

## DETECTION AND ESTIMATION OF MULTI-COMPONENT POLYNOMIAL PHASE SIGNALS BY CONSTRUCTING REGULAR CROSS TERMS

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**Abstract**—A regular cross terms algorithm is derived for the parameter estimation of the multi-component polynomial phase signals in additive white Gaussian noise. The basic idea is first to separate its phase parameters into two sets by nonlinear procedures and then each set has half of the parameters in its auto-terms. Furthermore, using two linear transforms to deal with the two signals respectively, the phase coefficients of cross terms can be regulated for the identification and elimination of false peaks caused by the cross terms. Simulations are presented to illustrate the performance of the proposed algorithm.

### 1. INTRODUCTION

Polynomial phase signal (PPS) is a very important model to deal with nonstationary signals, which have considerable technological applications in radar, wireless communications, seismology, neuroethology, etc. For example, in synthetic aperture radar (SAR) [1–3], the relative radar-target motion can cause a time-varying phase in the transmitted signal. The continuous variation of the distance between radar and target leads the instantaneous phase shift to a continuous function of time. Its phase estimation has received considerable attention in the field of signal processing, as the Weierstrass' theorem implies that the continuous instantaneous phase can be well approximated by a finite-order polynomial within a finite interval.

There are many methods to estimate the parameters for a PPS [4–15]. The maximum-likelihood (ML) [4], and nonlinear instantaneous

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least squares (NLS) [7] estimators can provide very high estimation accuracy, whose estimator variances achieve Cramér-Rao lower bound (CRLB) asymptotically under additive white Gaussian noise [8]. But ML and NLS methods result in a  $P$ -dimensional search with  $O(N^P \log_2 N)$  operations when the order of phase is  $P$ . In [9], the observed PPS with the order more than three is first converted to another sequence by a signal transformation procedure, and then the parameter search dimension is reduced by about half. The polynomial-phase transform or the high-order ambiguity function (HAF) [9, 10] or Polynomial Wigner-Ville distribution (PWVD) [14] can estimate the PPS parameters by one-dimensional search via multiple nonlinear operations on the received signal. For multi-component polynomial phase signals (mc-PPSs), the cross terms between the components give rise to undesired sinusoids in the high-order instantaneous moment (HIM), which is the main problem strongly to affect algorithms based on frequency estimation. The principle of demodulation of mono-component PPS no longer works with mc-PPSs [16].

Many methods for mc-PPSs analysis are derived from the previous methods for mono-component PPS or generalize the ones. The product high-order ambiguity function (PHAF) [13], which is the extension of HAF, improves the identification of the highest order polynomial phase coefficients by using high-order multiple transform technique and proper scaling. PWVD is also using high-order multiple transform similarly. There have been a number of methods to generalize existing classes of multilinear functions with a view to improving the power and flexibility of analysis for PPS [16], such as the generalized representation of phase derivatives (GRPD) [17], the generalized high-order phase functions (GHOPF) [18].

For the cross term interference problem of multi-component, the main methods used in the existing literatures are compressing the cross term to reduce the interference; however, it can't remove the interference completely. Particularly, when two components share high-order coefficients and other coefficients of the same order are different from each other, some of the cross terms can also be converted to sinusoids in HIM and they are indistinguishable from the auto-terms. Then, the identifiability problem occurs seriously.

In this paper, cross terms are constructed regularly in order to achieve completely the identification of auto-terms from cross terms. The phase parameters of mc-PPSs are separated into two sets by nonlinear procedures and then each set has half of the parameters in its auto-terms. And we introduce two linear transforms similar to chirp-z transform to estimate the parameters of the two sets respectively by searching the spectrum peak. If arbitrary two peaks satisfy some

regular characteristic, for example, the symmetry around arbitrary axis, they can be identified as cross terms. The proposed algorithm can well detect mc-PPSs without the cross term interference completely, especially for the case of 3rd or 4th order.

## 2. REGULAR CROSS TERMS

It will lead to serious cross term interference sometimes that high-order mc-PPSs have been deal with by nonlinear algorithm. For example, consider the estimation of multi-component cubic phase signals by HAF:

$$x(t) = \sum_{i=1}^N A_i e^{j(a_i + b_i t + c_i t^2 + d_i t^3)} + n(t), \quad -\Delta t/2 \leq t \leq \Delta t/2 \quad (1)$$

where  $A_i$ ,  $a_i$ ,  $b_i$ ,  $c_i$ ,  $d_i$ ,  $i = 1, 2, \dots, M$  denote the amplitude and phase parameters corresponding the  $i$ th-component and  $n(t)$  is the additive complex white Gaussian noise with zero-mean. The first order of HIM can be expressed as

$$P_1(t, \tau) = x(t + \tau)x^*(t - \tau) = \sum_{i=1}^N A_i^2 e^{j[(2b_i\tau + 2d_i\tau^3) + 4c_i\tau t + 6d_i\tau t^2]} + \sum_{1 \leq i, k \leq N, i \neq k} A_i A_k e^{j[\phi_0 + \phi_1 t + \phi_2 t^2 + (d_i - d_k)t^3]} + n'(t) \quad (2)$$

where the first summation part denotes the auto-terms and the second summation part denotes the cross terms. The first three orders of phase coefficients of cross terms can be expressed as

$$\begin{aligned} \phi_0 &= (a_i - a_k) + (b_i + b_k)\tau + (c_i - c_k)\tau^2 + (d_i + d_k)\tau^3 \\ \phi_1 &= b_i - b_k + 2c_i\tau + 2c_k\tau + 3d_i\tau^2 - 3d_k\tau^2 \\ \phi_2 &= c_i - c_k + 3d_i\tau + 3d_k\tau. \end{aligned}$$

From (2), it can appear that the number of the highest-order phase in cross term is same with the auto-terms when  $d_i - d_k = 0$ . As a result, this cross term will be treated as an auto-term in the next order of HAF and can not be identified.

We introduce a signal transformation procedure to convert mc-PPSs to new functions which have symmetric cross terms. It is worth mentioning that the similar signal transformation procedure is used to simplify the phase of mono-PPS in [7]. Consider two new functions  $x_1(t)$  and  $x_2(t)$ :

$$x_1(t) = x(t)x(-t) \quad (3)$$

$$x_2(t) = x(t)x^*(-t) \quad (4)$$

where  $*$  denotes the conjugate operator. Substitute (1) into (3) and (4), then

$$x_1(t) = \sum_{i=1}^N A_i^2 e^{2j(a_i+c_it^2)} + \sum_{1 \leq i, k \leq N, i \neq k} A_i A_k e^{j[a_i+a_k+(b_i-b_k)t+(c_i+c_k)t^2+(d_i-d_k)t^3]} + n_1(t) \quad (5)$$

$$x_2(t) = \sum_{i=1}^N A_i^2 e^{2j(b_it+d_it^3)} + \sum_{1 \leq i, k \leq N, i \neq k} A_i A_k e^{j[a_i-a_k+(b_i+b_k)t+(c_i-c_k)t^2+(d_i+d_k)t^3]} + n_2(t) \quad (6)$$

where

$$\begin{aligned} n_1(t) &= n(-t) \sum_{i=1}^N A_i e^{j(a_i+b_it+c_it^2+d_it^3)} \\ &\quad + n(t) \sum_{i=1}^N A_i e^{j(a_i-b_it+c_it^2-d_it^3)} + n(t)n(-t) \\ n_2(t) &= n^*(-t) \sum_{i=1}^N A_i e^{j(a_i+b_it+c_it^2+d_it^3)} \\ &\quad + n(t) \sum_{i=1}^N A_i e^{-j(a_i-b_it+c_it^2-d_it^3)} + n(t)n^*(-t). \end{aligned}$$

In Equation (5), the first summation includes  $N$  number of auto-terms and the second summation includes  $N(N-1)$  number of cross terms. In order to estimate the 0th order and second order phase coefficients, we divide the distribution of the cross terms into three cases according to the relationship of the first order and third order coefficients of different components.

(c1) All the third order coefficients are different from each other ( $d_k \neq d_i, \forall k, i = 1, \dots, N, k \neq i$ ). All cross terms in this case have a cubic phase that is different from the second order phase in the auto-terms. As a result, the cross terms would not affect the detection of auto-terms.

(c2) Some of the third order coefficients coincide ( $d_k = d_i, b_k \neq b_i, k \neq i$ ) and the corresponding second order coefficients are different ( $b_k \neq b_i, k \neq i$ ). In this case, the  $k-i$  cross terms have the same order phase with the auto-terms, which would affect the detection of auto-terms. Since the first order phase coefficients of the two the cross terms, i.e.,  $(b_k - b_i)$  and  $(b_i - b_k)$ , are opposite numbers of each other, the cross terms appears in pairs with respect to the first order phase, which can be removed by the symmetry.

(c3) Some of the highest order and the first order coefficients also coincide ( $d_k = d_i, b_k = b_i, k \neq i$ ). In this case, the two cross terms of  $k-i$  combine into one term and do not appear in the there is no

symmetrical property to use. However, the highest order coefficient of the  $k - i$  cross term is  $(c_i + c_k)$  that is the average of the highest order coefficients of the two auto-terms, so the identification of these cross terms need refer to the phase relationship of the  $k - i$  cross term and the corresponding auto-terms.

Similarly, three cases are considered to estimate the 1 order and 3rd order phase coefficients according to the 0th order and second order coefficients of different components.

(I) All the second order coefficients are different from each other ( $c_k \neq c_m, \forall k, m = 1, \dots, N, k \neq m$ ).

(II) Some of the second order coefficients coincide ( $c_k = c_m, k \neq m$ ) and the corresponding 0th order coefficients are different ( $a_k \neq a_m, k \neq m$ ).

(III) Some of the second order and the 0th order coefficients also coincide ( $c_k = c_r, a_k = a_r, k \neq r$ ).

For the case of (c2), (c3), (II) and (III), by using appropriate transforms, the relationships of the phase coefficients between auto-terms and cross terms can be converted into the position relationships of the corresponding peaks in the new transform domain. The interferences of cross terms can be completely eliminated by the symmetry and the estimation algorithm is discussed in the next section.

### 3. ESTIMATION ALGORITHMS

We introduce different transforms  $X_1(u, \alpha)$  and  $X_2(u, \beta)$  to deal with  $x_1(t)$  and  $x_2(t)$  respectively, which can be defined as

$$X_1(u, \alpha) = \int_{-\infty}^{+\infty} x_1(t) e^{j(ut+t^2 \cot \alpha)} dt \tag{7}$$

$$X_2(u, \beta) = \int_{-\infty}^{+\infty} x_2(t) e^{j(ut+t^3 \cot \beta)} dt \tag{8}$$

The kernel of  $X_1(u, \alpha)$  contains a quadratic phase that is suitable for estimate the parameters of auto-terms in  $x_1(t)$ . Substitute (5) into (7), and then

$$\begin{aligned} X_1(u, \alpha) = & \sum_{i=1}^N A_i^2 e^{2ja_i} \int_{-\infty}^{+\infty} e^{j[ut+(\cot \alpha+2c_i)t^2]} dt \\ & + \sum_{1 \leq i, k \leq N, i \neq k} A_i A_k \int_{-\infty}^{+\infty} e^{j\varphi(t, u, \alpha)} dt \\ & + \int_{-\infty}^{+\infty} n_1(t) e^{j(ut+t^2 \cot \alpha)} dt \end{aligned} \tag{9}$$

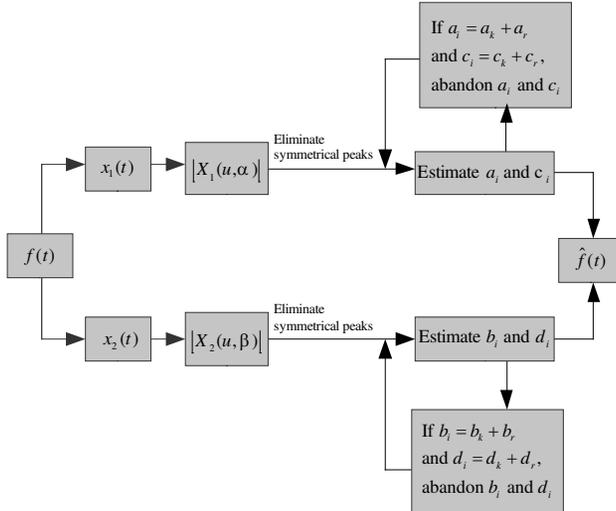
where  $\varphi(t, u, \alpha) = a_i + a_k + (b_i - b_k + u)t + (c_i + c_k + \cot \alpha)t^2 + (d_i - d_k)t^3$ . In (9), we can see that  $|X_1(u, \alpha)|$  would appear peaks in the place of  $\cot \alpha = -2c_i$ ,  $u = 0$ , which are corresponding the  $N$  number of auto-terms. The cross terms can not appear peaks on condition of (c1). There can appear peaks in pairs with respect to the  $u$  axis on condition of (c2). However, the pair of peaks combines into one peak on condition of (c3). If there exist two peaks whose parameters satisfy  $d_k = d_r$  and  $b_k = b_r$ , there must exist a cross term peak whose parameters satisfy  $c_i = (c_k + c_r)/2$ ,  $a_i = (a_k + a_r)/2$  and  $A_i = \sqrt{A_k A_r}$ .

The similar analysis for  $|X_2(u, \beta)|$  on condition of (I), (II) and (III) can be obtained by the following equation:

$$\begin{aligned} X_2(u, \beta) = & \sum_{i=1}^N A_i^2 \int_{-\infty}^{+\infty} e^{j[(u+2b_i)t + (\cot \beta + 2d_i)t^3]} dt \\ & + \sum_{1 \leq i, k \leq N, i \neq k} A_i A_k \int_{-\infty}^{+\infty} e^{j\varphi(t, u, \beta)} dt \\ & + \int_{-\infty}^{+\infty} n_2(t) e^{j(ut + t^3 \cot \beta)} dt \end{aligned} \quad (10)$$

where  $\varphi(t, u, \beta) = a_i - a_k + (b_i + b_k + u)t + (c_i - c_k)t^2 + (d_i + d_k + \cot \beta)t^3$ .

The flowchart of parameter estimation is shown in Fig. 1 as follows.



**Figure 1.** Parameter estimation flowchart.

For the 3rd or 4th order mc-PPS, the proposed algorithm needs a

2-dimensional maximization. When the order of mc-PPSs is  $N(N - 1)$ , the corresponding parameter search dimension is  $\lceil P/2 \rceil + 1$ , where  $\lceil \cdot \rceil$  denotes the ceiling operator.

#### 4. RESULT AND DISCUSSION

In this section, some simulations have been carried out to evaluate the multi-PPSs parameter estimation performance of the proposed algorithm in the presence of white Gaussian noise.

In the first experiment, we consider the multi-PPSs as sum of two cubic signals with different coefficients corresponding (c1) and (I)

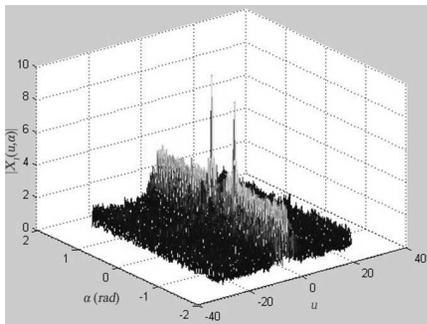
$$f_1(t) = \sum_{i=1}^2 A_i \exp [j (a_i + b_i t + c_i t^2 + d_i t^3)] + n(t) \quad (11)$$

where  $n(t)$  denotes complex white Gaussian noise with SNR = 0 dB. The parameters are

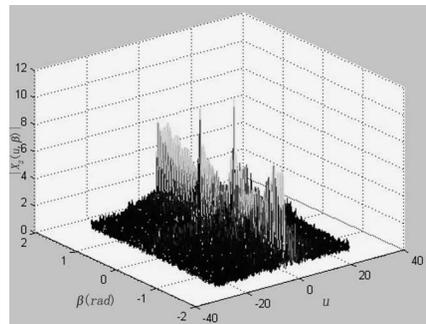
$$\begin{aligned} A_1 &= 1, a_1 = 0, b_1 = 1.5, c_1 = -1.98, d_1 = 2.8; \\ A_2 &= 1, a_2 = 0, b_2 = 0, c_2 = 1.58, d_2 = -1.8. \end{aligned}$$

The time range is  $(-1.25\pi, 1.25\pi)$  with 401 samples. Fig. 2 and Fig. 3 show the 3-D graphic of  $X_1$  and  $X_2$  respectively. As the highest order coefficients of cross terms in  $X_1$  or the 2nd order coefficients of cross terms in  $X_2$  are not zero, we can see that all peaks in Fig. 2 and Fig. 3 result from auto-terms.

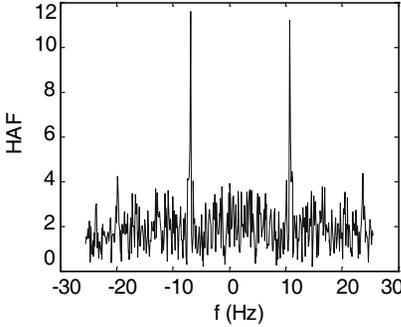
However, other methods like HAF, PHAF or PWVD, can not appear false peaks from cross terms effect. Fig. 4 shows the spectrum of the 2nd order HAF of  $f_1(t)$ , where  $\tau = 2$ . From Fig. 4, we can see that there are two peaks corresponding two auto-terms and no false peaks from cross terms. In Fig. 5, we compare the root mean square error (RMSE) performance of HAF and the proposed method



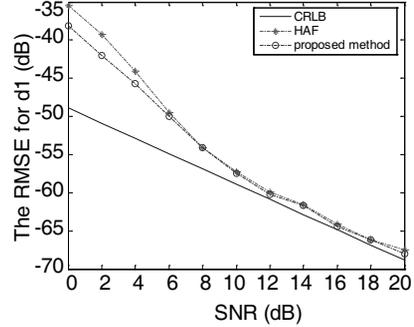
**Figure 2.** The 3-D graphics of  $X_1$ .



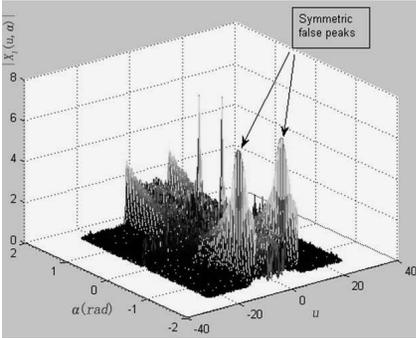
**Figure 3.** The 3-D graphics of  $X_2$ .



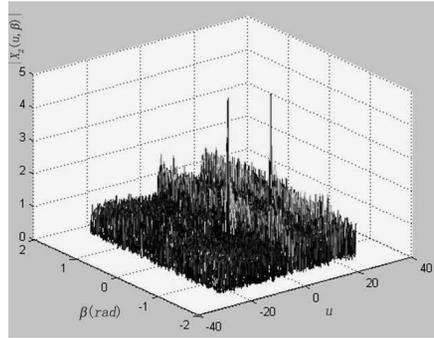
**Figure 4.** HAF of the sum of two cubic PPSs whose phase parameters have different third order phase.



**Figure 5.** RMSE versus SNR for the parameters of  $d_1$ .



**Figure 6.** The 3-D graphics of  $X_1$  of  $f_2(t)$ .



**Figure 7.** The 3-D graphics of  $X_2$  of  $f_2(t)$ .

in parameter estimation of  $d_1$ . As the proposed method involves only second-order nonlinearities that is less than the fourth- or higher order nonlinearities in the HAF algorithm, we can see that the performance of proposed method is better than HAF in low SNR.

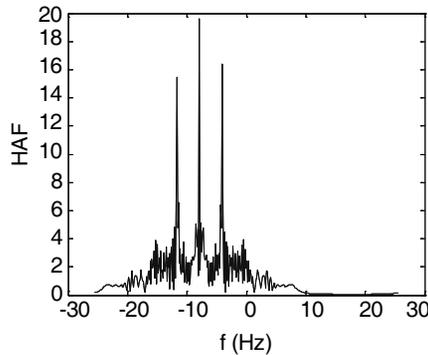
In the second experiment, we consider the multi-PPSs as sum of two cubic signals which their phase coefficients satisfy  $d_1 = d_2$  and  $b_1 \neq b_2$  corresponding (c2)

$$f_2(t) = \sum_{i=1}^2 A_i \exp [j (a_i + b_i t + c_i t^2 + d_i t^3)] + n(t). \quad (12)$$

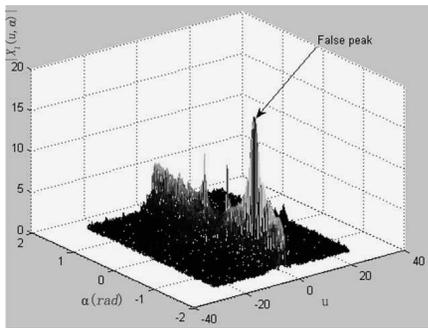
The parameters are

$$\begin{aligned} A_1 &= 1, \quad a_1 = 0, \quad b_1 = 50, \quad c_1 = 1.98, \quad d_1 = 3.9; \\ A_2 &= 1, \quad a_2 = 0, \quad b_2 = 0, \quad c_2 = -1.58, \quad d_2 = 3.9. \end{aligned}$$

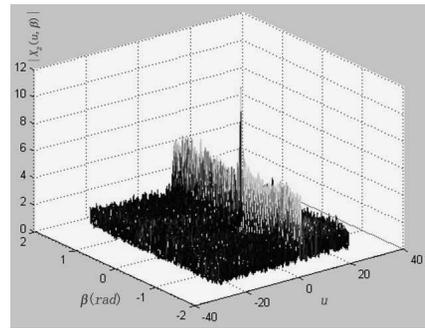
The 3-D graphic of  $X_1$  and  $X_2$  are shown in Fig. 6 and Fig. 7 respectively. In Fig. 6, we can see that there appear four peaks, two of



**Figure 8.** HAF of the sum of two cubic PPSs whose phase parameters have the same third order phase.



**Figure 9.** The 3-D graphics of  $X_1$  of  $f_3(t)$ .



**Figure 10.** The 3-D graphics of  $X_2$  of  $f_3(t)$ .

which are symmetrical about the axis. According to the symmetry of false peaks, the cross terms can be easily identified and the remaining two peaks are corresponding to the auto-terms.

Generally, HAF, PWVD, PHAF or other nonlinear algorithms have not obvious regular characteristic to identify the cross terms. Fig. 8 shows the spectrum of the 2nd order HAF of  $f_2(t)$  with  $\tau = 3.4$ , where we can see three peaks instead of one peak corresponding to the two overlapping auto-terms.

In the third experiment, we consider the multi-PPSs as sum of two cubic signals

$$f_3(t) = \sum_{i=1}^2 A_i \exp [j (a_i + b_i t + c_i t^2 + d_i t^3)] + n(t) \quad (13)$$

where the phase coefficients satisfy

$$A_1 = 1, a_1 = 0, b_1 = 20, c_1 = -1.38, d_1 = 3.9;$$

$$A_2 = 1, a_2 = 0, b_2 = 20, c_2 = 2.2, d_2 = 3.9.$$

The 3-D graphic of  $X_1$  and  $X_2$  are shown in Fig. 9 and Fig. 10 respectively. As the first order phase coefficients coincide, we can see that the two symmetric false peaks combine into a peak at the u-axis in Fig. 9. There is one peak in Fig. 10, for the 1st and 3rd order coefficients coincide.

## 5. CONCLUSION

A regular cross terms based method for the parameter estimation of multi-PPSs in white Gaussian noise has been proposed. The phase parameters are separated into two new signals by simple nonlinear procedures. Both the two new signals have regular cross terms that can be identified by two linear transform. In particular, the parameter estimation of multi-component cubic signals is analyzed in detail. Simulations illustrate that the proposed algorithm can well identify and eliminate the cross term interference in the parameter estimation.

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