Progress In Electromagnetics Research M, Vol. 23, 249–262, 2012

ELECTRIC AND MAGNETIC FIELD PROBLEMS WITH PERIODIC CIRCULAR CYLINDRICAL SYMMETRY AND THEIR CONNECTION WITH A NOVEL GEOMETRICAL INTERPRETATION OF THE ALGEBRAIC OPERATION $a^N \pm b^N$

J. A. Brandão Faria*

Instituto de Telecomunicações, Instituto Superior Técnico, Technical University of Lisbon, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Abstract—This paper deals with the evaluation of the electric and magnetic field generated by a set of N periodically distributed filamentary conductors, in a circular arrangement. The results obtained lead to the computation of a continuous product of distances. In close connection with the computation of such a continuous product, the general problem of the factorization of a sum or difference of two powers, $a^N \pm b^N$, where a and b are positive real numbers and N a positive integer, is addressed.

1. INTRODUCTION

Quoting Leonhard Euler: "*We have nothing particular to observe with regard to the addition and subtraction of powers*" [1].

In fact, by perusing the vast literature on number theory, high arithmetics, and algebraic geometry, very scarce references are found on the subject of the sum and difference of powers of type $a^N \pm b^N$, with the exception of the square and cubic cases, $N = 2$ and 3. However, when a and b are integers, abundant literature, connected with the Diophantine equation, is available, e.g., [2–7]. The major contribution of this work is to offer an absolutely new geometrical interpretation of the algebraic operations $a^N + b^N$ and $a^N - b^N$, where a and b are positive real numbers and N a positive integer. We show that such operations can be interpreted as the result of a continuous product of distances: the distances between a fixed point in a circumference and N points periodically distributed around a new circumference concentric

Received 22 December 2011, Accepted 9 February 2012, Scheduled 20 February 2012
* Corresponding author: J. A. Brandão Faria (brandao.faria@ist.utl.pt).

with the first. This mathematical result may find application in a number of electromagnetic problems with axisymmetric fields [8, 9], with periodically located parallel conductors (field sources), as, for example, in the case of bundle conductors in high-voltage overhead power lines [10, 11].

This paper is organized as follows. In Section 2, the electromagnetic potentials generated by a circular arrangement of N periodically distributed filamentary parallel conductors, with total charge q and total current intensity i, are evaluated, showing that both potentials are expressed in terms of a continuous product of distances. In Section 3, a heuristic approach to the subject of the sum and difference of powers is presented, giving rise to a conjecture about its geometrical interpretation (a product of distances); a formal proof of the conjecture being presented afterwards. In Section 4 we go back to the electromagnetic field problem posed in Section 2 and write its solution making use of the mathematical results of Section 3. An illustrative numerical example is provided in Section 5. Conclusions appear in Section 6.

2. POTENTIAL FUNCTIONS GENERATED BY *N* PARALLEL WIRES

Consider the cage-like structure depicted in Figure 1, made of N parallel filamentary wires periodically distributed along the periphery of a cylinder of radius R. In a cylindrical reference frame the position

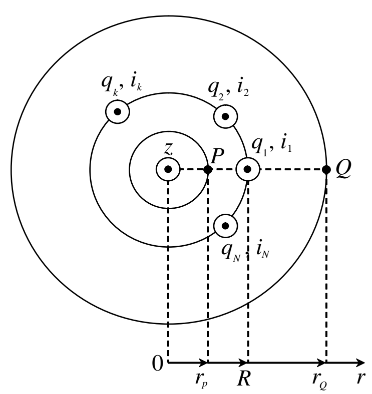

Figure 1. Cross sectional view of a cylindrical cage made of N filamentary wires uniformly distributed. Each wire has a per-unit-charge q/N and carries a z-oriented current intensity i/N.

of the generic kth wire is defined by $r = R$ and $\theta = 2\pi(k-1)/N$, with k an integer running from 1 to N. Each wire possesses a per-unit-length electric charge $q_k = q/N$ and, in addition, it carries a current intensity $i_k = i/N$. A stationary field regime is assumed.

The electric and magnetic fields originated by charges and currents can be determined from $\mathbf{E} = -\nabla\Phi$, $\mathbf{B} = \nabla \times \mathbf{A}$, where Φ is the electric scalar potential and $\mathbf{A}$ is the magnetic vector potential.

Fields are to be evaluated inside and outside the wire cage at discrete points P and Q, whose coordinates are

$$\text{Point } P \begin{cases} r = r_P, \ 0 \leq r_P < R \\ \theta = \theta_P = 2\pi(l-1)/N \end{cases}, \quad \text{Point } Q \begin{cases} r = r_Q, \ R < r_Q < \infty \\ \theta = \theta_Q = 2\pi(l-1)/N \end{cases} \tag{1}$$

where l is an integer number freely chosen in the interval 1 to N. The example in Figure 1 refers to the situation $l = 1$.

Our first goal is to determine the electric potential Φ_P and Φ_Q at points P and Q.

The contribution for the electric potential arising from the individual kth wire charge $q_k = q/N$ is given by [12, 13],

$$(\Phi_P)_k = -\frac{q}{2N\pi\varepsilon_0} \ln\left(r_{P_k}\right); \quad (\Phi_Q)_k = -\frac{q}{2N\pi\varepsilon_0} \ln\left(r_{Q_k}\right) \tag{2}$$

where ε_0 is the free-space permittivity, and r_{P_k} and r_{Q_k} respectively denote the distances from wire k to points P and Q.

The total electric potential at P and Q is obtained by summing all the q_k contributions from $k = 1$ to $k = N$.

$$\begin{cases} \Phi_P = -\frac{q}{2N\pi\varepsilon_0} \sum\limits_{k=1}^{N} \ln(r_{P_k}) = -\frac{q}{2N\pi\varepsilon_0} \ln\left(\prod\limits_{k=1}^{N} r_{P_k}\right) \\ \Phi_Q = -\frac{q}{2N\pi\varepsilon_0} \sum\limits_{k=1}^{N} \ln(r_{Q_k}) = -\frac{q}{2N\pi\varepsilon_0} \ln\left(\prod\limits_{k=1}^{N} r_{Q_k}\right) \end{cases} \tag{3}$$

Our second goal is to determine the magnetic vector potential $\mathbf{A}_P$ and $\mathbf{A}_Q$ at points P and Q. The contribution for the magnetic vector potential arising from the individual kth wire current $i_k = i/N$, oriented along z, is given by [12],

$$\begin{cases} (\mathbf{A}_P)_k = A_P\vec{e}_z, & A_P = -\frac{\mu_0 i}{2N\pi} \ln\left(r_{P_k}\right) \\ (\mathbf{A}_Q) = A_Q\vec{e}_z, & A_Q = -\frac{\mu_0 i}{2N\pi} \ln\left(r_{Q_k}\right) \end{cases} \tag{4}$$

where μ_0 is the free-space permittivity.

The total magnetic vector potential at P and Q is obtained by summing all the i_k contributions from $k = 1$ to $k = N$.

$$\begin{cases} A_P = -\frac{\mu_0 i}{2N\pi} \sum\limits_{k=1}^{N} \ln(r_{P_k}) = -\frac{\mu_0 i}{2N\pi} \ln\left(\prod\limits_{k=1}^{N} r_{P_k}\right) \\ A_Q = -\frac{\mu_0 i}{2N\pi} \sum\limits_{k=1}^{N} \ln(r_{Q_k}) = -\frac{\mu_0 i}{2N\pi} \ln\left(\prod\limits_{k=1}^{N} r_{Q_k}\right) \end{cases} \tag{5}$$

Note that the potential functions in (2)–(3) and (4)–(5) are defined apart from an arbitrary additive constant. As matter of fact, both potential functions are not univocally defined [12, 13]; one can add to Φ an arbitrarily chosen constant scalar and add to $\mathbf{A}$ an arbitrarily chosen constant vector, and, despite that, the associated $\mathbf{E}$ and $\mathbf{B}$ fields remain unaffected. In Section 5, we will come back to (3) and (5), and add the necessary constants in a manner such that the logarithm functions operate over dimensionless quantities.

3. A CONJECTURE ON THE FACTORIZATION OF POWERS

At the end of this section, we will conclude that the results shown in (3) and (5), which involve continuous products of distances, $\prod_{k=1}^{N} r_{P_k}$ and $\prod_{k=1}^{N} r_{Q_k}$, can be considerably simplified.

3.1. Heuristic Approach

Consider two concentric circumferences of radii, a and b, where $a > b$. On the inner circumference of radius b, two diametrically opposite points (called *foci*) are defined: point F_1 occupies the position $r = b$, $\theta = 0$, and point F_2 occupies the position $r = b$, $\theta = \pi$. Next, consider a set of N periodically distributed points, $S_1 \ldots S_k \ldots S_N$, called *sources* and placed along the outer circumference of radius a. The length of the line segment connecting the source S_k to the focus F_1 is denoted by r_k. Likewise, the length of the line segment connecting the source S_k to the focus F_2 is denoted by $\hat{r}_k$. See Figure 2.

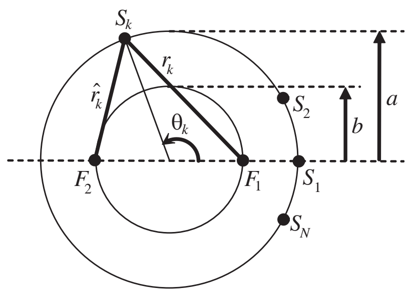

Figure 2. Concentric circumferences of radii a and b, showing foci F_1 and F_2, and periodically distributed sources S_1 to S_N.

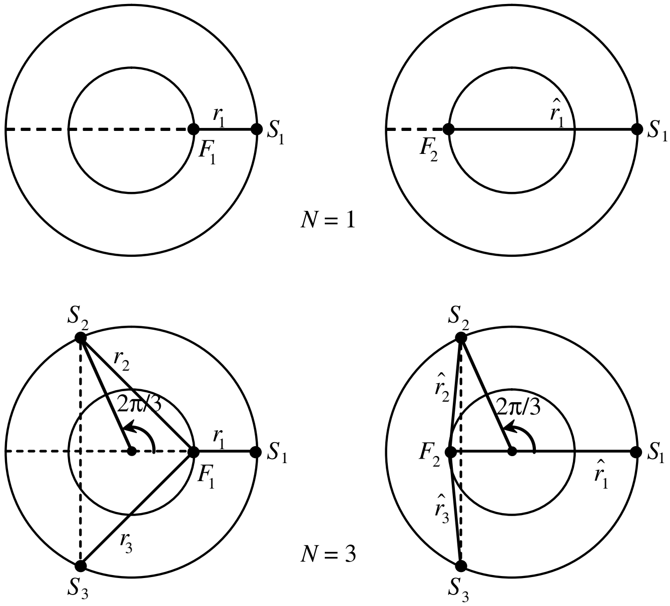

Figure 3. Factors r_k and $\hat{r}_k$ for $k = 1$ to N, with $N = 1$ and $N = 3$ (odd number of sources). Outer and inner circumferences have radii a and b, respectively.

Let us analyze the cases $N = 1$, $N = 3$ (odd numbers), and $N = 2$, $N = 4$ (even numbers).

<u>N odd</u> (Figure 3):

a) One source: S_1 ($r = a$, $\theta = 0$).

b) Three sources: S_1 ($r = a$, $\theta = 0$), S_2 ($r = a$, $\theta = 2\pi/3$), S_3 ($r = a$, $\theta = 4\pi/3$).

By examining Figure 3 we see that

$$\text{Case a)}\quad N = 1:\quad \begin{cases} r_1 = a - b \\ \hat{r}_1 = a + b \end{cases}$$

$$\text{Case b)}\quad N = 3:\quad \begin{cases} r_1 = a - b;\ r_2 = r_3 = (a^2 + ab + b^2)^{1/2} \\ \qquad \rightarrow r_1 r_2 r_3 = a^3 - b^3 \\ \hat{r}_1 = a + b;\ \hat{r}_2 = \hat{r}_3 = (a^2 - ab + b^2)^{1/2} \\ \qquad \rightarrow \hat{r}_1 \hat{r}_2 \hat{r}_3 = a^3 + b^3 \end{cases}$$

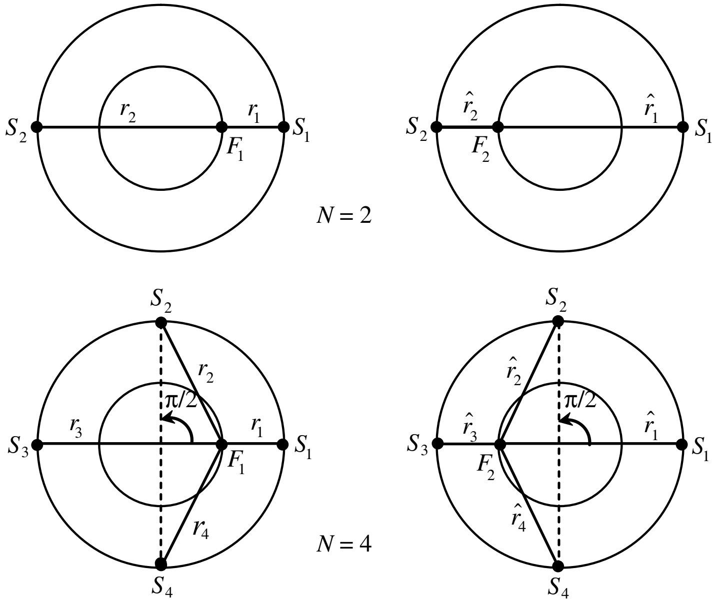

Figure 4. Factors r_k and $\hat{r}_k$ for $k = 1$ to N, with $N = 2$ and $N = 4$ (even number of sources). Outer and inner circumferences have radii a and b, respectively.

$\underline{N \text{ even}}$ (Figure 4):

c) Two sources: S_1 $(r = a,\ \theta = 0)$, S_2 $(r = a,\ \theta = \pi)$.

d) Four sources: S_1 $(r = a,\ \theta = 0)$, S_2 $(r = a,\ \theta = \pi/2)$, S_3 $(r = a,\ \theta = \pi)$, S_4 $(r = a,\ \theta = 3\pi/2)$.

By examining Figure 4 we see that

$$\text{Case c) } N = 2\text{:} \begin{cases} r_1 = a - b;\ r_2 = a + b \to r_1 r_2 = a^2 - b^2 \\ \hat{r}_1 = a + b;\ \hat{r}_2 = a - b \to \hat{r}_1 \hat{r}_2 = a^2 - b^2 \end{cases}$$

$$\text{Case d) } N = 4\text{:} \begin{cases} r_1 = a - b;\ r_2 = r_4 = (a^2 + b^2)^{1/2};\ r_3 = a + b \\ \qquad \to r_1 r_2 r_3 r_4 = a^4 - b^4 \\ \hat{r}_1 = a + b;\ \hat{r}_2 = \hat{r}_4 = (a^2 + b^2)^{1/2};\ \hat{r}_3 = a - b \\ \qquad \to \hat{r}_1 \hat{r}_2 \hat{r}_3 \hat{r}_4 = a^4 - b^4 \end{cases}$$

The above results suggest the following propositions concerning the factorization of the sum and difference of powers:

$$\text{For } N \text{ odd:} \begin{cases} \prod_{k=1}^{N} r_k = a^N - b^N \\ \prod_{k=1}^{N} \hat{r}_k = a^N + b^N \end{cases} \tag{6a}$$

$$\text{For } N \text{ even:} \prod_{k=1}^{N} r_k = \prod_{k=1}^{N} \hat{r}_k = a^N - b^N \tag{6b}$$

Therefore, the propositions that need to be proven are:

P1. For N an integer, the subtraction of powers $a^N - b^N$ is interpreted as the continuous product of the lengths r_k of the N line segments connecting the focus point F_1 to the N source points S_k.

P2. For N an even integer, the subtraction of powers $a^N - b^N$ is interpreted as the continuous product of the lengths $\hat{r}_k$ of the N line segments connecting the focus point F_2 to the N source points S_k.

P3. For N an odd integer, the addition of powers $a^N + b^N$ is interpreted as the continuous product of the lengths $\hat{r}_k$ of the N line segments connecting the focus point F_2 to the N source points S_k.

The propositions P2 and P3 are a natural consequence of the assertion in P1. In fact, we can convert the expressions of r_k into expressions of $\hat{r}_k$ by merely substituting $+b$ for $-b$. Therefore:

For N an even number, where $(-b)^N = b^N$, we find

$$\prod r_k = \prod \hat{r}_k = a^N - b^N \tag{7}$$

For N an odd number, where $(-b)^N = -b^N$, we find

$$\prod r_k = a^N - b^N \rightarrow \prod \hat{r}_k = a^N + b^N \tag{8}$$

This shows that P2 and P3 are corollaries of the proposition P1. In short: all we need to do is to prove the following conjecture, for any integer N:

$$\prod_{k=1}^{N} r_k = a^N - b^N \quad (\text{where } a > b) \tag{9}$$

3.2. Proof of the Conjecture

To prove the conjecture we analyze separately the situations N even and N odd. See Figure 5. The analysis takes into account the sequential numbering of the source points along the outer circumference and the existence of mirror-like symmetry.

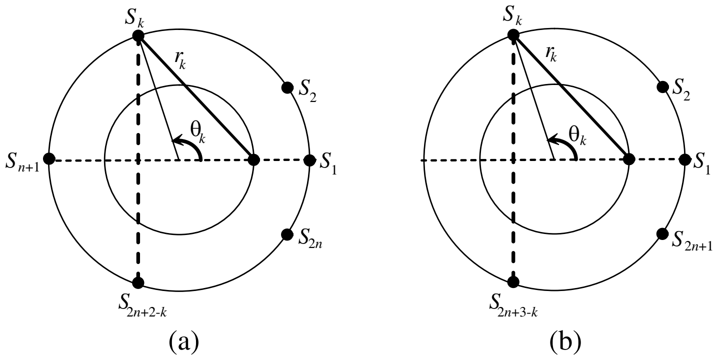

Figure 5. (a) Factor r_k for N even, $N = 2n$. (b) Factor r_k for N odd, $N = 2n + 1$.

3.2.1. N Even, (N = 2n)

Noting (Figure 5(a)) that the angular separation between adjacent sources is π/n we can determine the lengths r_k and r_{2n+2-k} by using the well-know law of cosines

$$r_{2n+2-k} = r_k = \left(a^2 + b^2 - 2ab\cos\theta_k\right)^{1/2}; \quad \theta_k = (k-1)\frac{\pi}{n} \tag{10}$$

Next we write

$$\begin{aligned}\prod_{k=1}^{N=2n} r_k &= r_1 r_{n+1}\left(\prod_{k=2}^{n} r_k\right)\left(\prod_{k=n+2}^{2n} r_k\right) = (a-b)(a+b)\left(\prod_{k=2}^{n} r_k\right)^2 \\ &= (a-b)(a+b)\prod_{k=2}^{n} r_k^2 \end{aligned} \tag{11}$$

Plugging (10) into (11) and making use of the formula 2.22 in [14] yields:

$$\begin{aligned}\prod_{k=1}^{N} r_k &= (a-b)(a+b)\left(\prod_{k=2}^{n}\left(a^2+b^2-2ab\cos\left((k-1)\frac{\pi}{n}\right)\right)\right) \\ &= (a-b)(a+b)\left(a^2+b^2-2ab\cos\frac{\pi}{n}\right)\left(a^2+b^2-2ab\cos\frac{2\pi}{n}\right) \\ &\quad \ldots\left(a^2+b^2-2ab\cos\frac{(n-1)\pi}{n}\right) \\ &= a^{2n}-b^{2n} = a^N - b^N. \end{aligned} \tag{12}$$

3.2.2. N Odd ($N = 2n+1$)

Noting (Figure 5(b)) that the angular separation between adjacent sources is $2\pi/(2n+1)$ we can determine the lengths r_k and r_{2n+3-k} by using the law of cosines

$$r_{2n+3-k} = r_k = \left(a^2 + b^2 - 2ab\cos\theta_k\right)^{1/2}; \quad \theta_k = (k-1)\frac{2\pi}{2n+1} \quad (13)$$

Next we write

$$\prod_{k=1}^{N=2n+1} r_k = r_1 \left(\prod_{k=2}^{n+1} r_k\right)\left(\prod_{k=n+2}^{2n+1} r_k\right) = (a-b)\left(\prod_{k=2}^{n+1} r_k\right)^2 = (a-b)\prod_{k=2}^{n+1} r_k^2 \quad (14)$$

Plugging (13) into (14) and making use of the formula 2.20 in [14] yields:

$$\begin{aligned}\prod_{k=1}^{N} r_k &= (a-b)\left(\prod_{k=2}^{n+1}\left(a^2+b^2-2ab\cos\left((k-1)\frac{2\pi}{2n+1}\right)\right)\right)\\ &= (a-b)\left(a^2+b^2-2ab\cos\frac{2\pi}{2n+1}\right)\left(a^2+b^2-2ab\cos\frac{4\pi}{2n+1}\right)\\ &\quad\ldots\left(a^2+b^2-2ab\cos\frac{2n\pi}{2n+1}\right)\\ &= a^{2n+1} - b^{2n+1} = a^N - b^N. \end{aligned} \quad (15)$$

The conclusions in (12) and (15) prove the assertion in (9).

The proof presented here considered the case when the focus point is internal to the source points. However, following the same procedure, the conjecture could as well be proven (commuting a and b) for the case when the focus point is external to the source points, in which case we would get

$$\prod_{k=1}^{N} r_k = b^N - a^N; \quad (\text{where } b > a) \quad (16)$$

The results in (6) are absolutely general and valid for any values of a, b, and N. The conclusions in (6) were obtained resorting to simple trigonometric operations, however, we believe that, the same results could as well be derived by using complex polar coordinates, as in [15].

To conclude this section, we would like to add a parenthetical remark concerning Fermat's Last Theorem, which Wiles demonstrated, in 1995, from the modularity conjecture for elliptic curves [16]. Fermat's Last Theorem states that the inequality $a^N + b^N \neq c^N$ is always insured for any a, b, c, and N positive integers, with $N > 2$. By using (8), substituting a/c for a and b/c for b we can rewrite the preceding inequality in the form

$$(a/c)^N + (b/c)^N = \prod_{k=1}^{N} \hat{r}_k \neq 1 \tag{17}$$

Therefore, if it could be proven that for any a, b, c, and N positive integers, with $N > 2$, the continuous product of distances $\hat{r}_k$ is always different from unity, one would be able to find a geometrical proof of Fermat's Last Theorem. Of course this is highly speculative, but if Fermat really managed (in 1637) to find a proof of his theorem he must have used simple math, probably based on a geometric rationale.

4. ELECTRIC AND MAGNETIC FIELDS GENERATED BY N PARALLEL WIRES

The electric scalar potentials Φ_P and Φ_Q in (3), and the magnetic vector potentials $\mathbf{A}_P$ and $\mathbf{A}_Q$ in (5), can now be greatly simplified by making use of the key results in (9) and (16):

$$\begin{cases} \Phi_P = -\frac{q}{2N\pi\varepsilon_0} \ln\left(\prod_{k=1}^{N} r_{P_k}\right) = -\frac{q}{2N\pi\varepsilon_0} \ln\left(R^N - r_P^N\right) \\ \Phi_Q = -\frac{q}{2N\pi\varepsilon_0} \ln\left(\prod_{k=1}^{N} r_{Q_k}\right) = -\frac{q}{2N\pi\varepsilon_0} \ln\left(r_Q^N - R^N\right) \end{cases} \tag{18}$$

$$\begin{cases} A_P = -\frac{\mu_0 i}{2N\pi} \ln\left(\prod_{k=1}^{N} r_{P_k}\right) = -\frac{\mu_0 i}{2N\pi} \ln\left(R^N - r_P^N\right) \\ A_Q = -\frac{\mu_0 i}{2N\pi} \ln\left(\prod_{k=1}^{N} r_{Q_k}\right) = -\frac{\mu_0 i}{2N\pi} \ln\left(r_Q^N - R^N\right) \end{cases} \tag{19}$$

From a physical point of view, the results in (18) and (19) are a little discomforting, for the logarithm functions are not operating over dimensionless quantities. To circumvent the problem, we can add to Φ an extra constant Φ' such that $\Phi_P = 0$ for $r_P = 0$, likewise, we can add to A an extra constant A' such that $A_P = 0$ for $r_P = 0$. In other words, we are arbitrarily setting to zero the potential functions on the longitudinal axis of the wire cage, $\Phi_0 = \mathbf{A}_0 = 0$.

Noting that $\Phi' = q \ln R^N/(2N\pi\varepsilon_0)$ and $A' = \mu_0 i \ln R^N/(2N\pi)$, the equations in (18) and (19) finally transform into

$$\begin{cases} \Phi_P = -\frac{q}{2N\pi\varepsilon_0} \ln\left(\frac{R^N - r_P^N}{R^N}\right) \\ \Phi_Q = -\frac{q}{2N\pi\varepsilon_0} \ln\left(\frac{r_Q^N - R^N}{R^N}\right) \end{cases} ; \begin{cases} A_P = -\frac{\mu_0 i}{2N\pi} \ln\left(\frac{R^N - r_P^N}{R^N}\right) \\ A_Q = -\frac{\mu_0 i}{2N\pi} \ln\left(\frac{r_Q^N - R^N}{R^N}\right) \end{cases} \tag{20}$$

The electric field strength and the magnetic induction field, at points P and Q, are obtained from $\mathbf{E} = -\nabla\Phi$, $\mathbf{B} = \nabla \times \mathbf{A}$, using (20):

$$\begin{cases} \mathbf{E}_P = -\frac{\partial \Phi_P}{\partial r_P} \vec{e}_r = \frac{q}{2\pi\varepsilon_0 r_P} \times \frac{r_P^N}{r_P^N - R^N} \vec{e}_r \\ \mathbf{B}_P = -\frac{\partial A_P}{\partial r_P} \vec{e}_\theta = \frac{\mu_0 i}{2\pi r_P} \times \frac{r_P^N}{r_P^N - R^N} \vec{e}_\theta \end{cases} \tag{21}$$

$$\begin{cases} \mathbf{E}_Q = -\frac{\partial \Phi_Q}{\partial r_Q} \vec{e}_r = \frac{q}{2\pi\varepsilon_0 r_Q} \times \frac{r_Q^N}{r_Q^N - R^N} \vec{e}_r \\ \mathbf{B}_Q = -\frac{\partial A_Q}{\partial r_Q} \vec{e}_\theta = \frac{\mu_0 i}{2\pi r_Q} \times \frac{r_Q^N}{r_Q^N - R^N} \vec{e}_\theta \end{cases} \tag{22}$$

To check and validate the above results, we employ (21) and (22) to analyze the particular problem of the electromagnetic field created by the charge and current of a cylindrical conducting sheet — a problem whose solution is quite well known [12, 13]. The results for the cylindrical sheet can be determined by considering that the wire cage is made of an infinite number of wire filaments. By making $N \to \infty$ in (20) and (21), we readily find

$$\begin{cases} \mathbf{E}_P = 0; \quad \mathbf{B}_P = 0 \\ \mathbf{E}_Q = \frac{q}{2\pi\varepsilon_0 r_Q} \vec{e}_r; \quad \mathbf{B}_Q = \frac{\mu_0 i}{2\pi r_Q} \vec{e}_\theta \end{cases} \tag{23}$$

The electromagnetic field is zero everywhere inside the region confined by the cylindrical conducting sheet. Conversely, outside the conducting sheet, the electromagnetic field decreases with the radial distance.

5. NUMERICAL EXAMPLE

To mitigate corona and breakdown phenomena in extra high voltage (EHV) overhead power lines, phase conductors are made of subconductor bundles. For exemplification purposes consider a bundle made of 4 subconductors in a circular arrangement (see Figure 6). The radius of each sub conductor is r_0 and the bundle radius is r_B. Assume that the phase conductor runs parallel to a good conducting soil at height $h \gg r_B$. For simplification, let us assume that losses are negligibly small and that the bundle is terminated by its characteristic impedance R_w. Further, assume that the phase-to-ground voltage at a given observation point of the line is a sinusoidal voltage of amplitude U.

Consider the following typical data: $r_0 = 14\,\text{mm}$, $r_B = 32\,\text{cm}$, $h = 15\,\text{m}$, $U = 500\,\text{kV}$.

The maximum values of the per unit length charge of the bundle and the current intensity of the bundle are respectively given by;

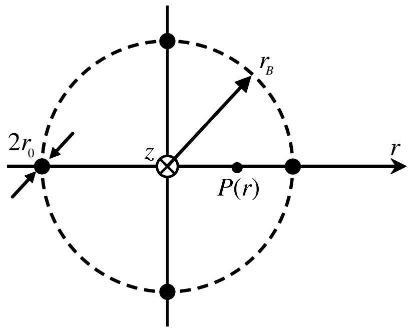

Figure 6. Bundle of 4 subconductors of an EHV power line.

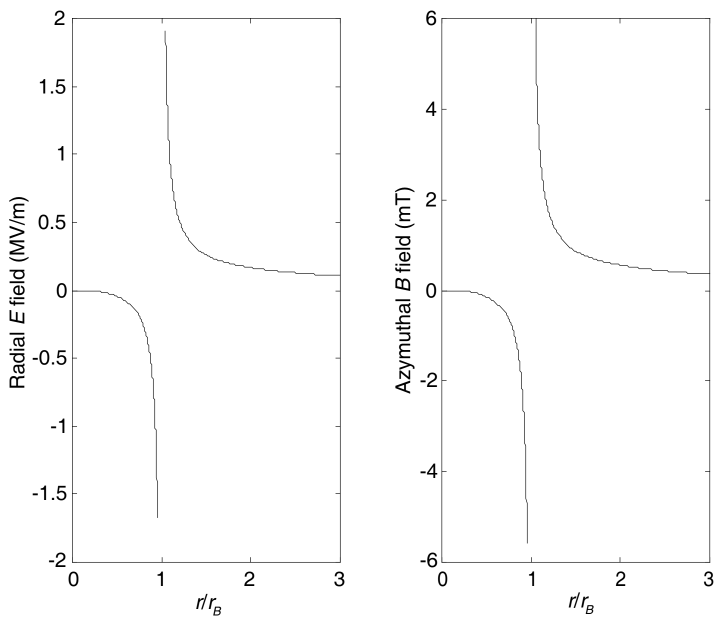

Figure 7. E and B fields against r/r_B.

$q = CU$, and $i = U/R_w$, where [10, 13],

$$C = \frac{2\pi\varepsilon_0}{\ln\left(\frac{2h}{\mathrm{GMR}}\right)}; \quad R_w = \frac{\sqrt{\varepsilon_0\mu_0}}{C}, \quad \mathrm{GMR} = r_B\left(\frac{Nr_0}{r_B}\right)^{1/N} \tag{24}$$

where GMR denotes the so-called geometric mean radius of the N bundled conductor system [17]. For $N = 4$, we obtain: GMR $\approx$ 207 mm, $C \approx 0.11$ pF/m and $R_w \approx 300\,\Omega$, which leads to $q \approx 5.58\,\mu$C/m and $i \approx 1.67$ kA.

Equations (21) and (22) were employed to determine the radial E field and the azymuthal B field at the point $P(r)$ in Figure 6, letting r vary in the range $r = 0$ to $r = 3r_B$. The functions $E(r)$ and $B(r)$ are plotted in Figure 7, where the horizontal axis is the normalized radial distance r/r_B.

6. CONCLUSIONS

Field solutions involving continuous products of distances frequently occur in the analysis of electromagnetic problems with periodic circular cylindrical symmetry. We have shown that those continuous product operations are equivalent to the algebraic operation of difference of powers. A novel geometrical interpretation for the sum and difference of powers $a^N \pm b^N$, for a and b real numbers, and N a positive integer was established in this paper, showing that the result of both operations correspond to a factorization process where each factor is the distance between two points belonging to two concentric circumferences of radii a and b. To the author's best knowledge, this novel interpretation is an original contribution, which can be very useful in a variety of electromagnetic field problems with periodic circular cylindrical symmetry.

ACKNOWLEDGMENT

The Portuguese Foundation for Science and Technology (FCT) sponsored this work.

REFERENCES

1. Euler, L., *Elements of Algebra*, 2nd edition, Vol. 1, J. Johnson & Co, London, UK, 1810.
2. Diananda, P., "On integers expressible as a sum of two powers," *Proc. Japan Acad.*, Vol. 43, 417–419, 1967.

3. Penney, D., "Multiplication relation for series of initial kth powers," *Amer. Math. Monthly*, Vol. 92, 729–731, 1985.
4. Gessel, I., "A formula for power series," *Amer. Math. Monthly*, Vol. 95, 961–962, 1988.
5. Owes, R., "Series of powers of integers," *Math. Magazine*, Vol. 65, 38–40, 1992.
6. Tuenter, H., "The Frobenius problem, sums of powers of integers, and recurrence for the Bernoulli numbers," *J. Number Theory*, Vol. 117, 376–386, 2006.
7. Shirali, S., "On sums of powers of integers," *Resonance*, Vol. 12, 27–43, 2007.
8. Gluck, F., "Axysymmetric electric field calculation with zonal harmonic expansion," *Progress In Electromagnetics Research B*, Vol. 32, 319–350, 2011.
9. Gluck, F., "Axysymmetric magnetic field calculation with zonal harmonic expansion," *Progress In Electromagnetics Research B*, Vol. 32, 351–388, 2011.
10. Faria, J., V. Machado, and D. van Dommelen, "Comparison of zeroth-order and harmonic expansion calculation of the electrostatic parameters of three-conductor bundles," *Electric Power Systems Research*, Vol. 81, 488–494, 2011.
11. Machado, V., M. Pedro, J. Faria, and D. van Dommelen, "Magnetic field analysis of three-conductor bundles in flat and triangular configurations with the inclusion of proximity and skin effects," *Electric Power Systems Research*, Vol. 81, 2005–2014, 2011.
12. Solymar, L., *Lectures on Electromagnetic Theory*, Oxford University Press, Oxford, UK, 1984.
13. Faria, J., *Electromagnetic Foundations of Electrical Engineering*, Wiley, Chichester, UK, 2008.
14. Spiegel, M. and L. Rapun, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, New York, USA, 1970.
15. González-Morales, M. J., R. Mahillo-Isla, E. Gago-Ribas, and C. Dehesa-Martínez, "Complex polar coordinates in electromagnetics," *Journal of Electromagnetic Waves and Applications*, Vol. 25, Nos. 2–3, 389–398, 2011.
16. Wiles, A., "Modular elliptic curves and Fermat's last theorem," *Annals of Mathematics*, Vol. 141, 443–551, 1995.
17. Barros, M., "Computation of line parameters," *European EMTP Short Course*, Belgium, Jul.–Aug. 1984.