

UNCONDITIONALLY STABLE LEAPFROG ADI-FDTD METHOD FOR LOSSY MEDIA

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Abstract—This paper presents an unconditionally stable leapfrog alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method for lossy media. Conductivity terms of lossy media are incorporated into the leapfrog ADI-FDTD method in an analogous manner as the conventional explicit FDTD method since the leapfrog ADI-FDTD method is a perturbation of the conventional explicit FDTD method. Implementation of the leapfrog ADI-FDTD method for lossy media with special consideration for boundary condition is provided. Numerical results and examples are presented to validate the formulation.

1. INTRODUCTION

Recently an unconditionally stable leapfrog alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method that is free from the Courant-Friedrich-Lewy (CFL) stability criterion has been developed [1]. The leapfrog ADI-FDTD method is derived from the ADI-FDTD method [2–4], and has similar numerical stability and dispersion properties as the ADI-FDTD method [5]. It is more memory efficient than the ADI-FDTD method as it is leapfrog staggered in time, and thus does not require the field variables of the intermediate time-step [6]. Alternatively, the leapfrog ADI-FDTD method may be considered as a perturbation of the conventional explicit FDTD method [5], and consequently it requires slightly more memory than the conventional explicit FDTD method [7, 8] due to the implementation of the tridiagonal matrix.

Several schemes for the formulation of lossy media have been recently unified for the fundamental ADI-FDTD method [9]. The

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fundamental ADI-FDTD method has right-hand-side free of matrix operator, and thus it is simpler and more efficient than the conventional ADI-FDTD method. It is found that the backward-backward scheme is not unconditionally stable, while the forward-forward scheme is only first-order temporal accurate. The time-averaging scheme is unconditionally stable and second-order temporal accurate. Furthermore, several methods have been proposed in [10] for the implementation of the perfect electric conductor (PEC) condition for the conventional ADI-FDTD method. The method is more complicated as the PEC condition is incorporated directly into the modified tridiagonal matrix. On the other hand, it can be verified that PEC conditions may also be realized by proper definition of loss terms in the media.

In [11], the leapfrog ADI-FDTD method was employed to model a two-dimensional electromagnetic bandgap waveguide filled with dielectric posts. However, the dielectric posts were modeled ideally. To further enhance the simulation model, we could incorporate more physical material parameter, such as the dielectric losses. Therefore, it is important to develop a simple and straightforward method to incorporate loss terms into the leapfrog ADI-FDTD method.

Intuitively, to formulate the leapfrog ADI-FDTD method for lossy media, one should approach such derivation by first considering the conventional ADI-FDTD method for lossy media [12, 13]. However, such an approach would not directly result in the proper formulation of the leapfrog ADI-FDTD method for lossy media.

Furthermore, the leapfrog ADI-FDTD method is a fully implicit FDTD method, and hence requires special consideration for the treatment of the boundary condition. The Mur absorbing boundary condition (ABC) for the leapfrog ADI-FDTD method was implemented in [14] by formulating the Mur update equations for both the implicit electric and magnetic field updates. Numerical instabilities may exist if the boundary conditions are not treated carefully during actual implementation.

In this paper, we present an unconditionally stable leapfrog ADI-FDTD method for lossy media. In Section 2, conductivity terms of lossy media are incorporated into the leapfrog ADI-FDTD method in an analogous manner as the conventional explicit FDTD method since the leapfrog ADI-FDTD method is a perturbation of the conventional explicit FDTD method. Implementation of the leapfrog ADI-FDTD method for lossy media with special consideration for boundary condition is provided. In Section 3, numerical results and examples are presented to validate the formulation.

2. FORMULATION FOR LOSSY MEDIA

In this section, we present the three-dimensional (3-D) leapfrog ADI-FDTD method for lossy media. Here, we assume wave propagation in a lossy medium with permittivity ϵ , permeability μ , electric conductivity σ and magnetic conductivity σ^* .

Since the leapfrog ADI-FDTD method is a perturbation of the conventional explicit FDTD method, we can incorporate the conductivity terms of lossy media for the leapfrog ADI-FDTD method in an analogous manner as the conventional explicit FDTD method [15]. Consequently, electric and magnetic losses are incorporated into the leapfrog ADI-FDTD method by means of introducing conductivity terms for lossy media at the right-hand-side of the update equations as follows (illustrated here for E_x and H_x field components).

(i) Implicit updating for $E_x^{n+\frac{1}{2}}$:

$$\left(1 - \frac{\Delta t^2}{4} \epsilon_x^{-1} \partial_y \mu_z^{-1} \partial_y\right) E_x^{n+\frac{1}{2}} = \left(1 - \frac{\Delta t^2}{4} \epsilon_x^{-1} \partial_y \mu_z^{-1} \partial_y\right) E_x^{n-\frac{1}{2}} + \Delta t \epsilon_x^{-1} \left(\partial_y H_z^n - \partial_z H_y^n - \sigma_x E_x^n\right) \quad (1)$$

(ii) Implicit updating for H_x^{n+1} :

$$\left(1 - \frac{\Delta t^2}{4} \mu_x^{-1} \partial_y \epsilon_z^{-1} \partial_y\right) H_x^{n+1} = \left(1 - \frac{\Delta t^2}{4} \mu_x^{-1} \partial_y \epsilon_z^{-1} \partial_y\right) H_x^n + \Delta t \mu_x^{-1} \left(\partial_z E_y^{n+\frac{1}{2}} - \partial_y E_z^{n+\frac{1}{2}} - \sigma_x^* H_x^{n+\frac{1}{2}}\right) \quad (2)$$

where ∂_x , ∂_y , ∂_z are the spatial difference operators for the first derivatives along x , y , z directions, respectively. Here, electric losses are implemented in (1), while magnetic losses are implemented in (2). Note that it is not proper to directly formulate the leapfrog ADI-FDTD method for lossy media by following strictly the previous derivation of the leapfrog ADI-FDTD method from the conventional ADI-FDTD method for lossy media. This is because of the additional (non-unity) coefficients incurred for the inclusion of the conductivity terms of lossy media. For the leapfrog ADI-FDTD method for lossless media, these coefficients are unity and permit the exact cancellation of the mixed derivative terms. However, the previous derivation of the leapfrog ADI-FDTD method [1] is no longer directly applicable for the leapfrog ADI-FDTD method for lossy media, as these coefficient are non-unity, and they do not eliminate the mixed derivative terms. Further discussion with detailed explanations of the above is provided in the Appendix.

By taking time-averaging of the conductivity term in (1), the update equation for E_x reads

$$\begin{aligned} \left(1 - \frac{\Delta t^2}{4} \epsilon_x^{-1} \partial_y \mu_z^{-1} \partial_y\right) E_x^{n+\frac{1}{2}} &= \left(1 - \frac{\Delta t^2}{4} \epsilon_x^{-1} \partial_y \mu_z^{-1} \partial_y\right) E_x^{n-\frac{1}{2}} \\ &+ \Delta t \epsilon_x^{-1} \left(\partial_y H_z^n - \partial_z H_y^n - \frac{\sigma_x}{2} E_x^{n+\frac{1}{2}} - \frac{\sigma_x}{2} E_x^{n-\frac{1}{2}}\right) \end{aligned} \quad (3a)$$

and after some manipulation, we arrive at

$$\begin{aligned} &\left(1 - \frac{\Delta t^2}{4} \epsilon_x^{-1} \partial_y \mu_z^{-1} \partial_y + \frac{\Delta t}{2} \epsilon_x^{-1} \sigma_x\right) E_x^{n+\frac{1}{2}} \\ &= \left(1 - \frac{\Delta t^2}{4} \epsilon_x^{-1} \partial_y \mu_z^{-1} \partial_y - \frac{\Delta t}{2} \epsilon_x^{-1} \sigma_x\right) E_x^{n-\frac{1}{2}} + \Delta t \epsilon_x^{-1} \left(\partial_y H_z^n - \partial_z H_y^n\right). \end{aligned} \quad (3b)$$

The update equations for the other field components can be obtained in an analogous manner. Similar to the conventional explicit FDTD method, time-averaging of the conductivity term for the leapfrog ADI-FDTD method is second-order temporal accurate.

Next, we proceed to provide the actual implementation of the 3-D leapfrog ADI-FDTD method for lossy media. The update equations for the leapfrog ADI-FDTD method for lossy media are given as

(i) Implicit updating for $E_x^{n+\frac{1}{2}}$:

$$\begin{aligned} &-\frac{\Delta t^2}{4\mu\epsilon\Delta y^2} E_x|_{i+\frac{1}{2},j-1,k}^{n+\frac{1}{2}} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2} + \frac{\Delta t\sigma}{2\epsilon}\right) E_x|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} \\ &-\frac{\Delta t^2}{4\mu\epsilon\Delta y^2} E_x|_{i+\frac{1}{2},j+1,k}^{n+\frac{1}{2}} = -\frac{\Delta t^2}{4\mu\epsilon\Delta y^2} E_x|_{i+\frac{1}{2},j-1,k}^{n-\frac{1}{2}} \\ &+ \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2} - \frac{\Delta t\sigma}{2\epsilon}\right) E_x|_{i+\frac{1}{2},j,k}^{n-\frac{1}{2}} - \frac{\Delta t^2}{4\mu\epsilon\Delta y^2} E_x|_{i+\frac{1}{2},j+1,k}^{n-\frac{1}{2}} \\ &+ \frac{\Delta t}{\epsilon\Delta y} \left(H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^n - H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^n\right) - \frac{\Delta t}{\epsilon\Delta z} \left(H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^n - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^n\right) \end{aligned} \quad (4)$$

(ii) Implicit updating for H_x^{n+1} :

$$\begin{aligned} &-\frac{\Delta t^2}{4\mu\epsilon\Delta y^2} H_x|_{i,j-\frac{1}{2},k+\frac{1}{2}}^{n+1} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2} + \frac{\Delta t\sigma^*}{2\mu}\right) H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+1} \\ &-\frac{\Delta t^2}{4\mu\epsilon\Delta y^2} H_x|_{i,j+\frac{3}{2},k+\frac{1}{2}}^{n+1} = -\frac{\Delta t^2}{4\mu\epsilon\Delta y^2} H_x|_{i,j-\frac{1}{2},k+\frac{1}{2}}^n \\ &+ \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2} - \frac{\Delta t\sigma^*}{2\mu}\right) H_x|_{i,j+\frac{1}{2},k+\frac{1}{2}}^n - \frac{\Delta t^2}{4\mu\epsilon\Delta y^2} H_x|_{i,j+\frac{3}{2},k+\frac{1}{2}}^n \\ &+ \frac{\Delta t}{\mu\Delta z} \left(E_y|_{i,j+\frac{1}{2},k+1}^{n+\frac{1}{2}} - E_y|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}\right) - \frac{\Delta t}{\mu\Delta y} \left(E_z|_{i,j+1,k+\frac{1}{2}}^{n+\frac{1}{2}} - E_z|_{i,j,k+\frac{1}{2}}^{n+\frac{1}{2}}\right) \end{aligned} \quad (5)$$

Referring to (5), the H_x field components are solved implicitly using a tridiagonal matrix, and involve terms that are out-of-domain (e.g., $H_x|_{i,j-\frac{1}{2},k+\frac{1}{2}}$). Therefore, special consideration must be taken for these out-of-domain terms when implementing the boundary condition [16].

As an illustration, we consider the commonly used PEC boundary condition. Implementation of more complex boundary condition follows in a similar manner.

It is straightforward to impose the PEC boundary condition for the electric fields. We can simply define the tangential electric fields at the PEC boundaries to be zero. However, as mentioned above, the out-of-domain terms for the tangential magnetic fields are non-zero, and this is critical for the proper implementation of the PEC boundary condition. Note that this issue does not exist for the conventional ADI-FDTD method as the tangential magnetic fields are (usually) not updated implicitly, and thus there are no such out-of-domain terms.

To circumvent this issue, we resort to the image theory. Since the H_x field components are updated in the y -direction, we consider the PEC at the $j = 0$ boundary. By applying the image theory, we can rewrite (5) as

$$\begin{aligned} & \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2} + \frac{\Delta t\sigma^*}{2\mu} - \frac{\Delta t^2}{4\mu\epsilon\Delta y^2} \right) H_x|_{i,\frac{1}{2},k+\frac{1}{2}}^{n+1} - \frac{\Delta t^2}{4\mu\epsilon\Delta y^2} H_x|_{i,\frac{3}{2},k+\frac{1}{2}}^{n+1} \\ &= \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2} - \frac{\Delta t\sigma^*}{2\mu} - \frac{\Delta t^2}{4\mu\epsilon\Delta y^2} \right) H_x|_{i,\frac{1}{2},k+\frac{1}{2}}^n - \frac{\Delta t^2}{4\mu\epsilon\Delta y^2} H_x|_{i,\frac{3}{2},k+\frac{1}{2}}^n \\ &+ \frac{\Delta t}{\mu\Delta z} \left(E_y|_{i,\frac{1}{2},k+1}^{n+\frac{1}{2}} - E_y|_{i,\frac{1}{2},k}^{n+\frac{1}{2}} \right) - \frac{\Delta t}{\mu\Delta y} \left(E_z|_{i,1,k+\frac{1}{2}}^{n+\frac{1}{2}} \right) \end{aligned} \quad (6)$$

Note that the tangential electric field components at the $j = 0$ boundary are defined as zero. The update equations for the other magnetic field components can be written down by permuting the subscript indices accordingly.

Finally, it is also worth highlighting that the out-of-domain terms of the magnetic field updates must be treated carefully especially during actual implementation. Improper indexing and/or material assignment may potentially cause numerical instability. In addition, it is necessary to implement the PEC boundary condition properly for the perfectly matched layer (PML) as it is eventually terminated by PEC walls [17].

3. NUMERICAL RESULTS

In this section, we validate the formulation of the leapfrog ADI-FDTD method for lossy media through numerical simulations. The simulation

setup for a PEC cavity meshed with $50 \times 30 \times 9$ uniform cells of size 1 mm is shown in Figure 1. It is excited by a line Gaussian pulse, extended from bottom to top passing through the cavity center as

$$J_z = e^{-\left(\frac{t-t_0}{\tau}\right)^2}, \quad \tau = 150 \text{ ps}, \quad t_0 = 3\tau \quad (7)$$

The Courant limit time step size is $\Delta t_{CFL} = \frac{\Delta}{v_{\max}\sqrt{3}}$, where v_{\max} is the maximum phase velocity of wave propagation in the medium. We also denote $CFLN = \frac{\Delta t}{\Delta t_{CFL}}$. The observation point is located at cell (15, 15, 5).

Figure 2(a) plots the time-domain E_z field component computed using the conventional explicit FDTD method, ADI-FDTD method and leapfrog ADI-FDTD method for a lossless cavity. The CFLN for the conventional explicit FDTD method is 1, while the CFLN for the ADI-FDTD method and leapfrog ADI-FDTD method is 8. It is filled with a lossless ($\sigma = 0 \text{ S/m}$) medium of relative permittivity $\epsilon_r = 2$. It can be observed that the E_z field component of the leapfrog ADI-FDTD method resonants in a similar manner as the conventional explicit FDTD method and ADI-FDTD method.

Next, to ascertain the formulation of the leapfrog ADI-FDTD method for the lossy media, we chose the electric conductivity σ as 0.02 S/m , while retaining the same relative permittivity ($\epsilon_r = 2$). Figure 2(b) shows the time-domain E_z field component computed using the conventional explicit FDTD method, ADI-FDTD method and leapfrog ADI-FDTD method for a lossy cavity. As before, the CFLN for the conventional explicit FDTD method is 1, while the CFLN for the ADI-FDTD method and leapfrog ADI-FDTD method is 8. The lossy nature of the medium is clearly depicted by the decaying E_z waveform.

From Figure 2, we can also observe that the E_z field component of

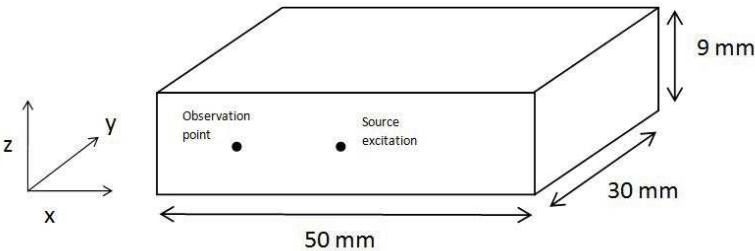


Figure 1. Simulation setup for PEC cavity.

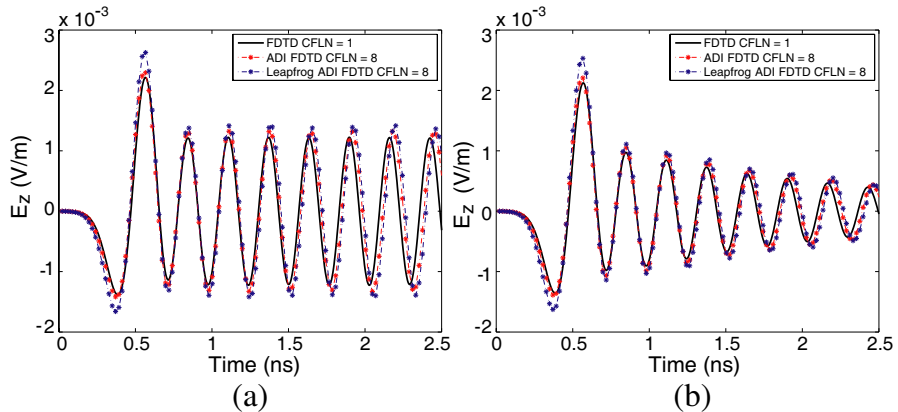


Figure 2. Time-domain E_z field component computed using conventional explicit FDTD, ADI-FDTD and leapfrog ADI-FDTD method for various CFLN. (a) Relative permittivity ϵ_r is 2 and electric conductivity σ is 0 S/m. (b) Relative permittivity ϵ_r is 2 and electric conductivity σ is 0.02 S/m. The lossy nature of the medium is clearly depicted by the decaying E_z waveform.

the leapfrog ADI-FDTD method differs slightly from the conventional explicit FDTD method and ADI-FDTD method during the source excitation period at time instant $t = 0.5 \text{ ns}$. This is due to the different source excitation scheme of the various FDTD methods. Figure 3 plots the time-domain E_z field component computed using the conventional explicit FDTD method, ADI-FDTD method and leapfrog ADI-FDTD method for CFLN = 1. It is clear that the various methods agree very well for CFLN = 1, and the discrepancies due to the different source excitation schemes are not longer visible. As mentioned previously, the leapfrog ADI-FDTD method is a fully implicit FDTD method. Therefore, the current source of the leapfrog ADI-FDTD method is excited implicitly at half time step.

For a practical example, we consider an improvised explosive device (IED) buried into a ground filled with soil. IED is a threat to humans as they may explode and incur injuries and even death to surrounding personnels. High power microwaves (HPM) may be employed to neutralize such threats by overloading the electronics of the the IED with high intensity electric fields [18]. Here, we examine the possible effects of such microwaves on an IED buried in soil.

The simulation setup for the IED buried in a ground filled with soil is shown in Figure 4. The IED is modeled as a rectangular PEC box with $\sigma = 1\text{E}300 \text{ S/m}$ and it is buried 5 cm below the surface of

the soil. The relative permittivity ϵ_r of the soil is 4.4, while the loss tangent is 0.046 [19]. Hence, the electric conductivity σ of the soil is 0.0338 S/m.

The source is excited in freespace by a sinusoidal modulated Gaussian pulse given by

$$J_z = \sin [2\pi f_0 (t - t_0)] e^{-\left(\frac{t-t_0}{\tau}\right)^2} \quad (8)$$

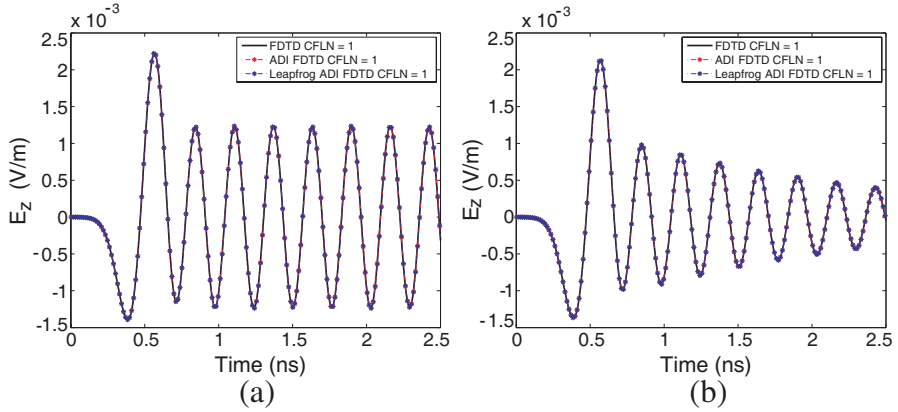


Figure 3. Time-domain E_z field component computed using conventional explicit FDTD, ADI-FDTD and leapfrog ADI-FDTD method for CFLN = 1. (a) Relative permittivity ϵ_r is 2 and electric conductivity σ is 0S/m. (b) Relative permittivity ϵ_r is 2 and electric conductivity σ is 0.02 S/m.

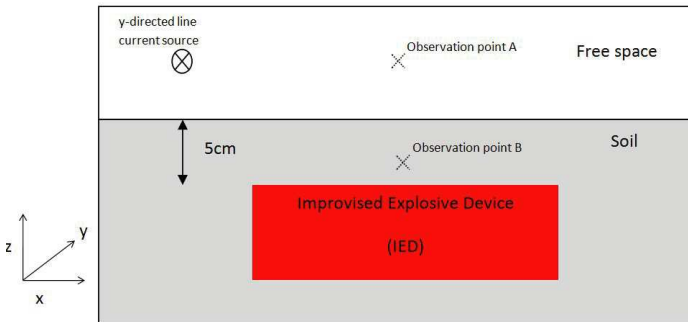


Figure 4. Simulation setup for improvised explosive device (IED) buried in ground filled with soil.

where $f_0 = 3\text{ GHz}$, $\tau = 0.5\text{ ns}$ and $t_0 = 3\tau$. The computation domain has a dimension of $70 \times 80 \times 80$ grids with uniform cell size of $\Delta = \Delta x = \Delta y = \Delta z = 2.5\text{ mm}$. It is truncated by a PML of 10 cell thickness [17, 20].

Figure 5(a) presents the received E_z field component located in freespace at observation point A (51, 40, 60), while Figure 5(b) shows the received E_z field component in the soil. It is located one cell above the IED at observation point B (35, 40, 36). The E_z field components in the two figures propagate almost the same distance away from the source, but in different media. Note the difference in scales between the two figures.

By comparing Figures 5(a) and 5(b), it can be observed that the electric field component in the soil has lower magnitude than the electric field component in freespace. There are mainly two physical mechanism involved in this phenomena. Firstly, reflection occurs at the freespace-soil interface due to the difference in refractive index of the two media. Therefore, only a portion of the microwave would penetrate into the soil. Furthermore, such reflection is a function of incidence angles. For safety and practical reasons, the distance between the HPM source and the IED is usually quite large. Consequently, the incidence angle is large and reflection at the material interface is usually not negligible.

Secondly, the microwaves that penetrate into the soil are further attenuated due to the lossy nature of soil. Along the propagation in the lossy soil, power is absorbed by the soil and not by the IED. Note the IED is only buried quite shallow in the soil in this example. Thus,

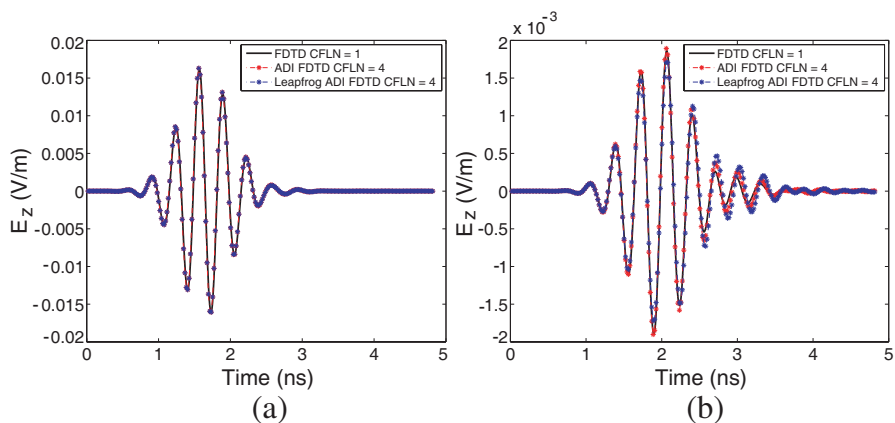


Figure 5. (a) Received E_z fields component in freespace. (b) Received E_z field component in soil.

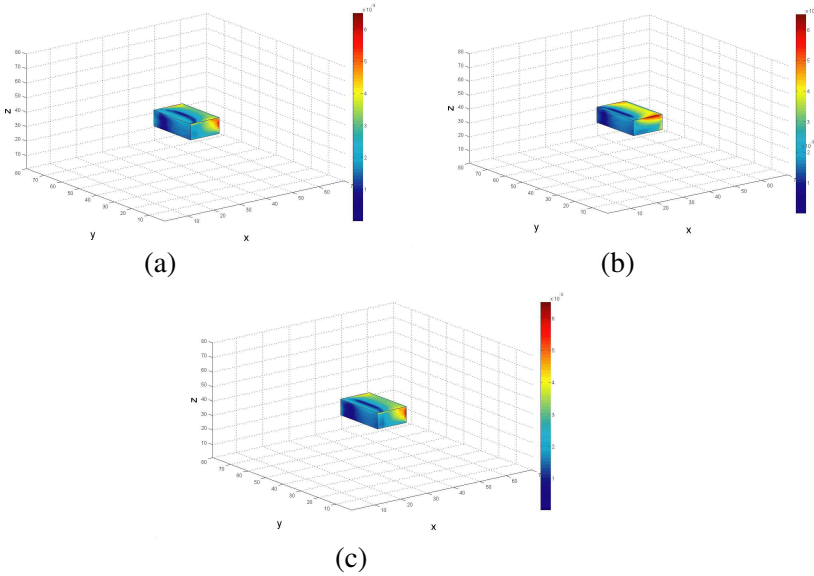


Figure 6. Magnitude of electric current density induced on the surface of the IED. (a) Conventional explicit FDTD method for CFLN = 1. (b) ADI-FDTD method for CFLN = 4. (c) Leapfrog ADI-FDTD method for CFLN = 4.

HPM systems might be more effective in neutralizing IED threats in freespace, rather than IED buried in soil or in general lossy media.

Figure 6 shows the magnitude of the electric current density induced on the surface of the IED for the conventional explicit FDTD method, ADI-FDTD method and leapfrog ADI-FDTD method. Fairly good agreement can be seen for the various methods. The maximum surface electric current density for the conventional explicit FDTD method, ADI-FDTD method and leapfrog ADI-FDTD method is $5.6\text{E-}3$ A/m, $6.0\text{E-}3$ A/m and $6.2\text{E-}3$ A/m, respectively.

4. CONCLUSION

This paper has presented an unconditionally stable leapfrog ADI-FDTD method for lossy media. Conductivity terms of lossy media have been incorporated into the leapfrog ADI-FDTD method in an analogous manner as the conventional explicit FDTD method since the leapfrog ADI-FDTD method is a perturbation of the conventional explicit FDTD method. Implementation of the leapfrog ADI-FDTD

method for lossy media with special consideration for boundary condition has been provided. Numerical results and examples have been presented to validate the formulation. This formulation might also suggest how we can readily extend the leapfrog ADI-FDTD method to model dispersive media as a future work.

The leapfrog ADI-FDTD method for lossy media can be used to efficiently model complex structures with practical material parameters. This is achieved by properly defining the conductivity terms of the media at the required locations in the computational domain [21].

APPENDIX A. DISCUSSION ON THE DERIVATION OF THE LEAPFROG-ADI FDTD METHOD FOR LOSSY MEDIA

In this section, we discuss the derivation of the leapfrog ADI-FDTD method for lossy media, by following strictly the previous derivation of the leapfrog ADI-FDTD method from the conventional ADI-FDTD method for lossy media.

For simplicity, we assume a medium with permittivity ϵ , permeability μ and electric conductivity σ . The magnetic conductivity σ^* is not considered here, but may be appended accordingly, if necessary. The update equation for the E_x field component of the conventional ADI-FDTD method in the first procedure for electrically lossy media is given by

$$E_x^{n+\frac{1}{2}} = \frac{\left(1 - \frac{\Delta t \sigma}{4\epsilon}\right)}{\left(1 + \frac{\Delta t \sigma}{4\epsilon}\right)} E_x^n + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t \sigma}{4\epsilon}\right)} \left(\partial_y H_z^{n+\frac{1}{2}} - \partial_z H_y^n\right) \quad (\text{A1})$$

where time-averaging is adopted. Likewise, the update equation for the H_z field component of the conventional ADI-FDTD method in the first procedure can be expressed as

$$H_z^{n+\frac{1}{2}} = H_z^n + \frac{\Delta t}{2\mu} \left(\partial_y E_x^{n+\frac{1}{2}} - \partial_x E_y^n\right) \quad (\text{A2})$$

The implicit update equation for E_x can be obtained by substituting (A2) into (A1) as

$$E_x^{n+\frac{1}{2}} = \frac{\left(1 - \frac{\Delta t \sigma}{4\epsilon}\right)}{\left(1 + \frac{\Delta t \sigma}{4\epsilon}\right)} E_x^n + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t \sigma}{4\epsilon}\right)} \partial_y \left[H_z^n + \frac{\Delta t}{2\mu} \left(\partial_y E_x^{n+\frac{1}{2}} - \partial_x E_y^n\right) \right] - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t \sigma}{4\epsilon}\right)} \left(\partial_z H_y^n\right) \quad (\text{A3})$$

and after some manipulations, the implicit E_x update equation for the conventional ADI-FDTD method for the first procedure reads

$$E_x^{n+\frac{1}{2}} - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \frac{\Delta t}{2\mu} \partial_y^2 E_x^{n+\frac{1}{2}} = \frac{\left(1 - \frac{\Delta t\sigma}{4\epsilon}\right)}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} E_x^n + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_y H_z^n - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \frac{\Delta t}{2\mu} \partial_y \partial_x E_y^n - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_z H_y^n \quad (\text{A4})$$

To formulate the leapfrog ADI-FDTD method for lossy media, we first consider the second procedure of the conventional ADI-FDTD method, but at the previous time step. The E_x update equation can then be written as

$$E_x^n = \frac{\left(1 - \frac{\Delta t\sigma}{4\epsilon}\right)}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} E_x^{n-\frac{1}{2}} + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \left(\partial_y H_z^{n-\frac{1}{2}} - \partial_x H_y^n \right) \quad (\text{A5})$$

while the H_z update equation can be expressed as

$$H_z^{n-\frac{1}{2}} = H_z^n - \frac{\Delta t}{2\mu} \left(\partial_y E_x^{n-\frac{1}{2}} - \partial_x E_y^n \right) \quad (\text{A6})$$

Next, we substitute (A6) into (A5) and obtain

$$E_x^n = \frac{\left(1 - \frac{\Delta t\sigma}{4\epsilon}\right)}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} E_x^{n-\frac{1}{2}} + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_y \left[H_z^n - \frac{\Delta t}{2\mu} \left(\partial_y E_x^{n-\frac{1}{2}} - \partial_x E_y^n \right) \right] - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_z H_y^n \quad (\text{A7})$$

Finally, we substitute (A7) into the implicit tridiagonal E_x update equation (A4) of the first procedure to arrive at

$$E_x^{n+\frac{1}{2}} - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \frac{\Delta t}{2\mu} \partial_y^2 E_x^{n+\frac{1}{2}} = \frac{\left(1 - \frac{\Delta t\sigma}{4\epsilon}\right)}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \left\{ \frac{\left(1 - \frac{\Delta t\sigma}{4\epsilon}\right)}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} E_x^{n-\frac{1}{2}} + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_y H_z^n - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \frac{\Delta t}{2\mu} \partial_y^2 E_x^{n-\frac{1}{2}} + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \frac{\Delta t}{2\mu} \partial_y \partial_x E_y^n - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_z H_y^n \right\} + \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_y H_z^n - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \frac{\Delta t}{2\mu} \partial_y \partial_x E_y^n - \frac{\frac{\Delta t}{2\epsilon}}{\left(1 + \frac{\Delta t\sigma}{4\epsilon}\right)} \partial_z H_y^n \quad (\text{A8})$$

Referring to (A8), it is clear that the mix partial derivative ($\partial_y \partial_x E_y^n$) terms do not cancel off due to the additional (non-unity) coefficient $\frac{1 - \frac{\Delta t\sigma}{4\epsilon}}{1 + \frac{\Delta t\sigma}{4\epsilon}}$. This cancellation is essential for the formulation of the leapfrog ADI-FDTD method. Thus, the aforementioned derivation in this section does not directly leads to the leapfrog ADI-FDTD method for lossy media.

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