}\Psi(-\phi)\left[\sin(\phi/2)-\cos(\phi_0/2)\right] \\ +\Psi(2\pi-\phi)\left[\sin(\phi/2)+\cos(\phi_0/2)\right]\end{array}}{\sin^2(\phi/2)-\cos^2(\phi_0/2)}. \tag{27}$$

Equation (27) reads

$$g(\phi,\,\phi_0) = -2\frac{\begin{array}{c}\Psi(-\phi)\left[\sin(\phi/2)-\cos(\phi_0/2)\right] \\ +\Psi(2\pi-\phi)\left[\sin(\phi/2)+\cos(\phi_0/2)\right]\end{array}}{\cos\phi+\cos\phi_0} \tag{28}$$

when the relation of

$$\cos\phi+\cos\phi_0 = 2\left[\cos^2(\phi_0/2)-\sin^2(\phi/2)\right] \tag{29}$$

is taken into account. We will defined the function of

$$\begin{aligned} h(\phi,\,\phi_0) \;=\; & \Psi(-\phi)\left[\sin(\phi/2)-\cos(\phi_0/2)\right] \\ & +\Psi(2\pi-\phi)\left[\sin(\phi/2)+\cos(\phi_0/2)\right] \end{aligned} \tag{30}$$

in Equation (28). As a result the diffracted field can be written as

$$u_d^M = \frac{\exp(-j\pi/4)}{\sqrt{2\pi}} \frac{\sin(\phi_0/2)}{\Psi(\pi-\phi_0)} \frac{h(\phi,\phi_0)}{\cos\phi+\cos\phi_0} \frac{\exp(-jk\rho)}{\sqrt{k\rho}}. \tag{31}$$

Equation (31) can be rewritten as

$$u_d^M = \frac{\sin(\phi_0/2)}{\Psi(\pi-\phi_0)\sin\phi_0} h(\phi,\, \phi_0)\,(u_{dr}-u_{di}) \tag{32}$$

where u_{dr} and u_{di} are equal to

$$u_{dr} = \frac{\exp(-j\pi/4)}{2\sqrt{2\pi}} \frac{\sin\frac{\phi+\phi_0}{2}}{\cos\frac{\phi+\phi_0}{2}} \frac{\exp(-jk\rho)}{\sqrt{k\rho}} \tag{33}$$

and

$$u_{di} = \frac{\exp(-j\pi/4)}{2\sqrt{2\pi}} \frac{\sin\frac{\phi-\phi_0}{2}}{\cos\frac{\phi-\phi_0}{2}} \frac{\exp(-jk\rho)}{\sqrt{k\rho}} \tag{34}$$

respectively. The uniform fields are expressed in Equations (22) and (23).

5. NUMERICAL COMPARISON

In this section, we will compare the scattered fields, evaluated by Raman-Krishnan, Senior and Maliuzhinets numerically. The parameters of ρ, ϕ_0 and $\sin\theta$ will be taken as in Fig. 2.

Figure 3 shows the variation of the total diffracted waves with respect to the observation angle. It can be observed from the figure that the solutions of Senior and Maliuzhinets are in harmony, but deviate from the field expression of Raman and Krishnan. Since the total diffracted fields of Senior and Maliuzhinets are equal to zero on the surfaces of the half-plane, they do not satisfy the boundary condition, given in Equation (2).

Figures 4 and 5 depict the variations of the total scattered waves in the neighborhoods of the reflection and shadow boundaries. At these points the diffracted wave compensates the discontinuity of the GO field. It can be seen from the graphics that the solutions of Raman/Krishnan and Maliuzhinets are smooth and continuous at the reflection and shadow boundaries. But the scattered field expressions of Senior are discontinuous at $\phi = 120°$ and $\phi = 240°$. Thus the diffracted field, given by Equation (18), is problematic at the transition regions.

Two important points are put forward in the plots, given above. The first point is the satisfaction of the impedance boundary condition on the surfaces of the half-plane. At least, this point requires that the total diffracted field must be different from zero on the impedance surfaces. Figure 3 shows the behavior of the three solutions for the impedance half-plane. As can be seen from Eqs. (11)–(13), the Raman/Krishnan approach leads to nonzero field expressions on $\phi = 0$ and $\phi = 2\pi$. However the solutions of Maliuzhinets and Senior are equal to zero on the surfaces. The reason of this behavior can be explained when Eqs. (14) and (24) are taken into account. The incident and reflected diffracted waves are multiplied by the same coefficient and the two fields eliminate each other on the impedance surface. Thus, the two solutions satisfy the Dirichlet boundary condition. In the Raman/Krishnan approach, only the reflected diffracted field is multiplied by the reflection coefficient of the impedance surface. For this reason, the total field is not equal to zero on the surfaces of the half-plane. The second point that must be mentioned is the discontinuous behavior of the Seniors solution at the transition regions as can be observed in Figs. 4 and 5. Since the same MATLAB code is used for the numerical evaluation of both of the solutions, the reason of the discontinuous behavior comes from the nature of the solution of Senior. Also note that the same codes are used for the definition of the Maliuzhinets functions. But the field expression of Maliuzhinets is continuous at the reflection and shadow boundaries. The approach of

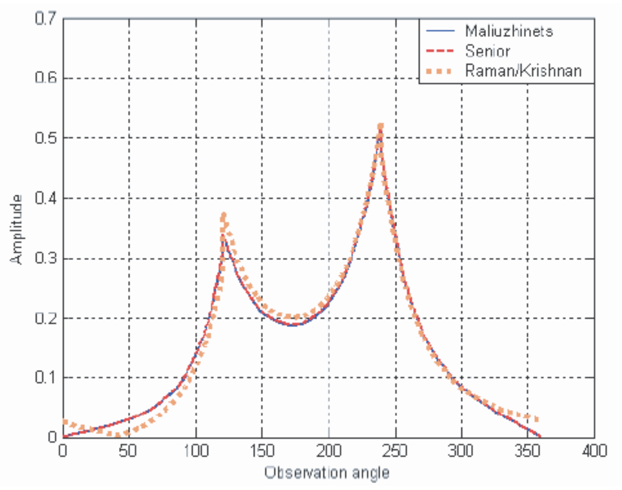

Figure 3. Comparison of the diffracted waves.

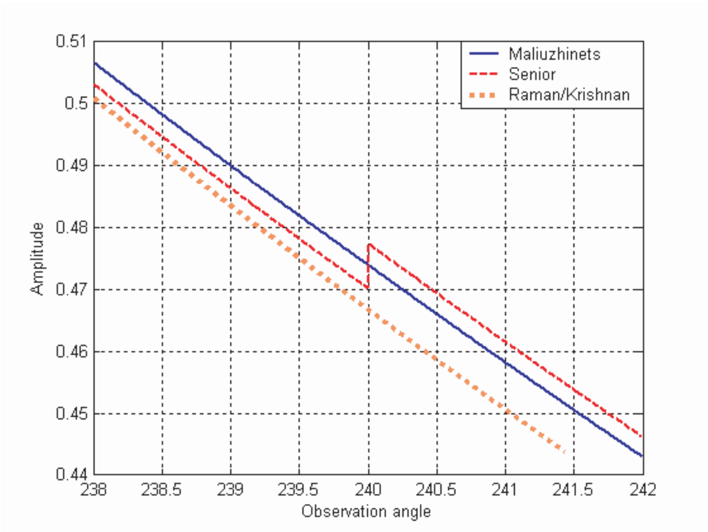

Figure 4. Comparison of the scattered waves at the shadow boundary.

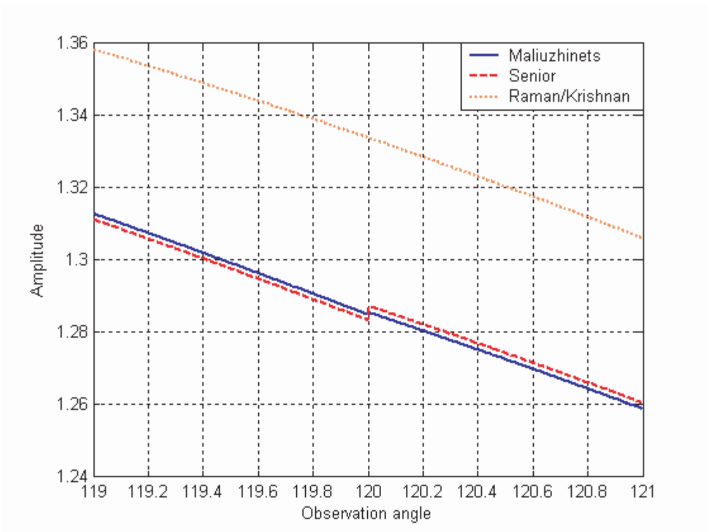

Figure 5. Comparison of the scattered waves at the reflection boundary.

Raman/Krishnan is based on the exact solution of Sommerfeld. For this reason the total scattered field is continuous everywhere.

6. CONCLUSION

In this study, we compared three solutions of the diffraction problem of plane waves by an impedance half-plane. It is seen that the field expressions of Senior and Maliuzhinets do not satisfy the impedance boundary condition on the surfaces of the half-plane. The diffracted field of Senior also does not compensate the discontinuity of the GO waves at the transition regions. However, the solution of Raman and Krishnan is free of these problems. In a recent paper, we showed that an impedance surface can be expressed in terms of soft and hard surfaces. Thus, our series solution was similar to the diffracted field of Raman and Krishnan. Note that the physical optics solution, derived in [6], also approaches to zero on the surfaces of the impedance half-plane.

This paper also provides a new uniform method for the non-uniform solutions of Senior and Maliuzhinets. The method is based on [9]. The difference of the approach from the uniform theory of diffraction [10] is the usage of the separation of the Fresnel function into its subfunctions instead of considering the square of the detour parameter. Also a similar approach is used for the uniform diffracted field expressions of Raman/Krishnan.

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